**Complex Analysis**
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**Module - 3**
**Complex Integrations Theory**
**Lecture - 3**
**Introduction to Cauchy's Theorem**

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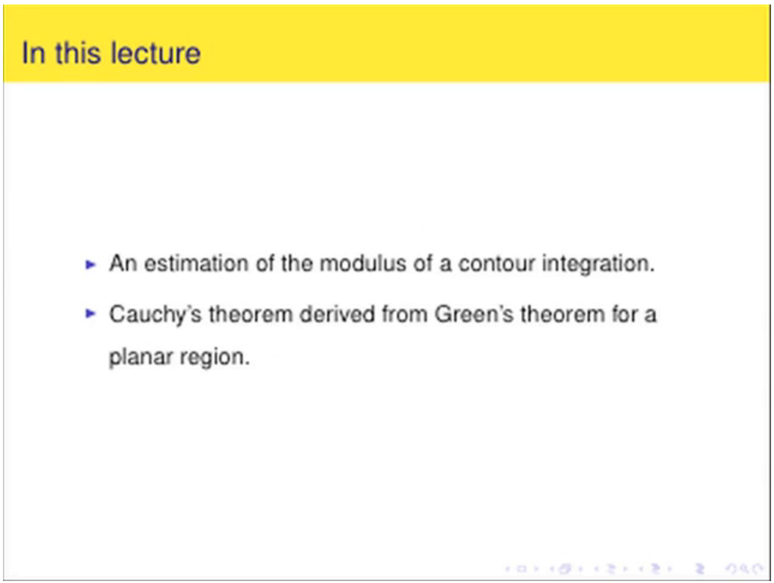


Hello viewers, in this session we will we will actually state and prove some estimation theorems, which will be useful for us to prove Cauchy's theorems. So, firstly recall from the last session, the fundamental theorem of calculus complex functions, which said that the anti derivative of for a function if it exists on all of the region, then the integration around a simple closed curve, which is contained in the region is equal to 0.

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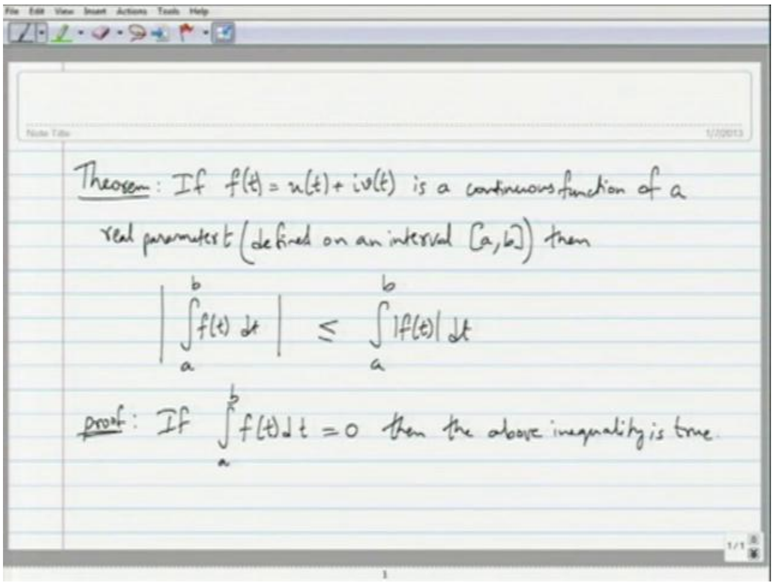


So, I will start with the following theorem, estimation theorem. So, f of t is u of t plus i times v of t. So, if it is written as its real and imaginary parts that way, is a continuous function of a real parameter t over defined on an interval a comma b. Let us say then the integration, so this is, I will put this in parenthesis. So, a b is in the domain of definition. So, then we have the following estimate, so integration from a to b f of t d t is less than or equal to the integration from a to t of the modulus of f of t d t.

So, at times when it is a difficult actually compute the complex integration itself, the contour integration itself, due to the complexity of the function involved or of or due to the complexity of the curve involved, such estimates are useful. We will put them to regular use and here is one estimation theorem, which tells that the modulus of integration from a to b of f of t d t is less than or equal to the integration of the modulus of f of t from a to b d t. So, here is a proof of this theorem. So, if integration from a to b f of t d t is 0.

So, this is one case suppose that this integration is equal to 0, then clearly the inequality is true. Then inequality, the above inequality is true because on the right have side. What you have is, the integrand is always a positive number. The modulus of a complex number f of t well its non negative number always. So, when you integrate a non negative number from a to b that is going to be greater than or equal to 0. So, if the integral itself is 0, then the inequality holds.

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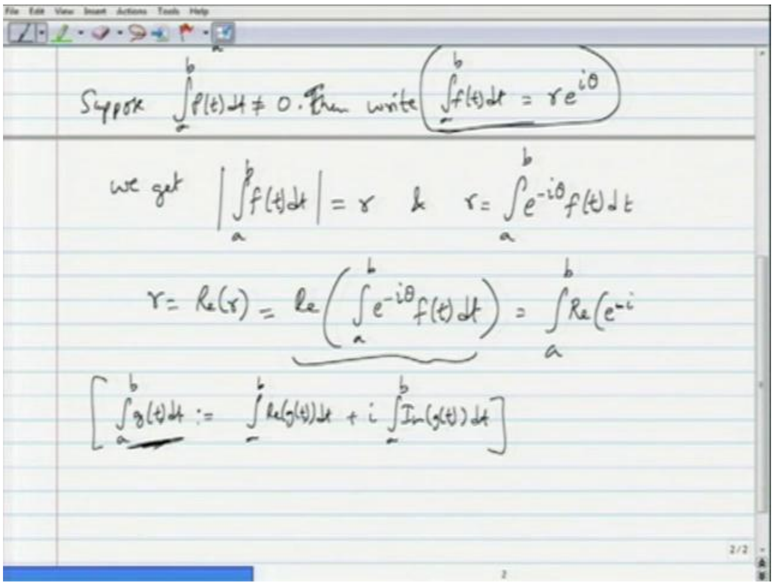


Now, suppose otherwise, so now suppose integration from a to b. So, suppose that integration from a to b of f of t d t is a non zero. So, then what we can do is, then we can write then write integration from a to b f of t d t, which is a complex number. This definite integral is a complex number write this as r e power i theta, where r is its modulus and theta is its argument. We get the modulus of integration from a to b f of t d t is equal to r and are hence is equal to integration from a to b e power minus i theta f of t d t because of this writing here.

So, since r e power i theta is this integral, you can multiply on both sides by e power minus i theta and push the e power minus i theta into the integrand because after all the integration is free of theta. So, you can push the e power minus i theta into the integrand to get this expression. Now, what we also know about this integral is as follows, so r is equal to of course, the of r itself, which is the real part now of this integral a to b e power minus i theta f of t d t.

So, r is this, so I am substituting r is this. We also know by the definition of this integral itself, so you will recall that the way this integral is defined integration from a to b sum g of t d t, what is this? This is nothing but a integration from a g to b real part of g of t d t plus i times integration from a to b, the imaginary part of g of t d t. This is how, we have defined this integral. This is a complex integral, so this so this can be written as the real part of this, is nothing but the integration from a to b of the real part of the function inside e power minus i theta f of t. So, the integrand here is the real part of e power minus i theta f of t d t.

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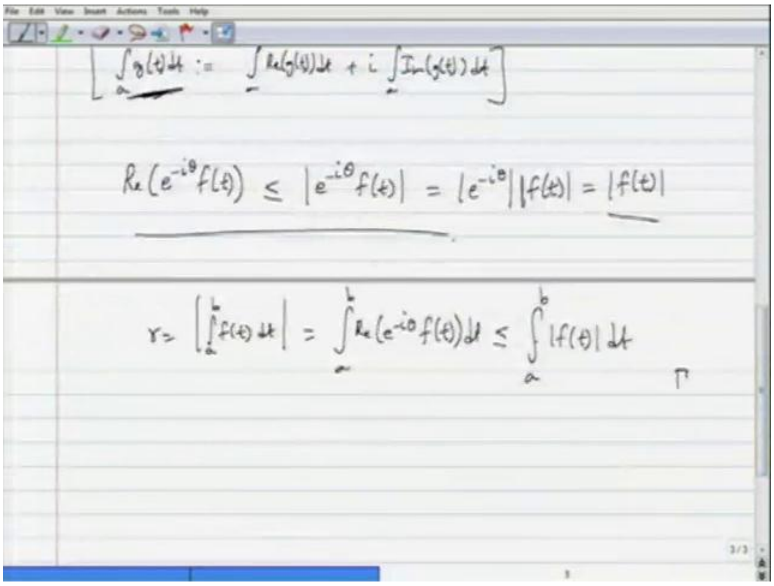

So, let makeshift arrangement isolate the integrand real part of e power minus i theta f of t. This is less than or equal to of course, the modulus of that complex number itself depending on t. So, this is less than or equal to the modulus of e power minus i theta f of t because the real part of a number is always less than or equal to its modulus. This in turn is equal to the modulus of e power minus i theta times the modulus of f of t, which is equal to the modulus of f of t, because the modulus of e power i minus i theta is always 1, so independent of theta.

So, this allows us to write this integral r is equal to integration, what is r? r is from here, r is what we want on the left hand side. So, r is equal to integration the modulus of integration from a to b of f of t d t. This is equal to this quantity. So, this is equal to the integration from a to b of the real part of e power minus i theta f of t d t, which is now by this estimate here is less than or equal to the integration from a to b of the modulus of f of t d t. This is what we want, so that shows that proves these prepositions. So, that is an estimate that we are going to use from time to time. These are simple theorem; so for example, we are going to put this to use in the very next proposition. So, is another estimation theorem; so here is another estimation theorem.

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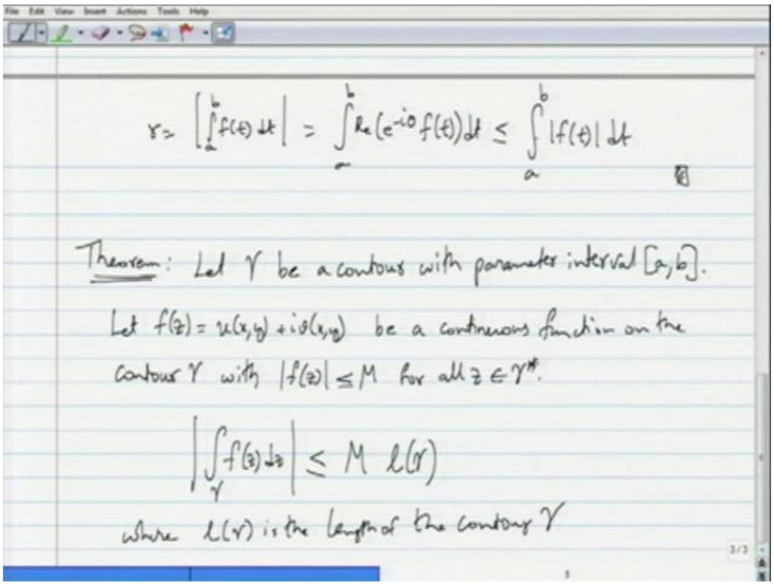


It is in this form that this theorem is mostly useful; the previous theorem is mostly useful. So, let gamma be a contour, with a parameter interval a b. Let f of z is equal to u of x y plus i times v of x y with the usual agreement at z is x i y. Let this be a continuous function be a continuous function on the contour gamma with the additional constrain that the modulus of f of z is less than or equal to m for for all z belongs to gamma star.

Whenever, z is on the trace of gamma, suppose that the modulus of f of z is less than or equal to m. So, with this condition the modulus of the integration over gamma the contour integration of f of z over gamma the modulus of that is less than or equal to m times l of gamma, where l of gamma is the length of the contour gamma given by integration from a to b of gamma prime of t modulus d t.

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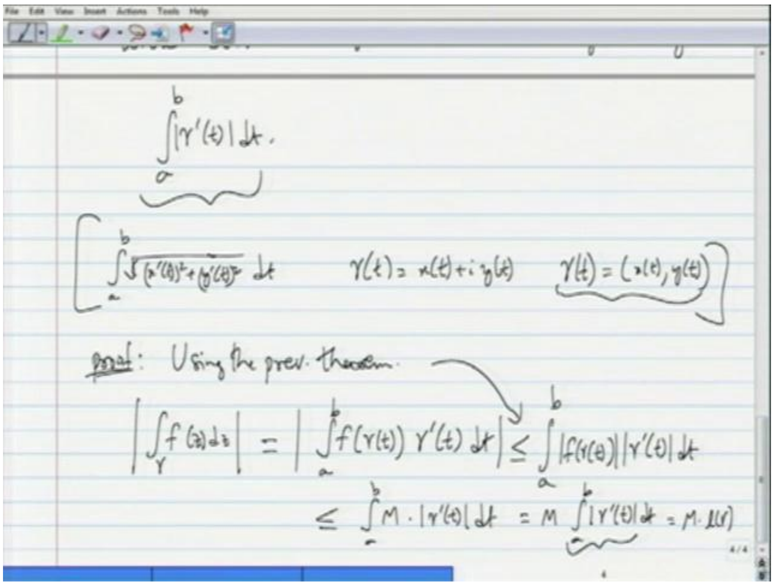

So, if you have a continuous complex value function defined on well its continuous function on the trace of gamma, then the integration, the contour integration of f on that contour is less than or equal to the maximum value of f or the or the bound of f the modulus of f times, the length of the curve gamma itself. So, this this length of gamma, this must be familiar to the viewer from multivariable calculus. So, recall that this is nothing but integration from a to b of square root of x prime of t squared plus y prime of t squared d t, if I write gamma of t as x of t plus i times y of t.

So, the viewer might have seen this in the context of functions of two variable or curves in r 2. So, if gamma 2 gamma of t is x of t comma y of t, which is a function from r to r 2, then the length of gamma under appropriate conditions gamma being differentiable smooth etcetera the length of gamma can be computed using this integral. So, that is the exact length we are reusing here. It is nothing but in this context gamma prime the modulus of gamma prime and put this in parenthesis.

So, the length of gamma is integration of modulus of gamma prime and we have this estimate, which will be used from time to time. So, the proof is once again easy, so we will use the previous theorem. So, using the previous result using the previous theorem, what we have is that the modulus of the contour integration of f of z d z on f of z on gamma is less than equal to the modulus of integration from a to b. I am just writing spelling out the contour integration is...

So, this is integration from a to be of f of gamma of t of gamma prime of t d t. This is less than or equal to this, is actually an equality. This is equal to this and then this is less than or equal to integration from a to b, this uses the previous theorem. This is less than or equal to the modulus of integrand. So, the modulus of f of gamma of t times the modulus of gamma prime of t d t. This is less than or equal to well the modulus of f of gamma of t is always less than or equal to m.

Whenever well, is always less than or equal to m because gamma of t is a point on the trace. So, this is less than or equal to integration from a to b of m times modulus of gamma prime of t d t, but that is equal to m times integration from a to b of the modulus of gamma prime of t d t. This quantity is nothing but the length of gamma. So, this is equal to m times the length of the curve, the contour gamma. So, what we have shown is that, the contour integration of f on gamma is always less than or equal to m times l of gamma, where m is the bound on the absolute value of f on the contour gamma. So, it shows this theorem and this is a very useful estimate, we will see all right?

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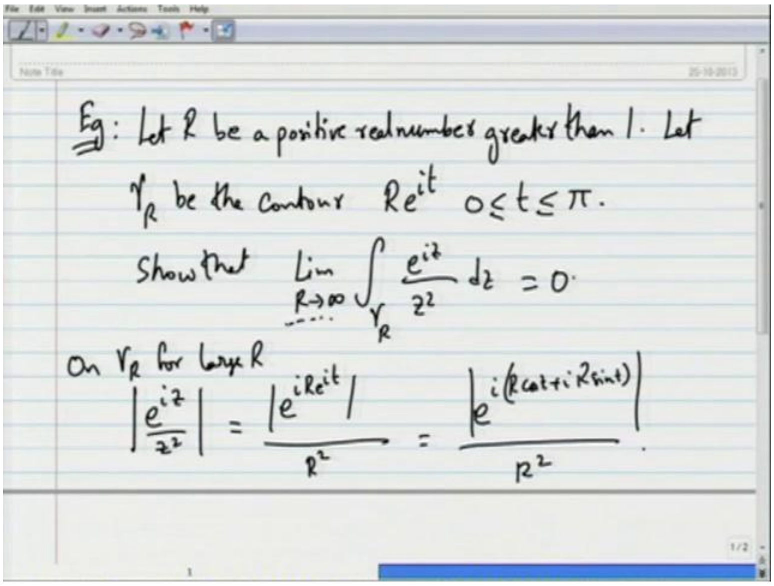


So, we will see one application of this particular estimate. So, here is an example which illustrates the use of the estimation theorem. So, examples is as follows, let r be a positive real number greater than 1. Let gamma r be the contour defined by r e power i t 0 less than or equal to t less than or equal to pi. So, that limit as r goes to infinity the integration over gamma r of e power i z pi z squared d z is equal to 0. So, what we will do is, we will take the function e power i z by z squared and we will estimate it on this given contour gamma r for r a large positive number, because ultimately we are interested in limit as r goes to infinity.

So, let us look at the modulus of this function e power i z by z squared. So, on gamma r for large r for large r on gamma r, what we have is, this is equal to e power i times. Well the contour is r e power i t the modulus of this and then of course, the modulus of r e power i t squared will be r squared in the denominator. So, this is equal to in the numerator, we have e power i r e power i t. I am going to write that as r cosine t plus i times r sine t in modulus divided by r squared.

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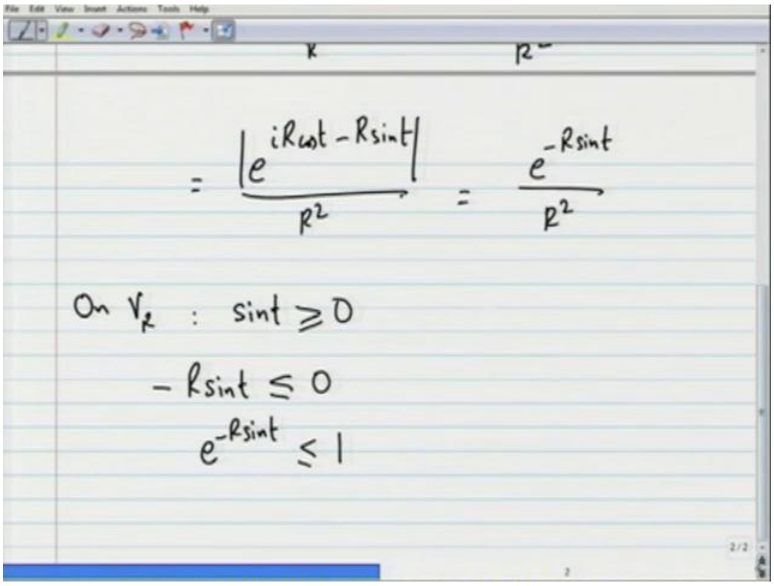


So, this is equal to e raised i r cosine t minus r sine t by r squared this is modulus. Of course, the modulus of e raised to something is e raised to the real part of that something. So, this is equal to e raised to minus r sine t by r squared because the modulus of i e raised to i r cosine t is 1. So, now notice that on gamma r which r e power i t, t goes from 0 to pi, so on gamma r sine t is always positive or equal to 0, sine is positive in the first and second quadrants, so what we have is minus r sine t; whatever be the value of r, this is going to be less than or equal to 0. So, e raised to minus r sin t is less than or equal to e raised to 0, which is 1 that is because the real exponential function is a strictly increasing function.

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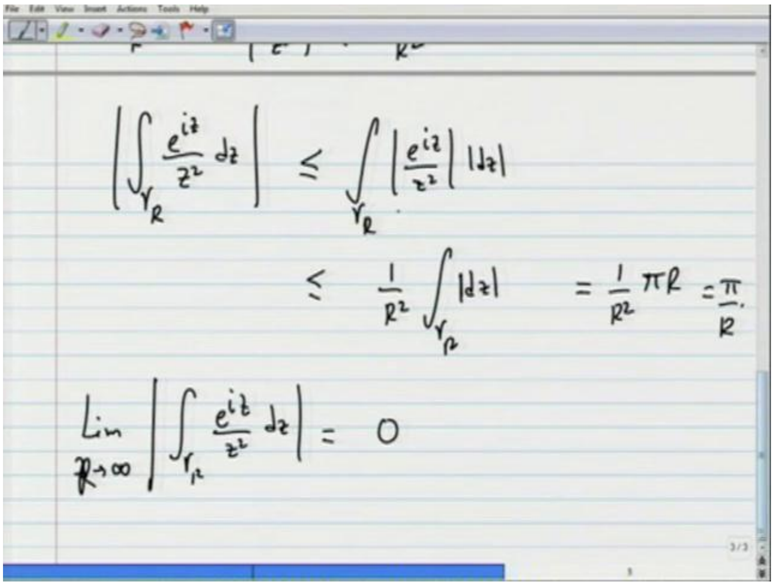

So, here this expression tells us that the modulus, so on gamma r for large r the modulus of e power i z by z squared is less than or equal to 1 by r squared. So, we will use this along with the estimation theorem. So, the integration the modulus of integration over gamma r of raised to i z by z squared d z by the estimation theorem, is less than or equal to integration over gamma r of the modulus of e power i z by z squared times mod d z, which is less than or equal to 1 by r squared on gamma r. This is bounded by 1 by r squared integration of over gamma r of mod d z, which we know is the length of the curve gamma r.

So, since gamma r is a semicircle of radius r, so this is less than or equal to or this is equal to 1 by r squared times pi r. This gives us pi by r. So, from here we know that limit as z goes to infinity of the modulus of integration over gamma r e power i z by z squared d z is equal to limit as z r goes to sorry, I apologise. This is r goes infinity, limit as r goes to infinity of pi by r, which is equal to 0. So, since if limit as r goes to infinity of some complex numbers is 0, then it has to be that the limit of those complex numbers is also 0. So, we conclude that limit as r goes to infinity of e power i z integral over gamma r of e power i z by z squared d z has to be equal to 0. So, this completes this example with these estimation theorems. What we will now do is that we will look at Cauchy's theorem, which is the fundamental theorem in complex analysis. So, it is it is central to complex analysis of functions of one variable.

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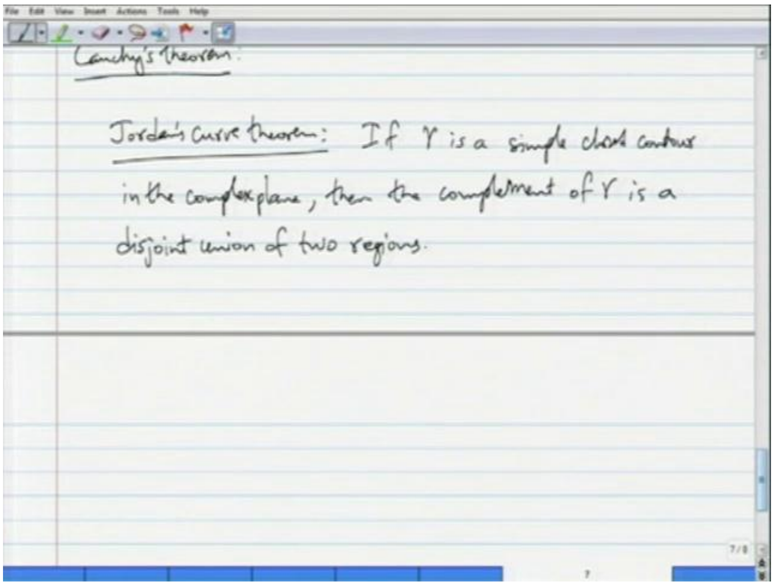


So, here is Cauchy’s theorem and a discussion of it. So, firstly before I state one or several versions of this theorem, I want to introduce the concept of the inside of a curve. So, here is here I will quote the Jordan’s curve theorem, so Jordan’s curve theorem. What it says is that if gamma, so I will as I said confuse between a contour and its trace in the complex plane. So, if gamma is a simple closed contour in the plane in our case in the complex plane, then the complement is a disjoint union of two regions. Recall that a region is a connected open side. So, the two regions are disjoint, they are disconnected by gamma and and that is the Jordan’s curve theorem for our purposes. So, intuitively this is very clear.

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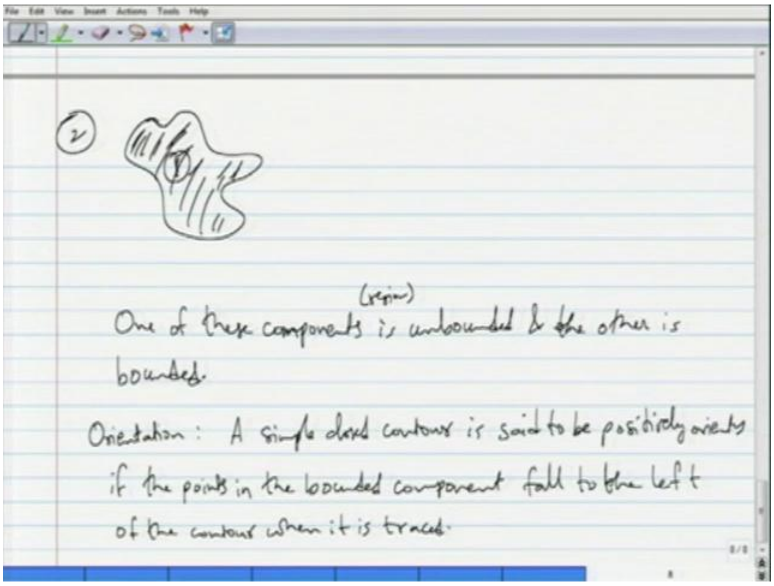


If you draw any simple minded curve like that simple closed contour like that. So, then there is this portion in the complement of this in the complex plane, you have this portion which we will call 1 and then that is called that will be the other region 2. One of these components is unbounded and one and the other is bounded component is the same as the region is a bounded.

So, then we will define orientation of of these regions of these curves. So, orientation in two ways, one is positive orientation and negative orientation. So, a curve, so a simple closed contour is said to be positively oriented. If the points on the inside on the bounded component or in the bounded component, in the bounded component fall to the left of the contour when it is traced. So, intuitively this is clear, once again.

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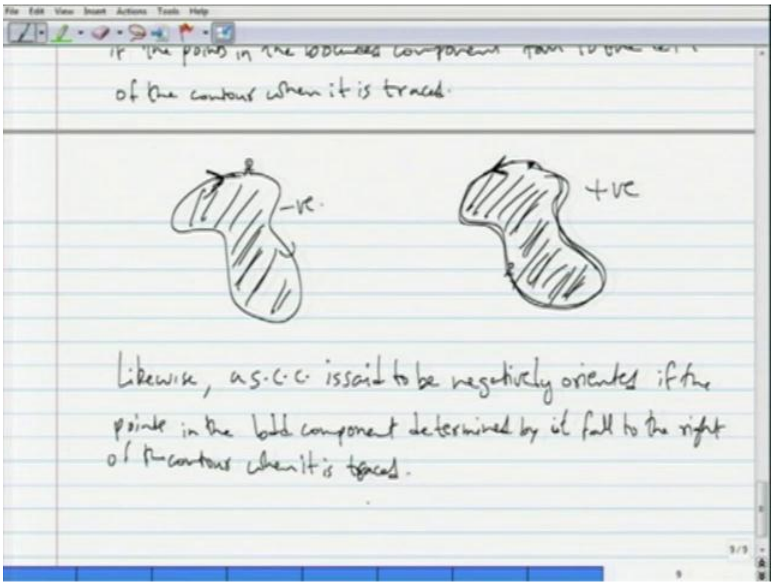


So, here is a contour and suppose this is the, this is the orientation on the contour determined by its parameter, parameterization. So, then you notice that when you keep tracing the the contour like that, imagine yourself walking on that contour then this bounded component appears to the left of you as you traverse the contour. All the points will appear always to the left whereas, if you if you have the other orientation of the same contour, namely...

Suppose, that the contour is oriented in that fashion, then the point in the bounded component will appear to the right, when you walk on this contour by the by the given orientation. So, this will be called a positive orientation of the contour and this will be called the negative orientation. So, this we will use later, but the, but this is what we will mean by positively oriented and negatively oriented simple closed contours. So, likewise likewise, likewise I will say that a simple closed contour is said to be negatively oriented, if the points in the bounded component determined by it it determines the bounded component, right?

So, if the points in the bounded component determined by it fall to the right of the contour, when it is traced so that is negatively and positively oriented simple closed contour. So, what we need now is that this this bounded component will always will will be called the inside of this a simple closed curve and this unbounded component will be called the outside of the exterior of the simple closed curve. So, the bounded component has a name by the Jordan's curve theorem. There are bounded and unbounded components.

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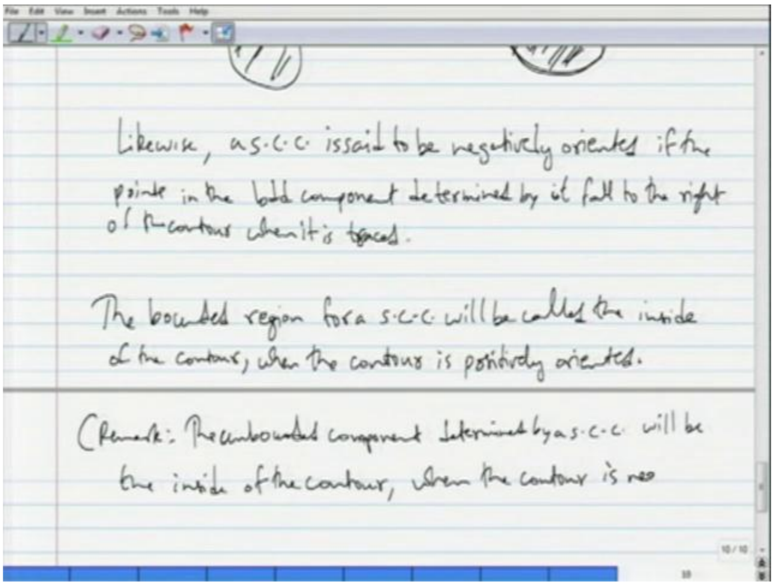


So the bounded region for a simple closed curve contour will be called the inside of the curve of the contour. When the contour is, this is important when the contour is positively oriented. So, only if the contour is positively oriented, so the unbounded component determined by the simple closed curve will become the inside, then then the contour is a negatively oriented. So, so just remark so here is a remark, the unbounded component. This is not, this is relatively unimportant, but I will state it, I will remark the following. So, the unbounded component determined by simple closed curve will be the inside of the contour, when the contour is negatively oriented with this.

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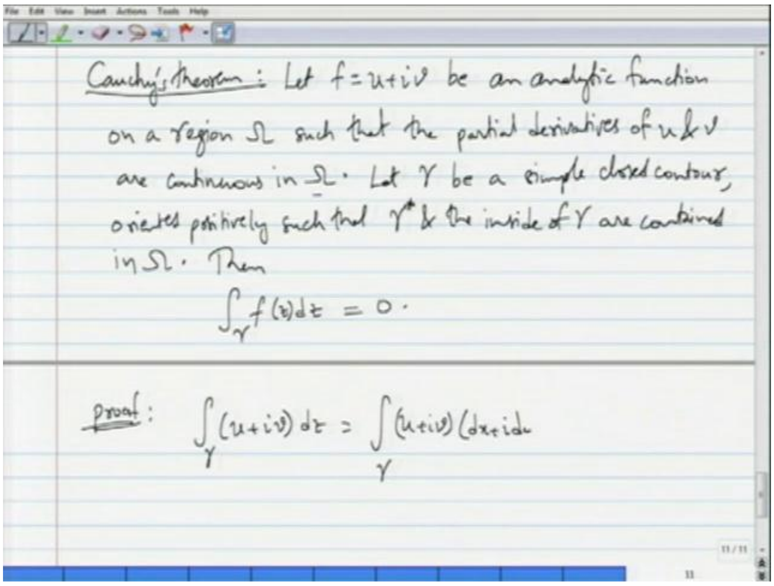


We will first see a very preliminary version of Cauchy's theorem and this is a direct consequence of Cauchy Riemann equations and and Greens theorem for a function, functions of two real variables. So, here is the statement, so let f equals u plus i v the I am writing f as its real and imaginary part like that u and i v be an analytic function on a domain or on a region. Open connected set recall a region is an open connected non empty set, on a region omega such that, so its analytic means its partial derivatives exist and the Cauchy Riemann equations are satisfied.

Now, let us also assume that such that, the partials, the partial derivatives of u and v are continuous in omega in all of omega. Now, if also I will say let let gamma be a simple closed contour oriented positively such that gamma and the inside of gamma. So, basically the trace of gamma and the inside of gamma are contained in omega. So, in that event the integration, then the integration over gamma of f of z d z the contour integration of f on gamma is equal to 0 and the proof of this version, where we assume that the partial derivatives are continuous in omega is easy its directly follows from Greens theorem. So, here is the proof, so the integration over gamma the contour integration over gamma of u plus i v d z at z d z is nothing but the integration over gamma of u plus i v. Well let us write z as x plus i, so that d z is d x plus i d y.

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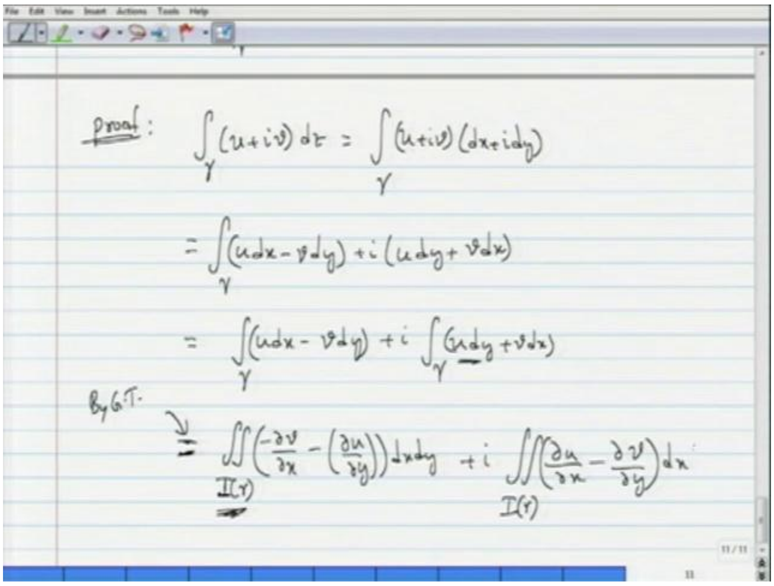

And this gives us integration over gamma u d x minus v d y upon multiplication plus i times u d y plus v d x. By the way, we define these integrals, these contour integrals this is nothing but integration over gamma of u d x minus v d y plus i times the integration over gamma of u d y plus v d x. Now, since we assume that u and v the partials of u and v exist and are continuous on omega. And and so being particular there continuous on the inside of gamma, we can apply Greens theorem by greens theorem.

This is a nothing but the double integral over closure of the inside of gamma of minus dou v by dou x minus dou u by dou y of k x times d x d y, when this inside is parameterized by x and y i times the double integral over the inside of gamma of dou v sorry, dou u by dou y or dou u by dou x. So, this is u d y so I will take its partial with respect to x minus dou v by dou y times d x d y.

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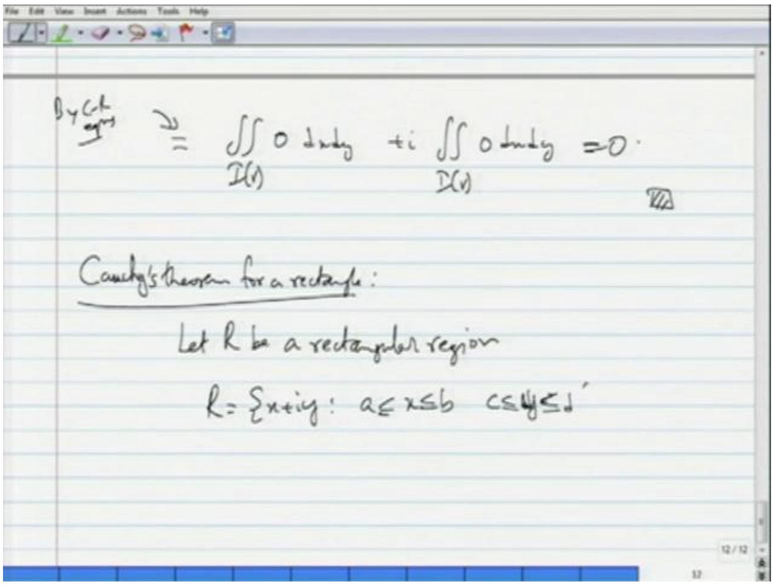


So, now one uses Cauchy Riemann equations since f is analytic at every point on inside inside this gamma on i of gamma. So, by Cauchy Riemann equations, you know that minus dou v by dou x is plus dou u by dou y. So, this is 0 and so likewise dou u by dou x is dou v by dou y, so this is 0 by the Cauchy Riemann equations. So, you have this is equal to double integral over i of gamma of 0 d x d y plus i times of gamma 0 d x d y, which gives you 0 by the Cauchy Riemann equations. And by the Greens theorem for regions, for plane regions you know that this contour integration is indeed 0.

So, this is the first version of Cauchy's theorem, which was known before before Goursat dropped mathematician by name Goursat dropped the assumption that u, the partial derivatives of u and v have to be continuous on the inside of gamma. So, we can indeed drop that assumption, so all we need is that f is analytic. If f is analytic on the region gamma and if gamma is a simple closed curve positively oriented, such that gamma star and inside of gamma are contained in omega. Then the contour integration of f is actually equal to 0.

So, that is Cauchy Goursat theorem. We will see that version of Cauchy's theorem in this statement. So, Cauchy's theorem, so this is the end of this proof Cauchy's theorem for a rectangle, r is x plus i y such that a less than or equal to x less than or equal to b c less than or equal to x y less than propellant equal to d. So, it is that kind of region in the complex plane.

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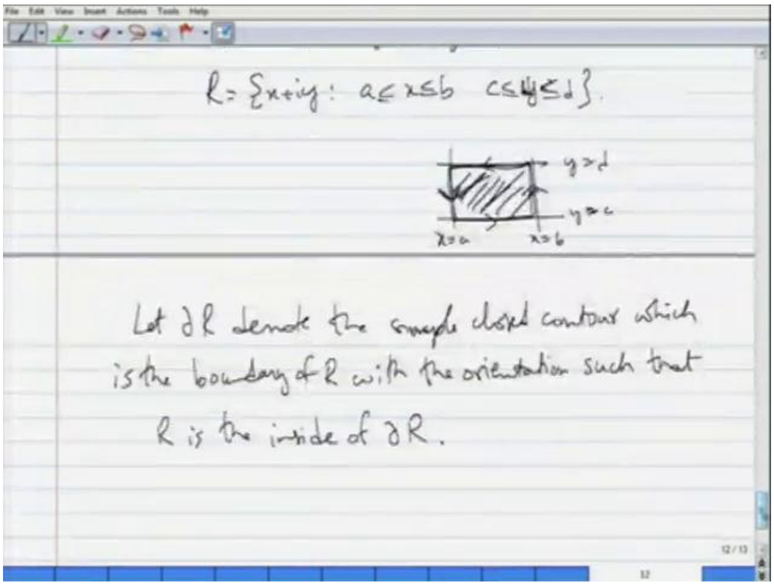


So, x equals a x equals b y equals c y equals d And let let dou r denote the the simple closed contour, which is the boundary of this region r. Namely this these four straight lines, which form a rectangle with the orientation, such that with the orientation, such that r is the inside of dou r. You consider the counter clockwise orientation on dou r, so that gives you, if you... When viewed from the top if you take the counter clockwise orientation on dou r then r becomes the inside of this contour dou r.

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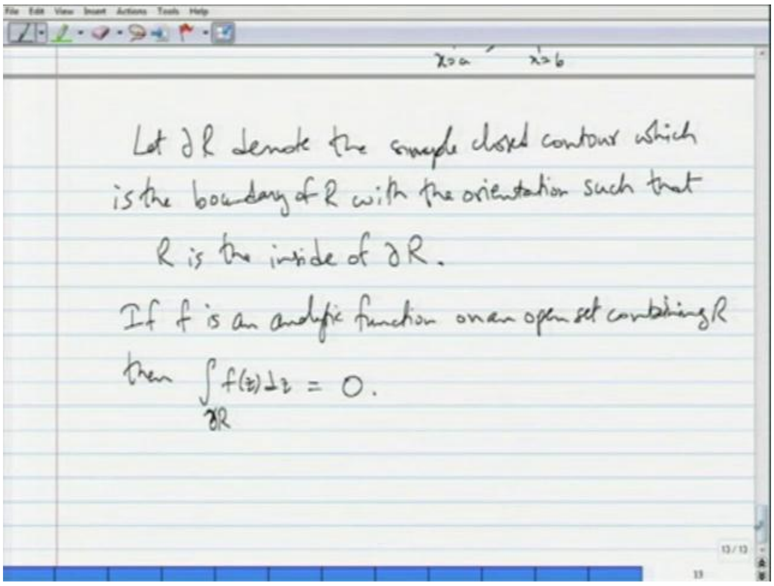

So, with this setup if f is an analytic function on an open set containing r then integration over gamma the contour integration or sorry, or dou r of f of z d z with the said orientation is equal to 0. So, notice that all we are assuming is that f is analytic and we are not demanding that the partial derivatives of the real and imaginary part be continuous on the inside. So, Cauchy’s theorem maintains that this is equal to 0, so we will prove this version of Cauchy’s theorem in the next session.