Complex Analysis
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Module - 3
Complex Integration Theory
Lecture - 2
Contour Integration

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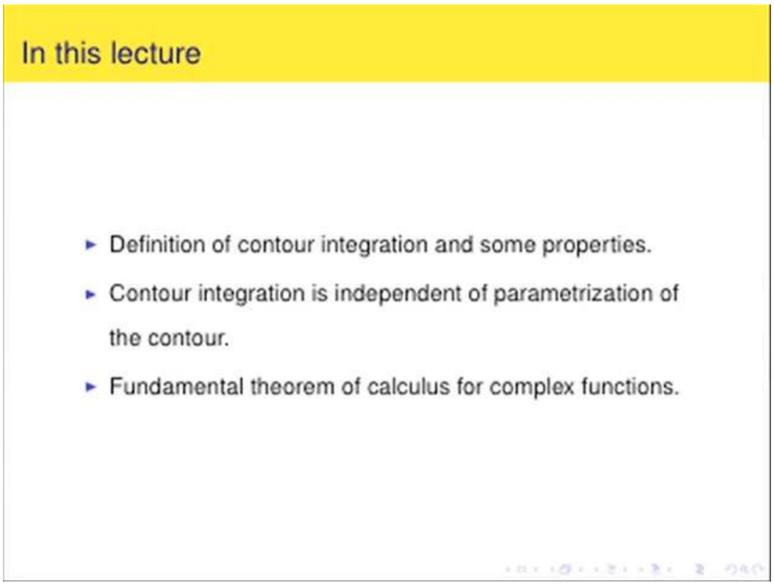

Hello viewers, we will discuss contour integration in the session. So, last time we defined what a contour is. It is a join of piecewise smooth curves over some parameter interval, and we are going to use those contours to define a line integral of some sort for complex valued functions. So, here is contour integration.

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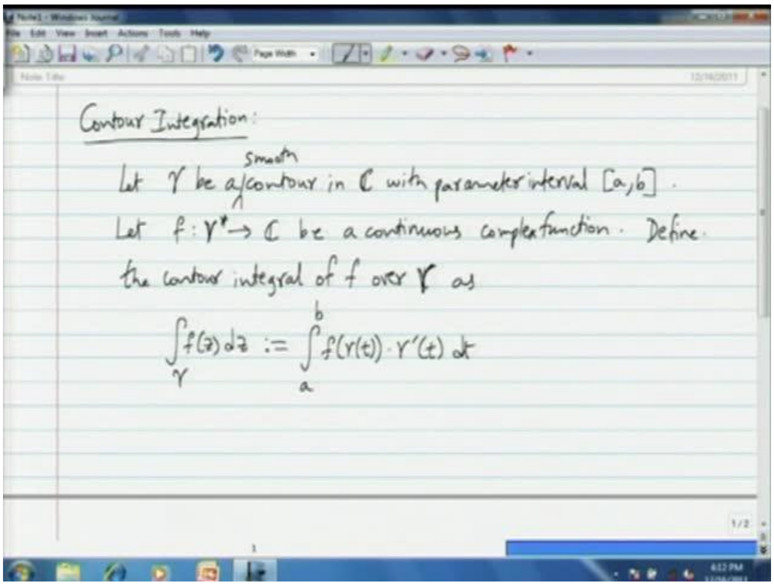

So, let gamma be a contour in C, with, in C, this is the complex plane C with a parameter interval a b; and let f from gamma star, remember that is the range of, the range set of the contour gamma. So, f from gamma star to C be a continuous complex function. So then define so the symbol, I will call this the contour integral of f over gamma as, well this is the notation the contour integration of f over gamma, is equal to the integration over this interval a b of f of gamma of t times gamma prime of t dt. The right hand side here is actually an integral of the sort we have seen in the last session, where f of gamma of t times gamma prime of t is essentially a complex valued function, in the, with a real parameter t. We have integrated such functions in the last session.

So the right hand side integral, in the right hand side integral gamma prime may not be defined at all places, so we will assume for this definition that gamma is this smooth contour. Please allow me to make the change at gamma be a smooth contour. If gamma is not smooth all over, if it is piecewise smooth, like we have in a general case, then we would define this integral to be summation over specific intervals and we break this interval a b into intervals where you know gamma is smooth.

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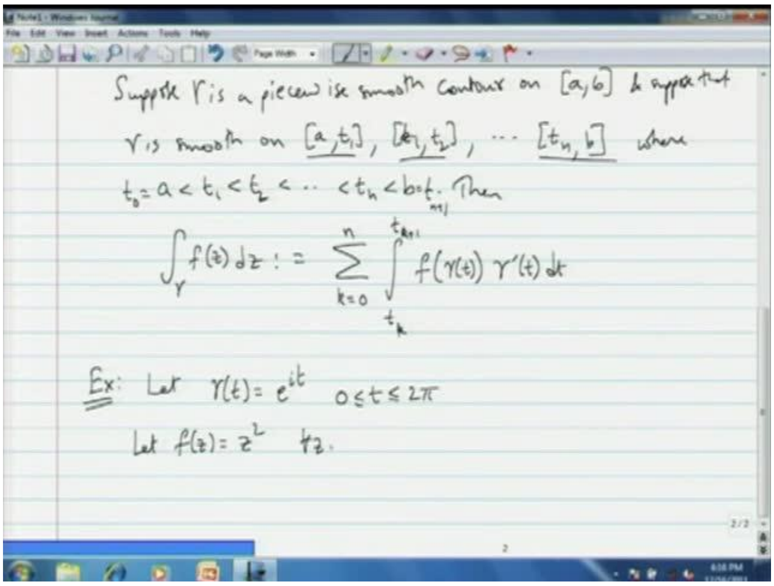

So what I mean by that is, suppose gamma is a piecewise smooth contour with the parameter interval a b, and suppose that there are n points. So, suppose that gamma is smooth on a t 1 comma t 1 t 2 etcetera t n comma b, where these points a strictly less than t1 strictly less than t 2, t n strictly less than b. So, this is how a piecewise smooth contour will look like; its smooth on pieces like that, pieces of this interval. Then, in this case define this integration, this contour integration f over gamma is defined as the summation, well I have written this yeah, so I will write t 0 to be a and t n plus 1 to be b for convenience.

So, then I can write summation k equals 0 to n plus 1 of the integration from t k. This summation will run only until n. t k to t k plus 1 of f of gamma of t, gamma prime of t dt. Now, we do not have problem because gamma prime is defined on all of these intervals, it is piecewise smooth so on each of these intervals gamma prime is smooth or gamma is smooth rather.

So, we can define the contour integration in a modified fashion like this for piecewise smooth contours. So let us see some examples. Let us see how to calculate contour integration, in some cases. So here is the first example. Let gamma of t be the curve, be the contour e power i t, 0 less than or equal to t less than or equal to 2 Pi. So this is the unit circle parameterised in that fashion, in the standard fashion, and let let f of Z equals Z squared, this is for all z. So, in particular this function is continuous on the range of the contour namely the unit circle, this is actually analytic function all over C. It is an entire function, so it is definitely continuous over this contour.

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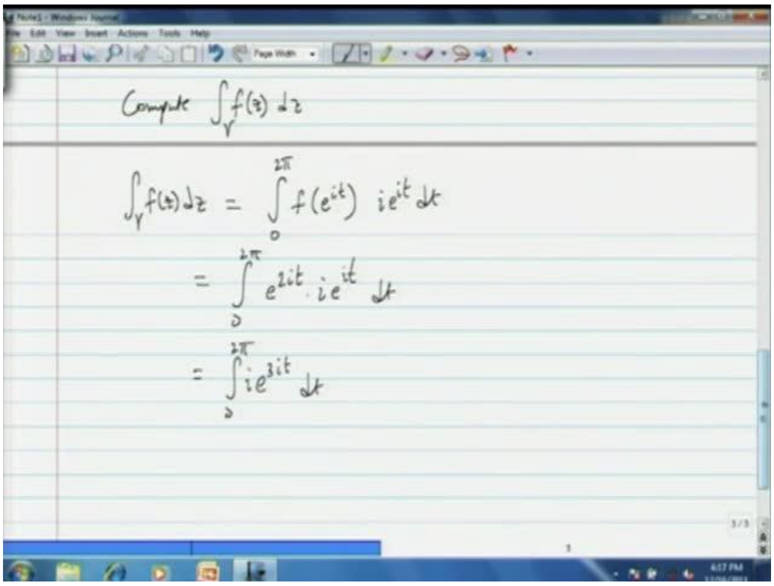

So we can compute the contour integration of f over the given contour. So, you can compute this. This by definition is the integration over this interval parametric interval 0 to 2 Pi of f of e power i t and gamma prime is i e power i t, the differentiation of e power i t is i e power i t and then we have dt. So, this gives us, well, f of e power i t, f is f of Z is Z squared. So, f of e power i t is e power i t squared, which is e power 2 i t times i e power i t dt.

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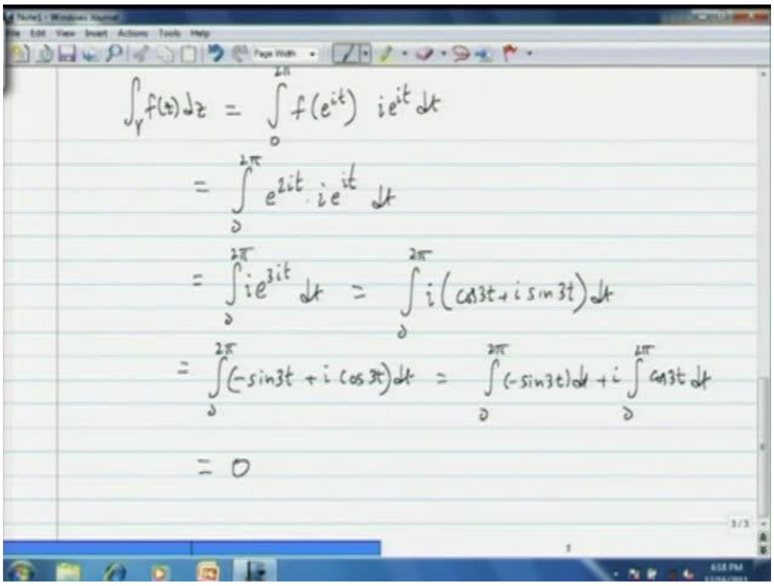

So, we get integration from 0 to 2 Pi of i e raise to 3i t dt, which I am going to write as integration from 0 to 2 Pi of i times Cos 3t plus i Sin 3t dt and multiply the i you get integration from 0 to 2 Pi of minus Sin 3t plus i times Cosine 3t dt. Now the integrand is a real parameter function and its complex value and we have seen how to integrate such a function, separate it into, it is integrated by separating it into the real and imaginary part. So you get integration from 0 to 2 Pi of minus Sin 3t dt plus i times integration from 0 to 2 Pi of Cos 3t dt, which is clearly 0. So, it is how we compute these integrals

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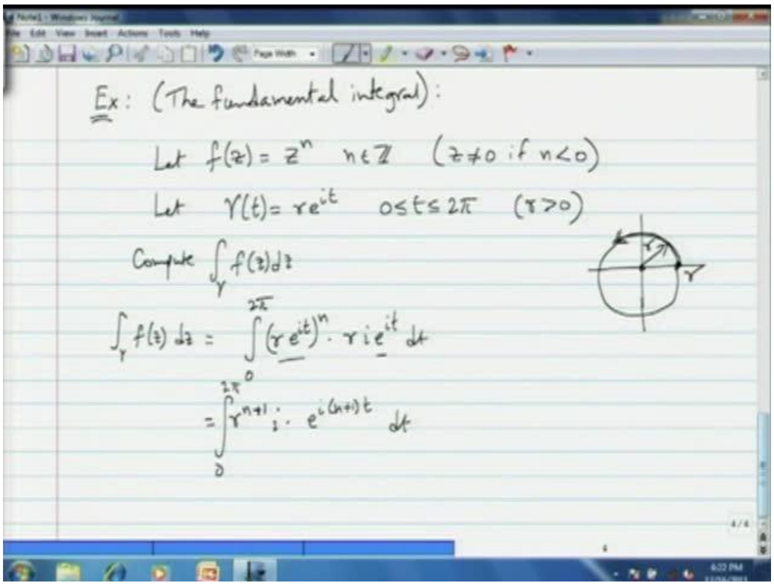

And there are, let us actually see one more example. So this is in some sense fundamental integral. This example is very important it keeps coming back to us again and again. We will use this very much, so it is an important example. Let f of z equal z power n. So essentially it is the previous example with z power n instead of Z squared. So, let f of z be z power n, n is any integer z not equal to 0 if n is strictly less than 0 but, that does not matter for us. Let gamma of t being your unit circle parameterised as before or we can actually take circle of radius r, where r is positive, 0 less than or equal to. Let me write this gamma of t is r e power i t 0 less than or equal t less than or equal to Pi.

So this is the circle of radius r; r is a positive real number. So compute. Well, since r is positive real number, f is definitely defined there on the range of gamma and also its continues there, it is actually analytic on gamma. So compute the contour integral of f over the contour gamma. So this is the integration of r raised to e power i t raise to n; z raise to n, z comes from the contour. So gamma of t raise to n and then times gamma prime of t, gamma prime is r i e raise to i t dt and the parameter interval is 0 to Pi. I apologise, this should be the circle 2 pi, 2 pi. So the contour visually is a circle of radius r centred at the origin it starts at r and ends at the same real number, it goes in the contour clockwise direction and ends with the same real number. The calculation is a similar. So you get r power n plus 1, r power n plus 1 integration from 0 to 2 Pi and then you have a i and then you have e power i times n plus 1 t dt, e power i n t from in here and then 1 e power i t from here.

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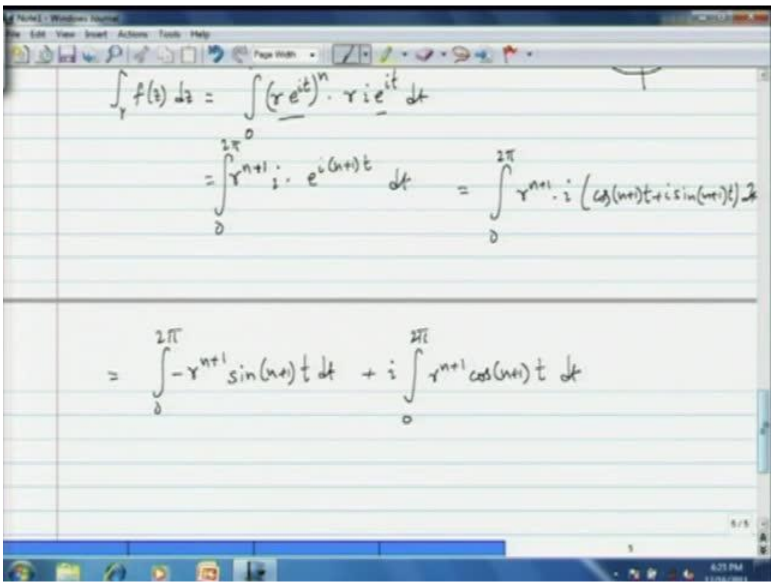

So you get a, what you get? Let me write e power i n plus 1 t as cos n plus 1 t plus i times Sin i plus 1 t, like before. so I get integration from 0 to 2 Pi of r power n plus 1 times i times Cos n plus 1 t plus i Sin n plus 1 t dt. So this gives me integration from 0 to 2 Pi and the real part is minus r power n plus 1 Sin n plus 1 t dt and then plus i times integration from 0 to Pi of r power n plus 1 Cosine n plus 1 t dt.

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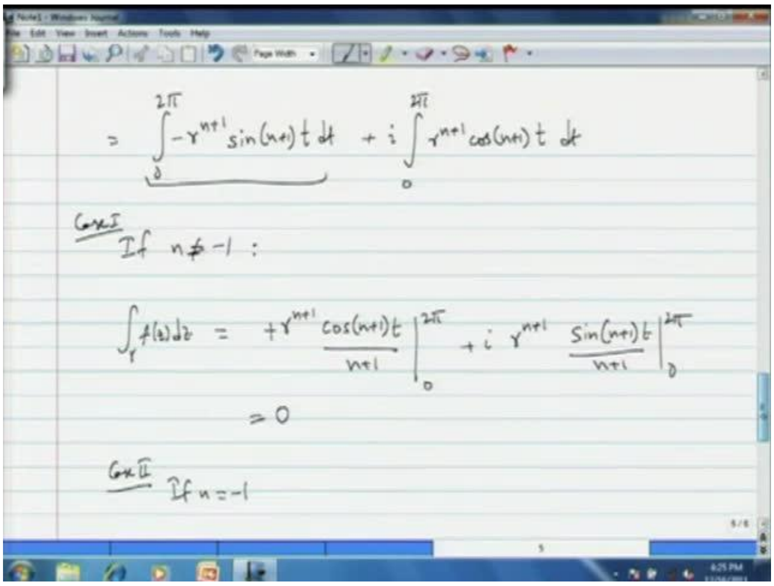

So, this is minus r power n plus 1. So I will split into the cases, if n is not equal to minus 1. So case 1: if n is not equal to minus 1, in this case you can integrate, you can integrate this piece to get minus r power n plus 1, the integration of Sin is negative Cosine, so plus Cosine n plus 1 t divided by n plus 1 between 0 and 2 Pi. That is clearly 0 plus i times, likewise, r power n plus 1 Sin integration of Cosine is Sin n plus 1 t divided by n plus 1 between 0 to 2 Pi that also 0. So the integration is 0. Case 2: if n is equal to minus 1, then of course, you cannot have n plus 1 in the denominator but, firstly the integrand is just 1 or sorry it is just.

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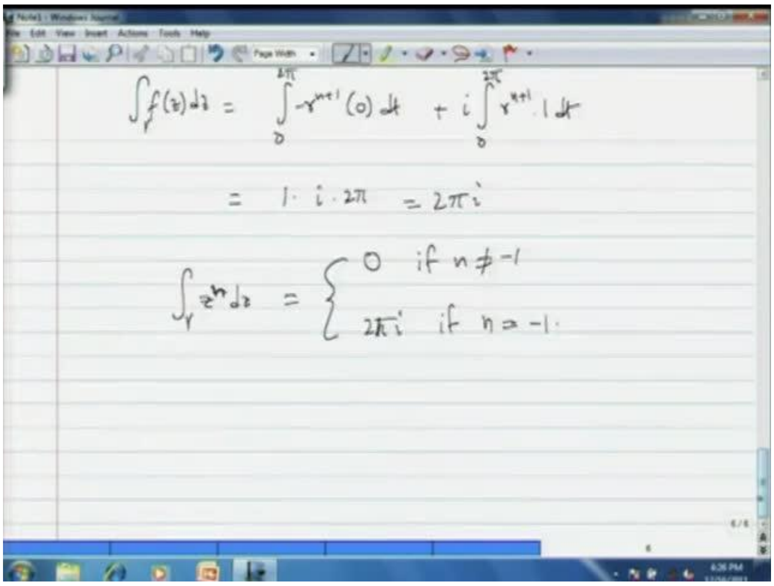

So the integrand is 0 to 2 pi minus r power n plus 1 times 0, for the real part because Sin of 0 is 0 and then plus i times integral 0 to 2 pi r power n plus 1 times cosine 0, which is 1 dt. So that you get r power n plus 1, well, n is minus 1. So I can substitute r power 0. So I just get 1 here, 1 times i times 2 pi, so which is 2 pi i. So the contour integral f of z dz where the contour is a circle of radius r centred at 0 and f is z power n is equal to, so let me write z power n instead of f z power n dz, is equal to 0 if n is not equal to minus 1 and its equal to 2 pi i if n equals minus 1. So, this is very important integral this is fundamental integral.

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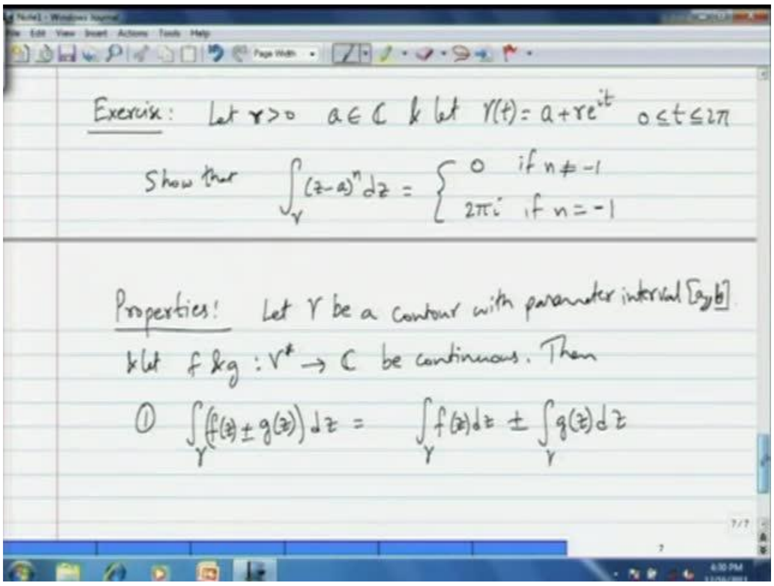

We can actually, there is another form of this, we can actually extend this to the following and this is an exercise to the viewer. Let r be greater than 0, like in the previous example, and a be any complex number; a be a given complex number. Let gamma of t is a plus r e to the i t 0 less than or equal to t less than or equal to 2 Pi. Show that the contour integration of the function Z minus a power n dz is 0 if n is not equal to minus 1 and 2 Pi i if n is equal to minus 1.

So this is an extended fundamental integral. So, it is very useful and important for us. Gamma is a circle of radius r around the point a, r is any positive number. Then, the contour integration of z minus a power n has exactly 2 values 0 or 2 Pi r. So that is an exercise. Now we have some properties of this contour integral, some easy properties, and some properties that we will prove these easy properties quite easy to see, so I would not prove them. The first of them I want to say is that the let I have to state my assumptions, let gamma be a contour with parameter interval a b.

Let f and g from gamma star to C be continuous. Firstly the contour integral over gamma of f of z plus g of z actually plus or minus g of z dz is the contour integration of f over gamma plus or minus the contour integration of g per gamma this is something we expect.

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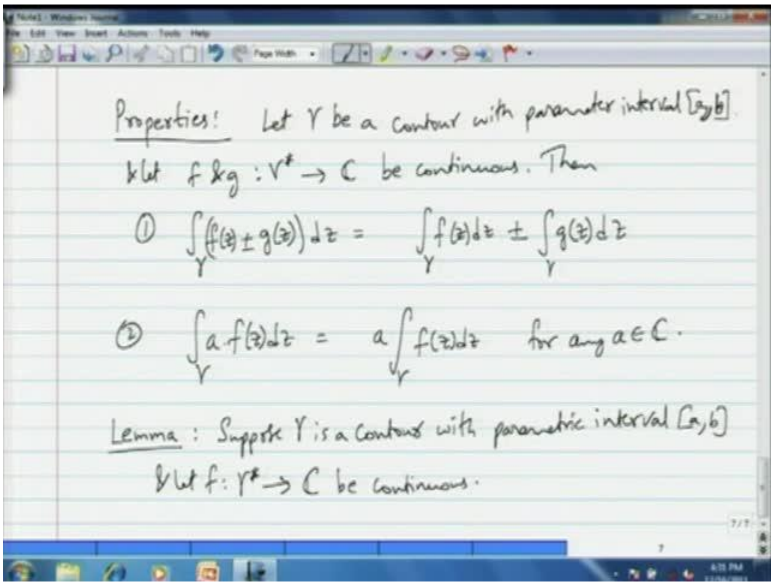

To prove it is very easy, just use the definition of contour integration and likewise if you have a constant times f of z dz and you want to do contour integration, the constant can scale out of the contour integration this gamma, for any a belongs to C. So if a is a constant like that its scales out the contour integration.

Next, suppose that, so here I want to state further properties in a Lemma, suppose that gamma is a path or a contour, so I am saying contour, so I will just say contour, with parametric interval a comma b and let f from gamma star to C be continuous, like above, then we have 3 properties.

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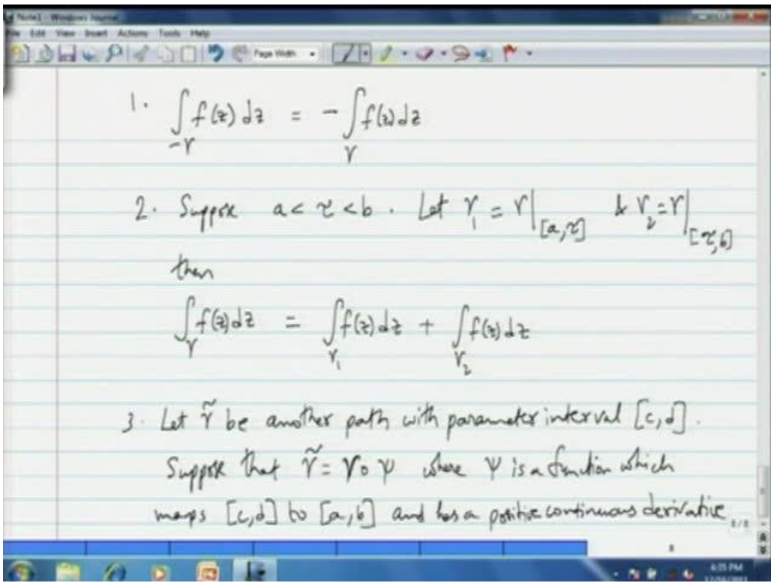

The first 1 is that the integration over minus gamma, remember what that is; that is the opposite path to gamma, of f of z dz the contour integral of f over the path minus gamma is the negative of the integral of f over gamma. So that tells us or that is the motivation for why we name that opposite path as minus gamma. So, it is a first property and the second of the properties is that suppose that a is strictly less than Tau strictly less than b, so Tau is some value between a and b. Let gamma 1 be the restriction of gamma to the closed interval a tau and gamma 2 be the restriction of gamma to the interval Tau b. What that means is that you just define gamma 1 of t equals gamma of t, on the interval a coma Tau, you do not care for the definition of gamma 1 outside of this interval a comma Tau.

So likewise, gamma 2 is gamma of t on the interval Tau b. Then, the integration, the contour integration of f of z on gamma is actually equal to the contour integration over gamma 1 of f of z dz plus the contour integration of f on the contour gamma 2. So you can take a parametric interval and split it up into pieces and restrict the original curve to each of these pieces and find the contour integral on each of these pieces and add up to get the original contour integral, mind you can only slice the interval into finite number of pieces.

So in this case we just split it up into 2 pieces and it works. So that is the second part of this lemma. In the third part of this lemma is that if you have what is called as re parameterisation of a contour. So let gamma tilde be another path with parameter interval c comma d. Suppose that gamma tilde is gamma circle Psi. So it is the composition of Psi with gamma where Psi is a function, which maps the interval c d to interval a b and has a positive continuous derivative.

So Psi is a function from c d to a b. It is a real value, real variable function and Psi prime is continuous function and also Psi prime is always positive, that means Psi is an increasing function and not only that it is also 1 to 1. It is a bisection actually and it is a smooth function. It is a smooth bisection from c d to a b and it is invertible as well, so automatically the inverse function will be smooth as well, smooth according to the definition, restricted definition we give.

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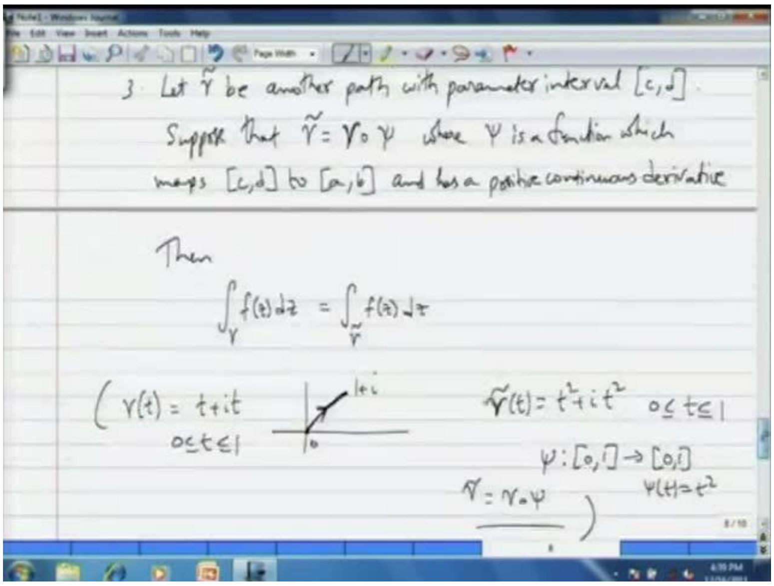

Then the integration of the contour integration of f on gamma will be the same as contour integration on gamma tilde of f. So that is the conclusion. So what this re-parameterisation really is that you take a contour and then you look at its range in the complex plane, so it is a its a curve of certain sort and it has a range, you try to come up with yet another smooth function, smooth curve, such that the range is the same. So you know to quickly exemplify, you look at gamma of t equals, let us say, t plus i t. So this looks like it starts at 0 and ends at 1 plus i; that is the range of this function 0 less than or equal to t less than or equal to 1. You could also look at gamma tilde of t equals t squared plus i t squared. This is just a motivational example, curves can be much more complicated of course, contours can be much more complicated.

So 0 less than or equal to t less than or equal to 1 then, gamma tilde is a re-parameterisation of gamma, because you can easily construct a function Psi from 0 1 to 0 1, such that gamma tilde is gamma circle Psi and what might that be: Psi of t is t squared, Psi of t is t squared. So gamma tilde is gamma circle Psi, that is clear, that is easy to verify. So, this gamma tilde is a re-parameterisation. So you can describe that particular curve you see, the range of gamma that you see, you can re-parameterise it in terms of some other function. Then there will, you know you can possibly come up with such a function. When you can, the Lemma says that the contour integral of f tallies with the contour integral of f on gamma tilde.

So this is very need because now it turns out that contour integral really depends only on the range of gamma, namely the gamma star and not on how you parameterise it, as long as gamma satisfies some smoothness conditions etcetera. So under very mild conditions your integration contour integration is really a property of the range gamma star.

So having said that we will see the proof of property 3 in this lemma, proof of property 1 and 2 are exercise; they are fairly easy, one has to use the definitions of contour integral. So, 1 and 2 are left as exercises for the viewer. Let us try to see the proof of property 3 here and so it is as follows.

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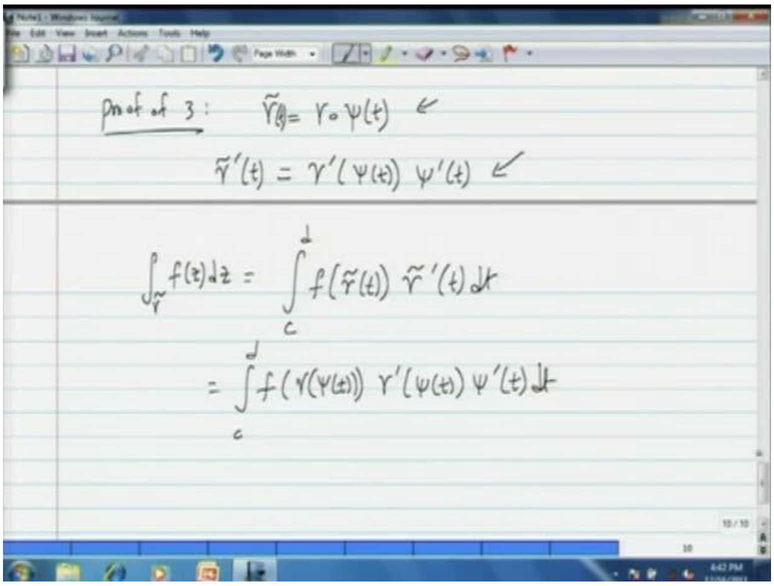

Proof of 3: So gamma tilde is gamma circle Psi, gamma tilde of t so suppressing the t here. Since psi is differentiable, gamma is differentiable, the composition is differentiable and so gamma tilde prime of t by using the chain rule is gamma prime of Psi of t times Psi prime of t that is the chain rule.

So using the chain rule you get this. We will use this to do the following. So the contour integral over gamma tilde of f of z dz is by the definition c to d, that is parameter interval for gamma tilde, of f of gamma tilde of t times gamma tilde prime of t dt. That is the definition of, that is by the definition of, contour integration and gamma tilde of t and gamma tilde prime of t are here. So I am going to substitute them here. So this is integration from c to d of f of gamma circle Psi of t. So, gamma circle Psi of t is gamma of Psi of t; that is the definition of composition. Then gamma tilde prime is gamma prime of Psi of t times Psi prime of t dt.

At this stage I have eliminated gamma tilde and brought in gamma just using the definition or just using the equality right here.

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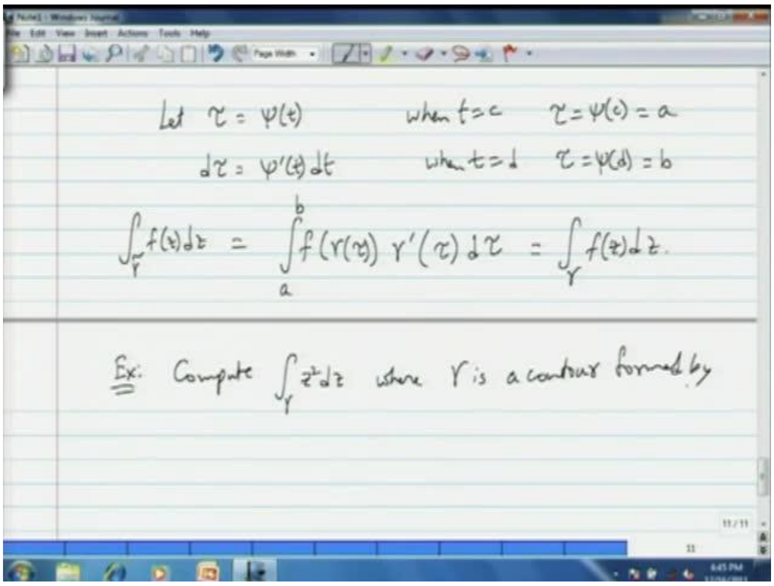

Then now I will make a substitution, let Tau be equal to Psi of t. So, that d tau is psi prime of t dt. When t equals c, Tau is Psi of c, which will be a and when t equals d for upper limit of integration Tau will be Psi of d or rather psi of d, which will be equal to b. Recall Psi is a monotone function, monotonously increasing function and it maps a to c to a and d to b ok.

So then the contour integration over gamma tilde of f of z dz will equal, with the substitution, it will equal the integration from a to b. Now, we will write everything in terms of Tau. So you get f of gamma of Tau because Psi of t is Tau, now and then gamma prime of Psi of t is Tau and Psi prime of t d t is your d Tau. This now exactly looks like well, it is the contour integration over gamma of f of z dz. That proves that these two integrals are one and the same. With that we can really talk about computing contour integrations on certain contour spoken off as curves in the complex plane or as ranges in the complex plane.

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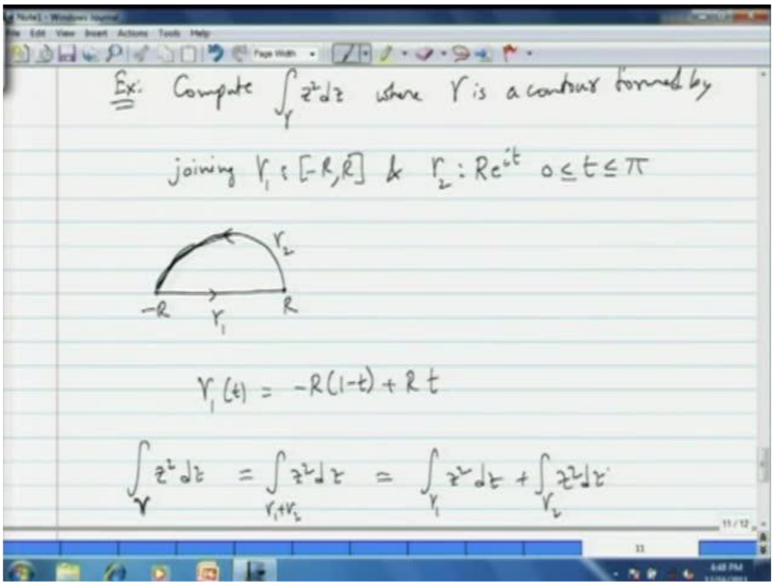

So here is an example. So compute the contour integral z squared dz, where gamma is a contour formed by joining gamma 1 the interval minus r comma r and gamma 2 which is r e power i t 0 less than or equal to t less than or equal to pi. So, compute the contour integration where only the range of gamma 1 is given, here is your minus r, here is your capital r and r e power i t is semicircle in the counter clockwise direction and this is your gamma 1 and this is your gamma 2 and this is the join.

So if you observe the parameterisation of gamma 1 has not been given to us, it is just, you know, it can be parameterised in any way you please owing to the lemma property 3 of this lemma above because, any way you parameterise or any smooth way you parameterise, gamma 1 is going to give you the same result. So, the contour integration over gamma of z squared dz; let us first write a parameterisation for gamma 1. Gamma 1 can be described as minus r times 1 minus t plus r times t.

So then we are almost all set, I am saying almost for a reason, the integration to compute the contour integration of z squared dz over the contour gamma. Now if I can split this into, well gamma is the join of gamma 1 and gamma 2. So, it would be nice if I can split this into the contour integration over gamma 1 plus contour integration over gamma 2 of z squared dz. So, let us see that 1 can indeed do this when gamma 1 and gamma 2 are smooth at least.

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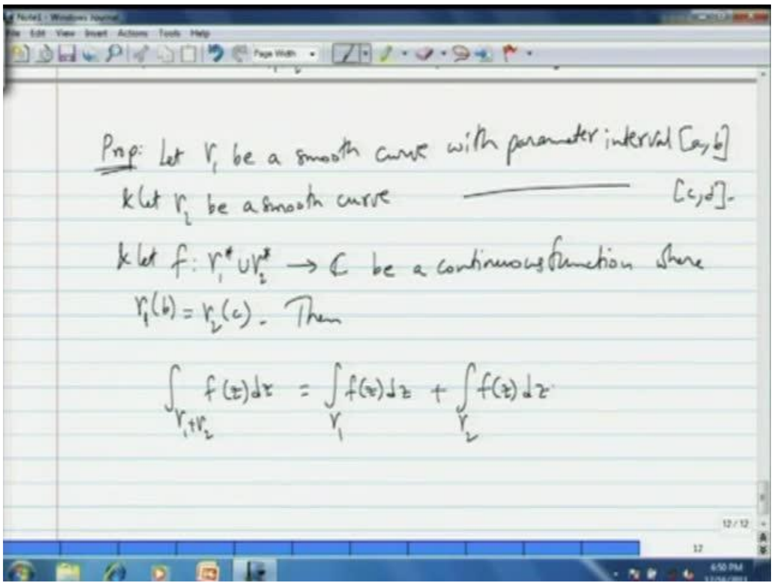

So here is the proposition, like in the above lemma let gamma 1 and gamma 2 also let gamma 1 be a smooth curve with parameter interval a b and let gamma 2 be a smooth curve with the parameter interval c d. Let f from gamma 1 star union gamma 2 star to C be a continuous function where gamma 1 of b is equal to gamma 2 of c.

So what that means is that the end point of gamma 1 is equal to the initial point, the final point of gamma 1 is equal to the initial point of gamma 2, so that I can form the join of gamma 1 and gamma 2. Then the conclusion is that the contour integral of f on the join of gamma 1 and gamma 2 is indeed equal to the contour integral of f on gamma 1 plus the contour integral of f on gamma 2. The proof involves the fact that you can shift the parameter interval and yet get the same value of the contour integration.

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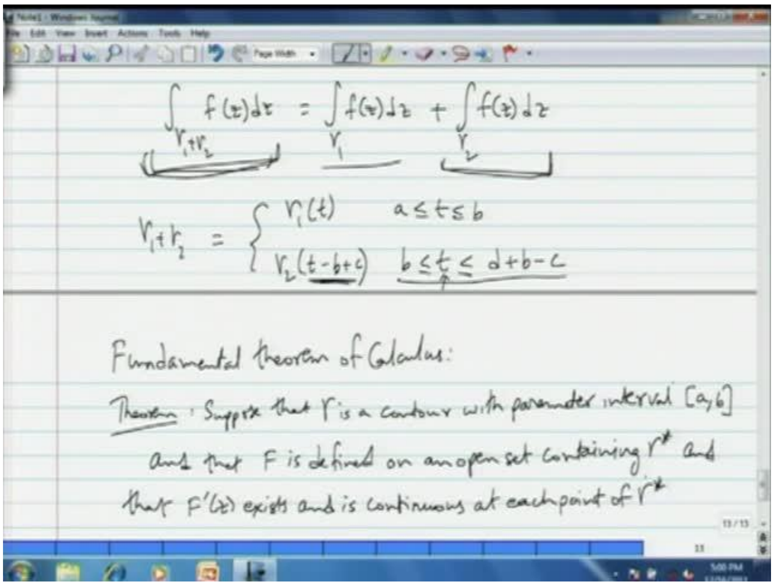

So recall gamma 1 plus gamma 2 is defined in certain way you go from a to b for gamma 1 and you go from b to not exactly d but, you go until d plus b minus c, for tracing gamma 2; where gamma 2 is t minus b plus c. Gamma 2 the parameter for gamma 2 will be t minus b plus c. So there is a shift here we shifted the interval c d to b comma d plus b minus c, so we have made a shift, but we have anyway balanced for that using this parameter t minus b plus c .

So that will take care of that and you can show that, using the definition of contour integral you can show that, this turns out, the first piece is not a problem but, the second piece turns out due to this balancing act. So that is easy to prove that is left to the viewer as an exercise. Next I want to talk about a fundamental theorem of calculus for complex integrals for contour integrals of this sort.

So the fundamental theorem of calculus, so here is the theorems, statement of the theorem. So suppose that gamma is a path, a contour with parameter interval a comma b and that capital f is defined on an open set containing gamma star and that F prime of z exists and is continuous at each point of gamma star.

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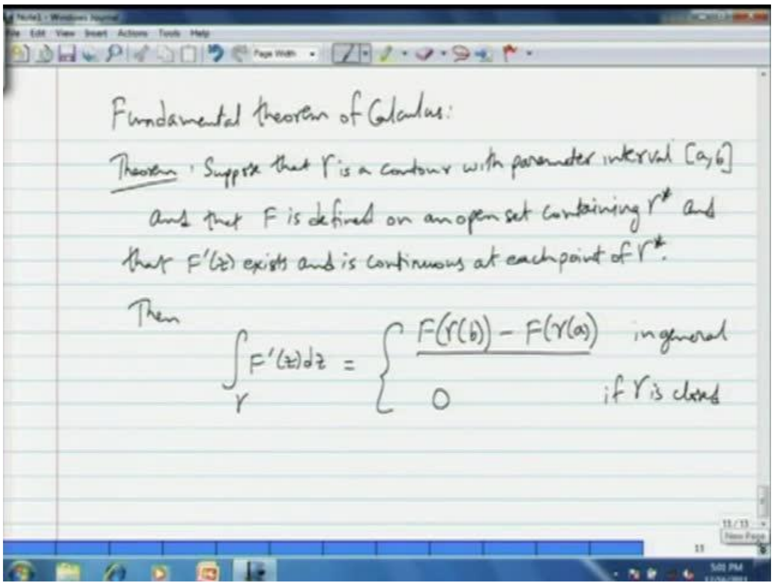

So then the contour integral over gamma of F prime of z dz is equal to F of gamma of b minus capital f of gamma of a in general and 0 in particular when if gamma is closed. So there is no real need of bifurcation. If gamma is closed, gamma of p is equal to gamma of a. So the subtraction is automatically 0. So, this is the fundamental theorem of calculus in this context and we will see the proof of this.

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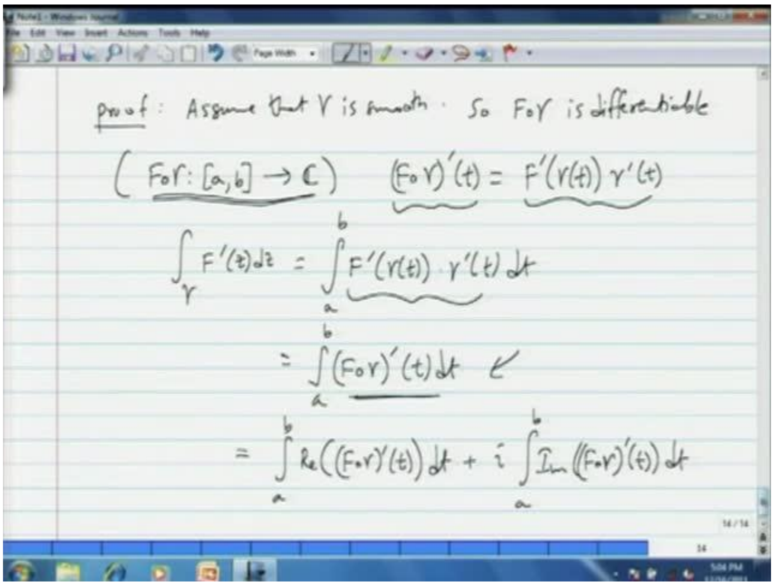

It is really easy. So assume that gamma is smooth to begin with. So we will relax this for the time being let us assume that it is smooth. Now the hypothesis on f that it is differentiable on an open set containing gamma star allows us to say that. So F circle gamma is differentiable. Notice F circle gamma it is a function from this interval parameter interval a b to the complex plane.

So F circle gamma; note, F circle gamma is a function from the parameter interval a b to the complex plane. It is a composition of a smooth function and a differentiable function so it is differentiable, of course. So let us compute the derivative using the chain rule F circle gamma prime of t is equal to F prime of gamma of t, F prime exists at every point gamma of t. So it is a F prime of gamma of t and then times gamma prime of t gamma smooth by assumption. So gamma prime exists as well. So that is the derivative.

So in order to compute the contour integral gamma F prime of z dz on the contour gamma, we need to use the definition. This is integration from a to b of F prime of gamma of t times gamma prime of t dt. But, the integrand is nothing but, the derivative of F circle gamma of t. So this is the integration from a to b of F circle gamma prime of t dt.

So F circle gamma is a function from the interval a b to c and its differentiable. So the derivative, and the derivative is continuous as well, so the derivative function here is being integrated and we define this kind of integral to be the integration of the real part of this function F circle gamma prime of t, which is a continuous function, dt plus i times the integration from a to b of the imaginary part of F circle gamma prime of t dt, that is the definition of an integral of this kind.

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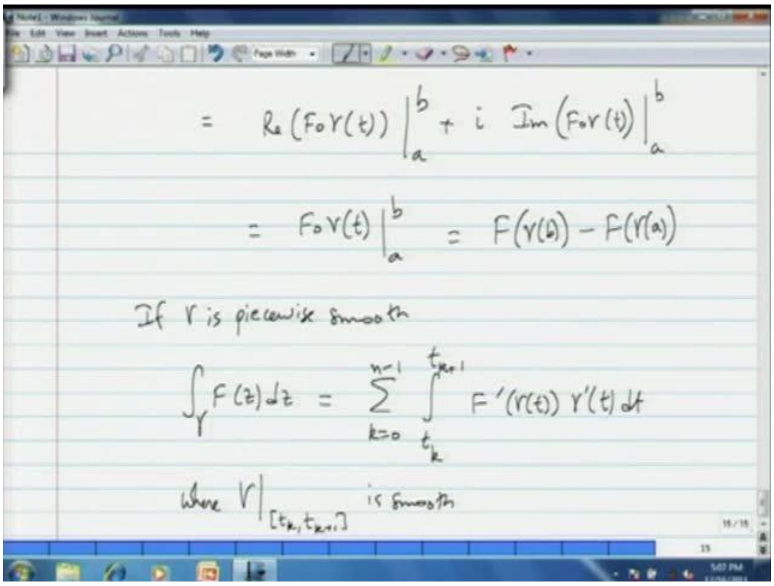

So that is by the real fundamental theorem of calculus, the fundamental theorem of calculus for a real functions, this is nothing but the real part of f circle gamma of t between the limits a and b plus i times the imaginary part of F circle gamma of t between the limits a and b. When we add, of course, we get F circle gamma of t between a and b. So that is F of gamma of b minus F of gamma of a and that proves the theorem. Of course, when gamma is a closed curve, well let me add something but, when gamma is a closed curve of course, this difference is 0 because gamma of b is equal to gamma of a. If gamma is piecewise smooth, all you have to do is split this contour integration like one of the properties in Lemma, we have seen earlier. We have to split this into pieces k equals 0 to n minus 1 the integration from t k to t k plus 1 of F prime of gamma of t gamma prime of t dt where gamma restricted to any of these intervals t k comma t k plus 1.

So its piecewise smooth, so you can separate the interval, parameter interval, a b into pieces such that it is smooth on each of these piece. These are finitely many pieces. So what I am doing is adding up the integrals on those pieces. Now you can apply what we have done before.

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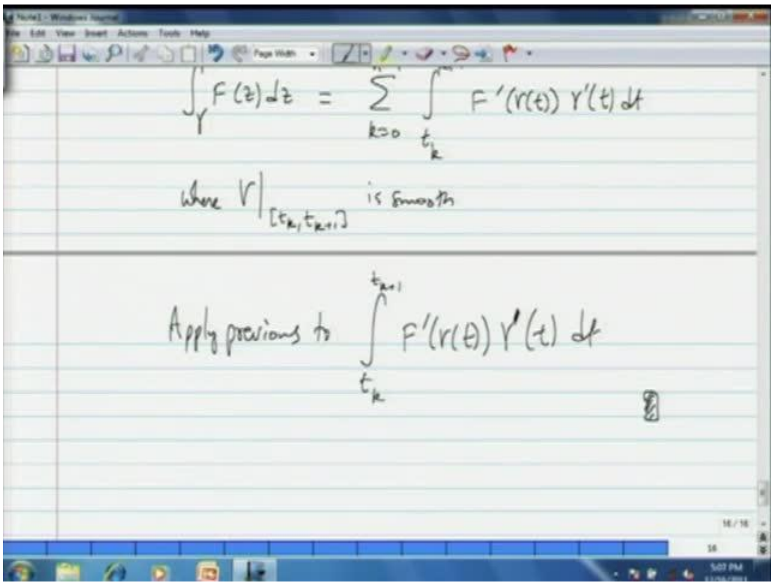

So apply previous to what we have done before to each of these t k t k plus 1 F prime of gamma of t gamma prime of t dt. Then what you get is the telescoping kind of sum and you end up with F of gamma of b minus F of gamma of a. So that piece I will leave to you, it is easy. So that is the end of this proof.