Mathematical Complex Analysis
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Module - 3
Complex Integration Theory
Lecture - 1
Integration and Contours

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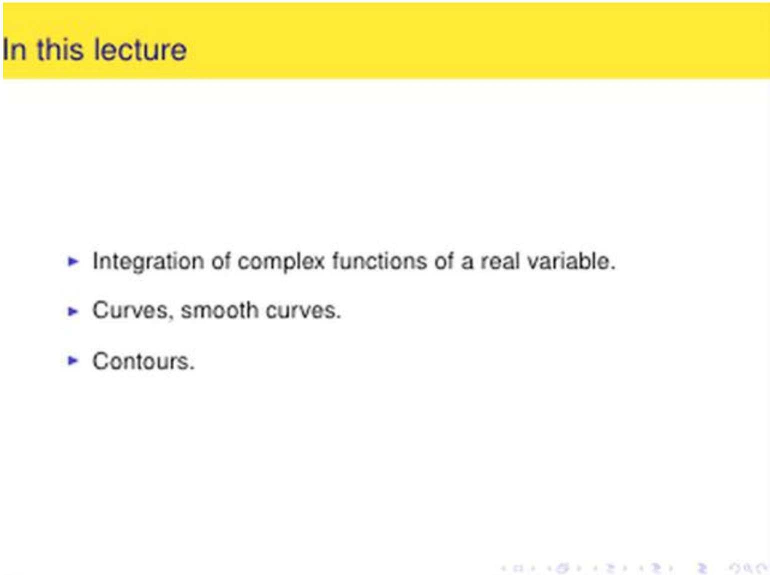

Hello viewers, we will begin complex integration. So, the the properties of analytic functions are very much related to integration of complex functions.

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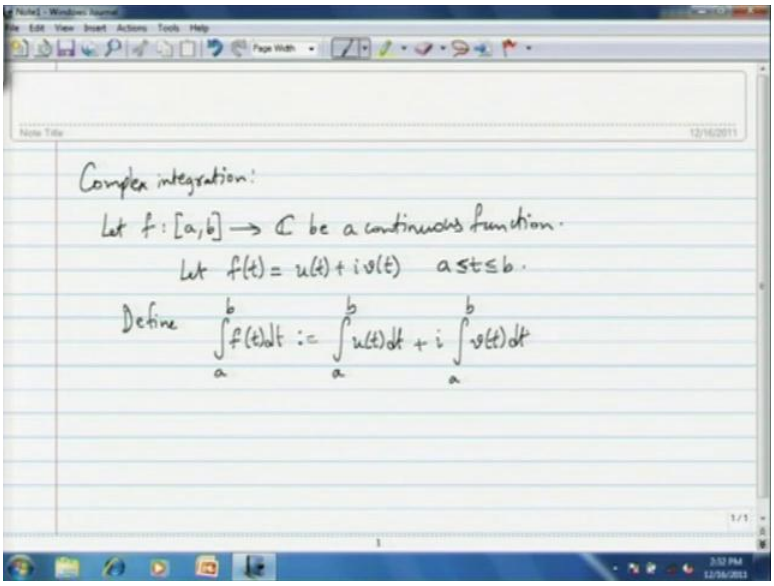

So, the complex integration actually brings out the properties of analytic functions. So, to begin with, we will consider an integral of certain type. So, complex integration, let f from let say a b to C, here a b is a closed and bounded interval. It is a subset of a real numbers of course. Let this be continuous function. Let f of t now, f is a function of one real parameter t, so let f of t be separated into its real and imaginary parts in the following way.

It is a complex valued function, so let it be u of t plus i v of t for t between a and b. Then let us define the definite integral of f of t d t in terms of the definite integral that we already know from one variable calculus. So, let this be define as the integral from a to b of u of d t plus i times the integral, the definite integral a to b v of t d t. So, it is define in terms of the Riemann integral that we already know.

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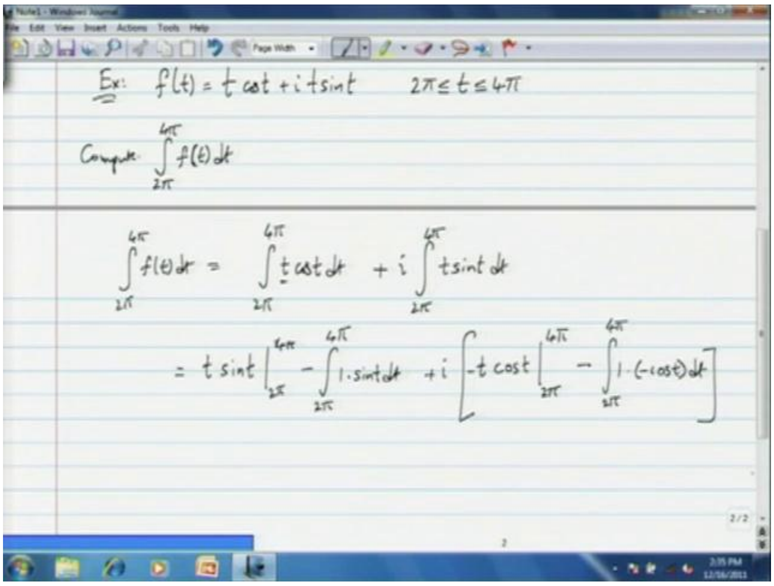

Let us see some quick simple minded examples. So, consider f of t equals t cosine t plus i times t sine t. t ranges let us say between 2 pi and 4 pi. So, then let us calculate the integration 2 pi to 4 pi of f of t d t. So, compute this, so this integral by definition is the integral from 2 pi to 4 pi of the real part of the f, which is t cosine t d t plus i times the integration of the imaginary part of f, which is t sine d t. We can integrate this functions the by using the parts, so t cosine t the integration of this will be t sine t the integration of cosine t is sine t.

So, between the limits 2 pi and 4 pi minus the integration of the differentiation of t is 1 times the integration of cosine t is once again sine t d t between the limits 2 pi and 4 pi. That is for the real part plus i times the imaginary parts, the integration of imaginary parts is once again by using integration by parts. So, the integration of t sin t is t times negative cosine t, which is integral of sine t. So, you have minus t cosine t between the limits 2 pi and 4 pi minus the integration of the differentiation of t is 1 and the integration of sine t minus cosine t d t well between 2 pi and 4 pi. So, the integral from 2 pi to 4 pi.

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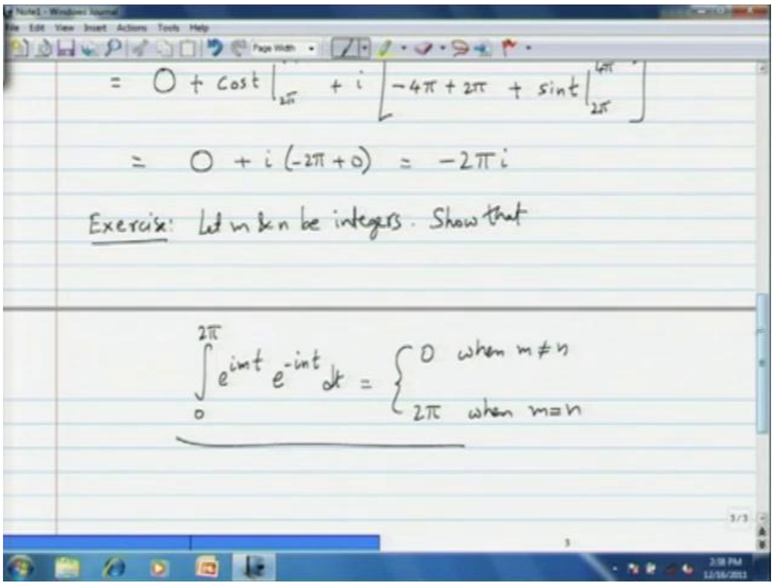

So this gives you well sine t is 0 either at 4 pi or 2 pi, so the so this piece vanishes. Then minus the integration of sine t is negative cosine t, so you get positive cosine t minus times minus, so to the plus so that between 2 pi and 4 pi plus i times. Once again here you get, minus 4 pi cosine 4 pi as 1 minus minus, so you get a plus 2 pi times cosine 2 pi as 1. Then minus the integration, well this is plus integration of cosine t is sine t between 2 pi and 4 pi. That is the zero because sine is zero at 4 pi or 2 pi.

So, you get cosine t between 4 pi and 2 pi well that gives you 0 again. So, this is again a 0 plus i times minus 2 pi plus 0. So, gives you minus 2 pi i. So, using integration by parts we can find out or compute this definite integral. So, here is another example, let me give this as an exercise to the viewer. So, let m and n be integers, let m and n be integers show that the integral 0 to to 2 pi e power i m t times e power minus i n t d t is 0 or 2 pi 0 when m is not equal to n and 2 pi when m is equal to n, okay?

So, it is an easy exercise using the definition. This property actually makes this functions, e power i m t the orthonormal basis modulo, some constant. It makes this this are orthonormal basis for a certain vector space. I mean this particular property makes it an orthonormal basis. But that is an s i that is not directly related to this topic here. So, next what we want do is, we have seen this type of integral. Next what you want to do is define, what is called a control integral.

So, roughly speaking if we have a complex function, a complex valued function of a complex variable, then you would like to define its integral over a piece of string or curve in the complex plane, where the curve has certain properties. So, it is a directed curve and there are some other properties. So, what we really formally call this curve is a contours. So, we define integration of complex function on an entity called a contour. So, in order to make that definition, first I need to discuss what a contours is.

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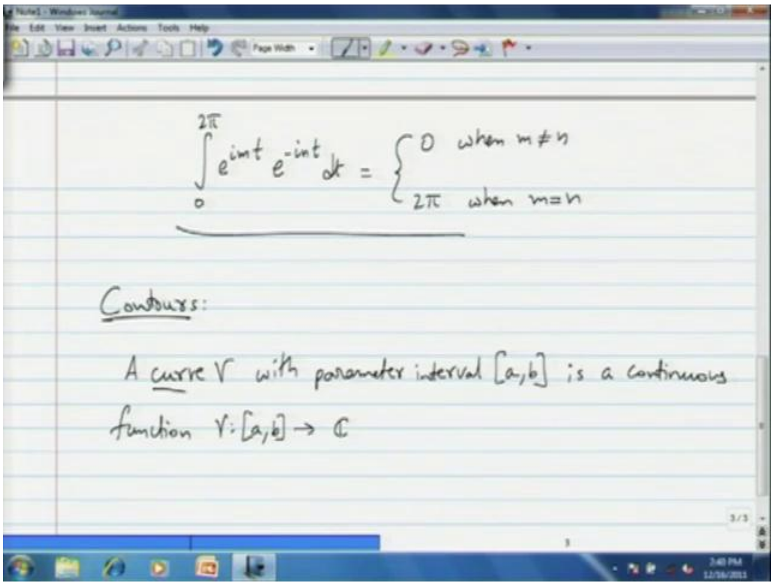

So, here is a contour. So, a contour say n C, so a curve. So, we will start with, what is a curve? A curve gamma with parameter interval a comma b is a continuous function gamma from a b to C.

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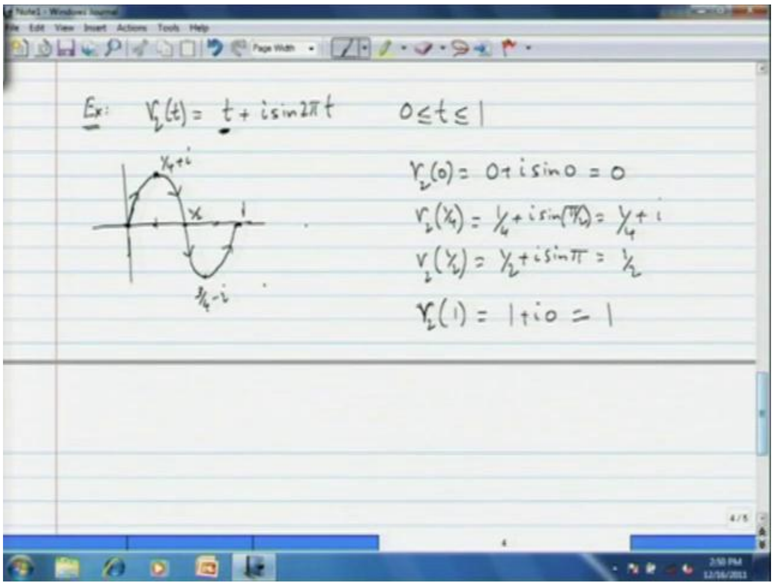

So, you are defining a curve with the… So, let us look at another example gamma 2 of t equals t plus i sine 2 pi t t ranges, let us say between 0 and 1. So, well gamma 2 of 0 is 0 plus i sine 0, so you get 0. So, this is, so this starts at 0 and gamma 2 traces traces the sine curve in the complex plane, where the real axis acting as the x axis for the graph of the sine curve. So, you get sine curve like that, when t equals for example, one-fourth you get one-fourth plus i sine pi by 2 and you reach one-fourth plus high and gamma 2 at half for example, gives you half plus i sine pi.

So, you are at the point half on the real line etcetera. So, gamma 2 at at 1 will give you 1 plus i times 0, which is 1, okay? So, gamma 2 starts at the origin in the complex plane and at at at time t equals to half you are at this point here, this is or one-forth rather, you are at this point one-forth plus i and the time t is equal to half, you are at this point half plus 0, I mean half on the real number. At time three-fourths, for example, at this point three-forth or rather minus i three-forth minus i and finally, at the time 1 you reach at this point. So, the particle starts here and travels as dictated by gamma 2 and this way and it ends up at this point 1.

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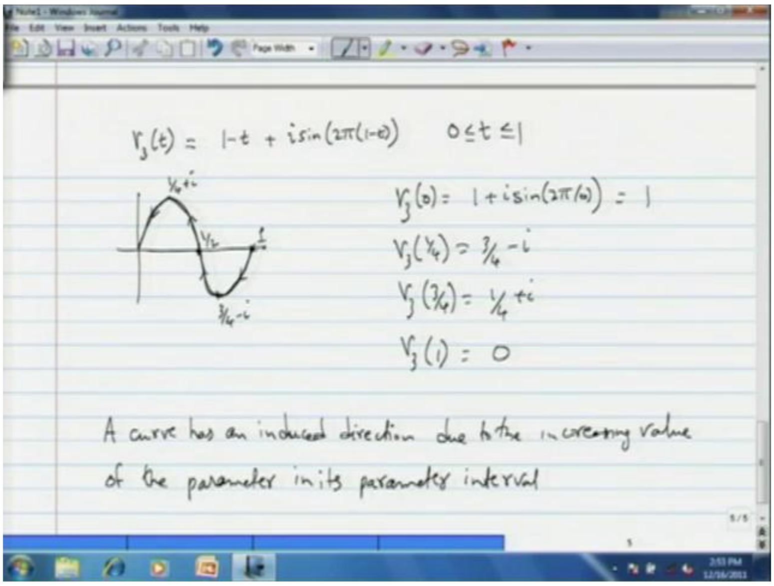

Using this very example, we can construct yet another example, gamma 3 of t, let us say is 1 minus t plus i sine 2 pi times on minus t t ranges from 0 to 1 again. So, when we try to trace gamma 3, gamma 3 by this definition is a is a curve with a parameter well 0 1 already. But when we try to look at the picture, when we try to visualize this gamma 3, we get this very sin plane curve except that, it now starts at gamma 3 of 0 is now 1 plus i sine 2 pi times 1 minus 1, which is 0. So, this you get is 1. Now, it starts at 1 and when gamma 3 is at one-forth, you get three-forth minus i.

So, it goes this way, this is three-forth minus i at one-fourth at time t equal to one-forth etcetera. Then you go backwards, at time half you are here again half. Then at time three-fourths, you are at one-forth plus i. So, gamma 3 of three-fourths is now one-forth plus i. Then finally, at t equals 1 you are at 0, so gamma 3 is I mean has the same range as the gamma 2, in the complex plane, but it is trace in in the opposite direction. Gamma dictates a particle to travel in the opposite direction, to that of gamma 2 of the curve gamma 2. So, we already see that, when you when you give a curve with a parameter interval, it comes with a direction associated with it.

When you look at the when you look at the range of the function in the, when you look at the range of the curve gamma in the complex plane. So, let us say in that words, a curve has an induced curve over a parameter interval has an induced direction, due to the increasing volume of the parameter in its parameter interval. So, you have a parameter interval for a curve and when the parameter increases in that parametric interval, that gives an induce direction on the curve.

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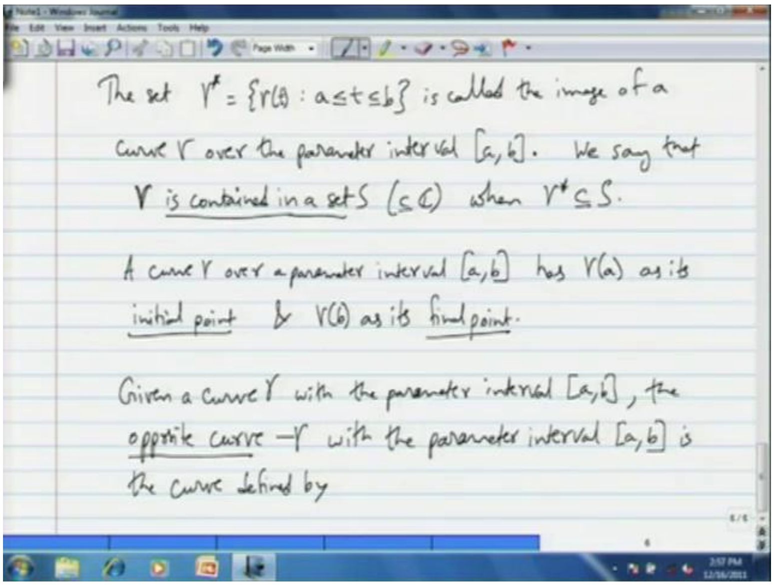

So, here graphically or visually, the direction prefers to the direction of the curve, of the locus or the range in the complex plane. The set, which is the range, the set gamma star equals set of all gamma t such that, a less than or equal to t less than or equal to b, is called the image of a curve gamma over the parameter interval a b. We say that gamma is contained in a set S. Set S, which is the subset of complex plane, so we say gamma is contained, gamma is a a function remember, but we use the terminology gamma as contained in a set to mean when gamma star, which is its range its contained in a S.

In case if the range is contained in S, we say gamma itself is contained in S. Let us just way of saying it and then a curve gamma over a parameter interval a b has gamma of a, which is complex number, which is in the complex plane, as its initial point and gamma of b as its final point. So, will use this words initial point and final points to mean gamma of a and gamma of b given a curve gamma. Because there is an induced direction, given a curve gamma with the parameters interval a comma b, the opposite curve minus gamma. So, we are we are calling the following as opposite curve and we denote it by minus gamma with the parameter interval.

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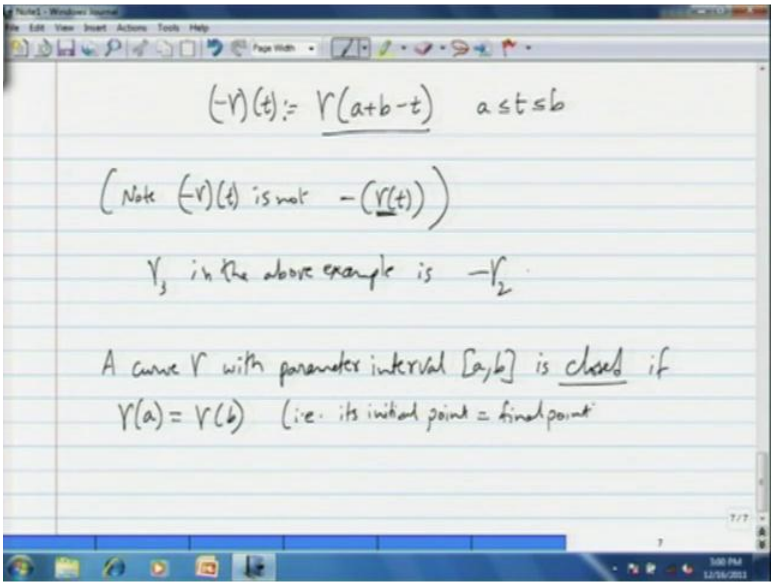

The same parameter interval is the curve defined by minus gamma of t, is gamma of a plus b minus t where t ranges from a to b. So, minus gamma is defined as gamma of a plus b minus t. So, in the above example, let me go back sorry, so in this example here gamma 3 was the opposite of gamma 2 because gamma 2 is traced in this direction, where as gamma 3 is the same, has the same range and is traced in the opposite direction. So, due to that geometric significance, we want to define the opposite curve in this fashion. Please note that minus gamma of t is not minus gamma, minus of gamma of t.

If gamma of t is complex number, you should consider that complex number gamma of t, it is not the same as minus gamma of t is an new function. It is not the same as minus of gamma of t. It is actually defined here, it is gamma of a plus b minus t. So, you have to be careful. So, essentially visually it is clear what we want to say, it is essentially the same range traced in the opposite direction. So, it has the opposite inclined direction to that of gamma. The parametric interval states the same. So, a gamma 3 in the above example is minus gamma 2.

Now, we want to make some more definition here, some more terminology is dew here. A curve a gamma with parameter interval a b is closed, we want to call it closed. If a gamma of a is equal to gamma of b, i e its initial point is in the complex plane is equal to the final point.

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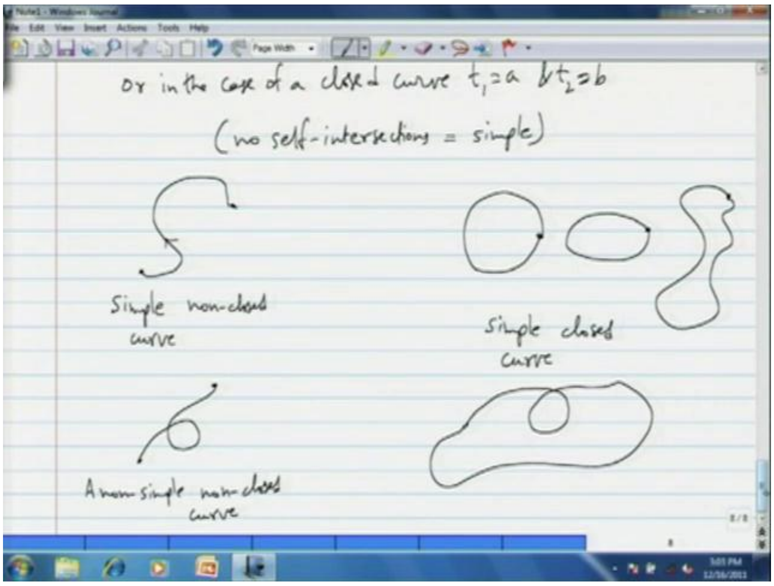

It is called simple, the curve the curve gamma with parameter interval a b is called simple. If gamma 1 of t or rather gamma of t 1 is not equal to gamma of t 2 unless t 1 is equal to t 2 or in the case of a closed curve of course, the end points are equal, the the initial and final points are equal. So, you want to allow this gamma of t 1 equal to gamma of t 2 when t 1 equals to a and t 2 equals to b. So, apart from this cases gamma of t 1 should not equal gamma of t 2.

What this means essentially is that, if you look at the range of gamma a curve is called a simple, if it does not have self intersections, the range does not self intersections. Except that the only intersection you will allow is, possibly that the initial point is equal to the final point. that is that is called simple curve. So, let me let me say no self intersections accepted the end points is simple. So, here is the visually a simple closed curve or rather simple non closed curve. What this really is in the complex plane is, it is the range of a simple non closed curve, but i am abusing the notation and terminology here and I am calling it the range itself as a curve.

So, maybe there is a direction. So, likewise if you have range is something like this, a circle or a ellipse or some kind of curve like this, where the initial point meets the final point, this is a simple closed curve. Something like this where the initial point is not equal to the final point and there are self intersections is a non simple, non closed curve.

It is non closed is clear, the initial and final points are not equal. It is not simple because there is a point of rather point of intersection.

Likewise here is a curve, which is which is closed and non simple closed curve. So, all this are actually ranges of certain curve with some parameter interval. So, there come with some direction, but yes I am going to call the visualization of them as curves now, just for motivating the definition here of simple and closed.

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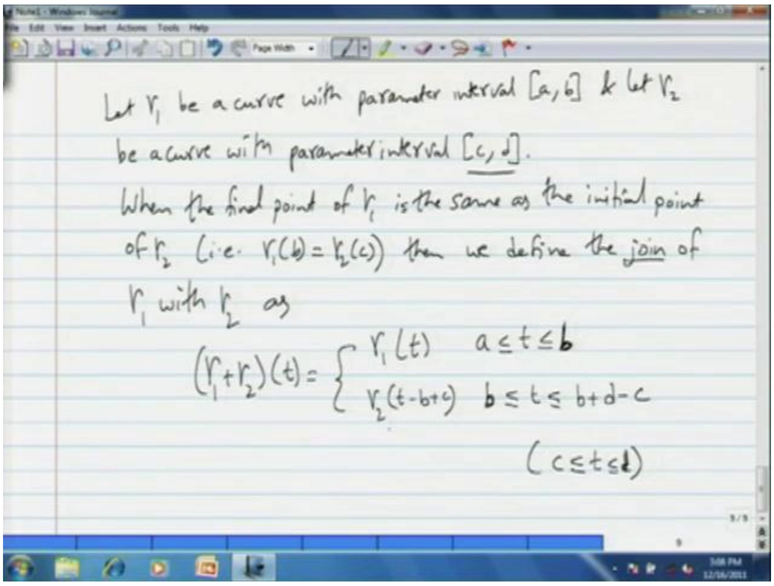

Now, let gamma 1 be a curve with parameter interval a comma b and let gamma 2 be a curve with parameter interval c d less us say. Now, what we want do is sort of join this two curves, when it makes sense. So, suppose you have a range of gamma 1, which is some piece of string sitting inside the complex plane and gamma 2 starts where gamma 1ends and suppose that it continuous from there. It is another piece of string, what you want to do is define a join of this two.

So, when so here is what we will do to define the join. When the final point of gamma 1 is the same as the initial point of gamma 2 i e gamma 1 of b in this case is equal to gamma 2 of c. Then we define we firstly here, we define the join of gamma 1 with gamma 2 as, so this is the notation gamma 1 plus gamma 2 to mean the join of gamma 1 and gamma 2. So, gamma 1 plus 2 gamma 2 of t this defined as, gamma 1 of t for t ranging from a to b. So, for the first part of the time t is like that for the time parameter for the first part of the time a to b, we describe gamma 1 and for the rest of the time we will decide, what that is. For the rest of the time we will trace gamma 2.

So, we will so we should we want start at b, now time has to continue where it has left of, so we want to start at b and and where, and where well we have the parameter interval for gamma 2 to be of length d minus c. So, we want to end it b plus d minus c. So, here is an interval of length d minus c, which starts at b. So, we have actually added, so what we have done is we have added d minus c to both sides of this kind of equations or rather we have added d minus c to both both sides of this kind of inequality.

So, to compensate we will, we will do the following, we will subtract b and add c, to the parameter of gamma 2. So, then for this time interval which is actually a translation of the interval c d, which is the translation of the interval c d a gamma 2 will be traced. So, the join is just a slight modification, you first travel along gamma 1 the range of gamma 1 and then you trace gamma 2, from there only. So, that is, that is a join.

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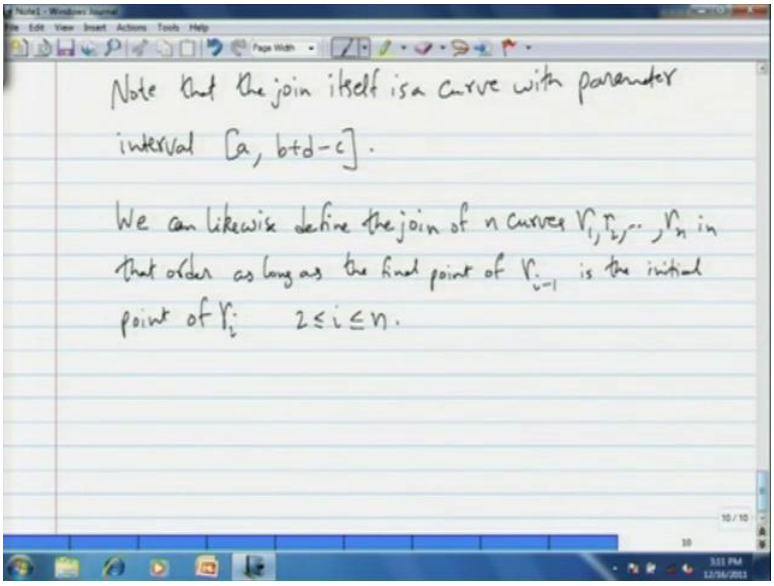

So, note that the join itself is a curve with parameter interval. Now, a to b plus d minus c. Then we have done this for two curves, there is no reason why we should stop that a two curves. If you have n curves satisfying the requirement that the final point of the curve now is equal to the initial point of the next curve, we can construct the join of this n curves. So, we can likewise define the join, the join of n curves or n curves gamma 1, gamma 2 so on until gamma n, in that order.

As long as the final point of gamma i minus 1 is the initial point of gamma i here, two less than or equal to i less than or equal to n. So, it is okay. So, we what is important is not to get lost in the technicality here, in the symbols here. What is important is to know, what this represent in the complex plane? What you want to do is essentially now want want to define the integration of complex function along this kind of joins. So, we will do an example here.

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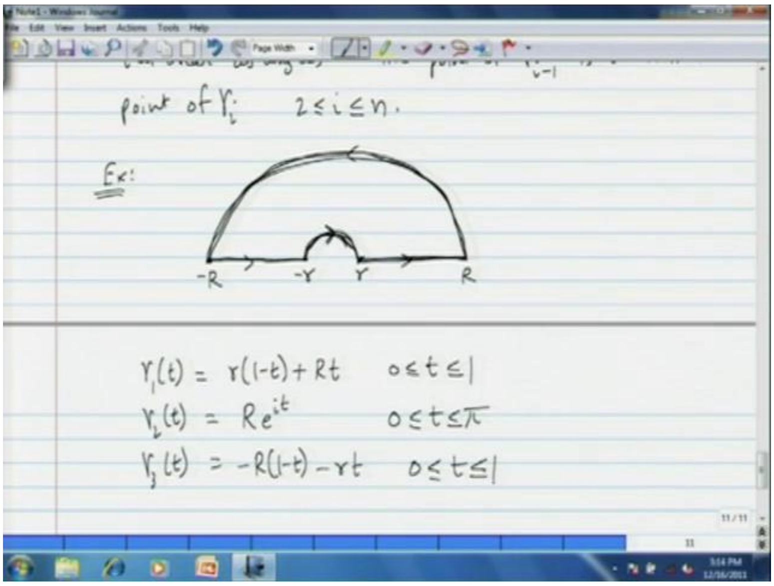

We will try to join four curves. So, here I have let us say, this is the real line. So, the real line I have minus capital R, R is some constant some positive real number and little r is some positive real number. Then here is r little r again and then capital R again. So, I am starting at minus R going until minus little r and then tracing this little circle here and going along this straight line from little r to capital R. Then back to here, via this semicircle in the anti clockwise direction from capital R to the capital minus R.

So, let us say, let me start here actually, does not matter. Let me start here and then go this way, this semicircle, this straight line in this in this semicircle in the anti clock in the clockwise direction. So, this is the curve, I want to describe. So, I can realize this as a join of four curves. So, gamma 1 of t, I will start here from little r and gone to capital R. Gamma 1 of t, I can write this as r times one minus t plus r t, this is a linear equation where goes from 0 to 1 and that describes a straight line. Of course, then gamma 2 of t is now, this semicircle in the anti clockwise direction.

So, gamma 2 of t let me write that as capital R e power i t, where t goes from 0 to pi. 0 to pi gives me half semicircle and in anti clockwise direction, we have R e power i t. Gamma 3 of t is the straight line, this is the straight line there, which starts at minus R and ends at minus little r. So, t goes from 0 to 1.

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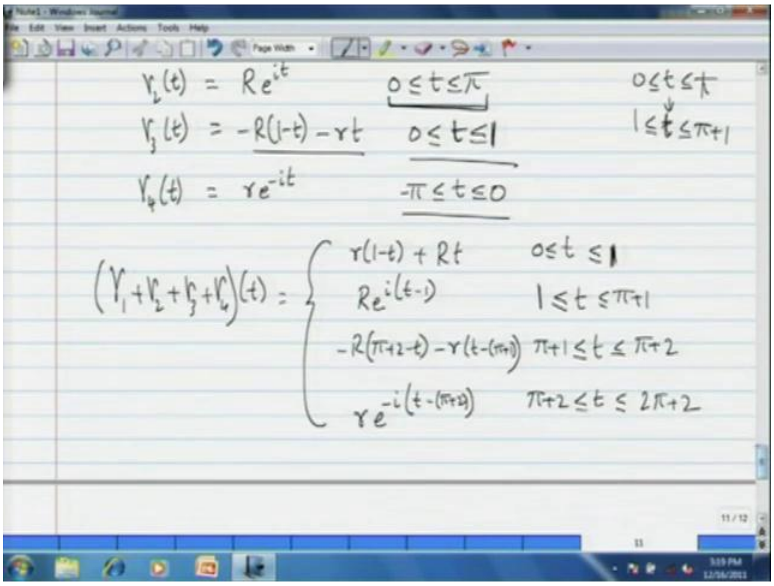

Gamma 4 of t is that other little semicircle in the clockwise direction. So, to trace that I have R e raise to minus i t. t going from minus pi sorry, t going from 0 to pi or rather minus pi to 0. Let us write, so we can see that gamma 1 the the final point of gamma 1 is the initial point of gamma 2 and etcetera. So, final point of gamma 2 is the initial point of gamma 3 etcetera. So, we can construct the join of this four curves gamma, 1 plus gamma 2 plus gamma 3 plus gamma 4 of t, we can construct the join by actually translating these parameter intervals appropriately.

So, let us, let us trace gamma 1, so this is equal to r times 1 minus t plus R t. So, let us trace gamma 1 for the time from t equals 0 to 1. So, for tracing gamma 2, I need to start at time 1, so from time 1. So, how should, how long should I go? Well I should go for a length pi because for gamma 2, the parameter interval has a length a pi. So, let me go until pi plus 1. So, then the description of gamma 2 will be R e power i times t minus 1, to adjust for the 1 I have added to this inequality on both sides.

So, when I add 1 to both sides, I get 1 less than or equal to t. well t plus pi, so I am substituting replacing that with a new t pi plus 1. That new t now for compensation, I will write that as t minus 1 here. So, let me erase and write that as i times t minus 1. Then likewise for gamma 3, i should start now at pi plus 1 and then go on for a length 1, for a interval length 1. So, t travels from pi plus 1 to pi plus 2. So, how do we describe gamma 3 now? We will compensate, will take this equation or rather this expression in compensate the addition of gamma pi plus 1.

So, then we get 1 minus t minus pi plus 1. So, we get pi plus 2 minus t minus r times, well t minus pi plus 1. Finally, we need to trace gamma 4, for that we should start now at pi plus 2 and peak an interval of length of pi. So, we should start at pi plus 2 and then we should end it 2 pi plus 2 because pi plus 2 plus pi is 2 pi plus 2 and then we trace R e raise to minus i. Then we adjust by subtracting pi plus 2. So, that is your join. So, in practice fortunately, we would not have constructed these joins again and again. We will see there is something, which will help us to deal with this join practically. But for now we understand join four different curves or an different curves.

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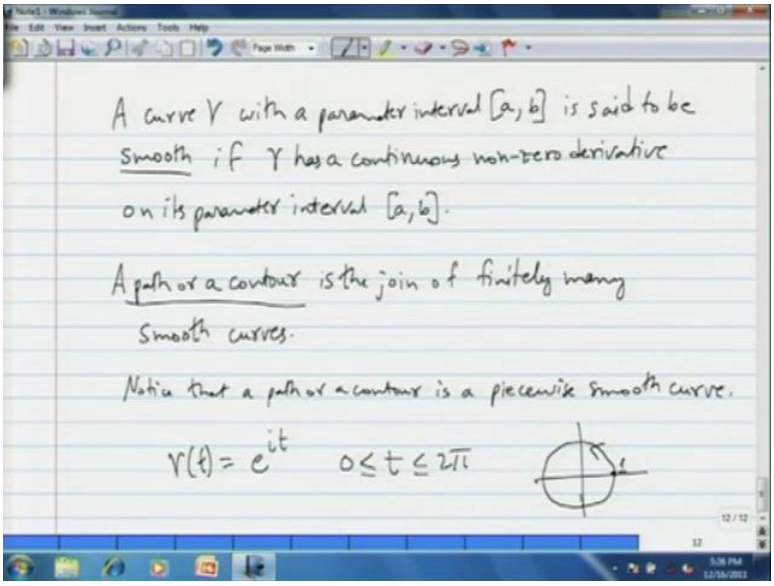

Gamma with a parameter interval a b, I want to say when it is considered smooth, I will define smooth is said to be smooth, if gamma has a continuous non zero derivative on its parameter interval a b. So, to avoid the pathologies, we need smooth curves. So, the requirement for a curve to be smooth is a three-forth. So, firstly gamma, if gamma is a curve it is a, it is a function. So, gamma primes we define at every point in the interval a b closed interval a b. So, at the points the the definition of derivative needs modification because you take one sided limits for for a or for b.

In the definition of the derivative you take one sided limits and define the derivative, but nevertheless that gamma should be differentiable at every point in the interval. Not only that, the the function gamma prime, you considered gamma prime as a function that should at all points in a b. Finally, gamma prime should not be 0 at any point in the interval a b. So, these three requirements will make a curve smooth curve. Also know that this definition of a smooth curve is a difference with a standard definition of smooth function.

Normally smooth function refers to function, which has derivatives of all orders. So, here we have differed from that kind definition of smooth function. Normally what we have defined here is called regular C 1 curve, where C 1 here refers to the class of function, which are differentiable and which have continuous derivative. Regular functions are functions, which have a non zero derivative. But to sum all the up, we have used the smooth and said that smooth curve, we redefine the word smooth and said that smooth curve is a curve, which has continuous derivative and a non zero a derivative as well at each point.

So, please make a note of this. So, that will actually clear many pathologies because functions gamma as we defined it, the curve as we defined it is just a continuous function from an interval to the complex plane. Now, they can a continuous function can be really weird of... For example, there are curve called space filling curves, which actually feel up whole disc if you wish starting in a interval a b. So, continuous functions can be unwarily, so we insist that this curves this smooth, so that we can play with them. Then we will define a path or a contour now.

So, at last we reached where we wanted to be. A path or contour is the join of finitely many smooth curves. So, contour is a essentially the join of finitely many smooth curves. Now, when you consider the path or contour itself as a curve that may not be smooth because the points where you join… Now you get a function a new function join is a function, remember? So the new function itself may not be differentiable at the points where you join, but in the mean time where you have this, this finitely mini smooth curve, in the mean time depending upon the time parameter is smooth. So, so path or contour is called a piece vise smooth curve.

So, it is it is a join of finitely any smooth curves. So, I will write that in the words, notice that the path or a path or a contour a path or a contour is a piece vise smooth curve, with certain parameter interval. So, depending on how many or depending on the parameter intervals for the smooth curves that we are picking to join. So, we often assign the standard geometrical terms such as, triangle or a circle etcetera to to these contours, depending on the range of the contour itself. So, for example, let me go back to the first example we chose sorry, so here is a first example that we chose gamma 1 is 3 plus 2 e power i t.

I will call this curve it is a piece vice smooth curve sure, well it is a smooth curve. So, I will, I will prefer to call this a semicircular contour or if you pick the, if you pick the following, so gamma of t equals e power i t, which is a circles 0 less than or equal to t less than or equal to 2 pi. So, it traces the unit circle, gamma gamma traces the units circle.

It starts at 1 goes around in an anti clockwise direction and ends at 1, this is a circular contour for example. So, we will associate the word circular etcetera to this contour, depending on the, on the geometric shape of the range of this contour. So, we can sue these contours to define the contour integral of a complex function.