Complex Analysis
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Module - 2
Complex Functions: Limits, Continuity and Differentiation
Lecture - 8
Problem Solving Session

Hello viewers, in this session we will solve some problems based on the theory that we have covered so far. I will write the questions and the viewer is encouraged to solve the question before he or she looks at the solution. So, I will give a brief pause after each question and try to answer the question yourself, before looking at those solutions. So, let me start with first question here.

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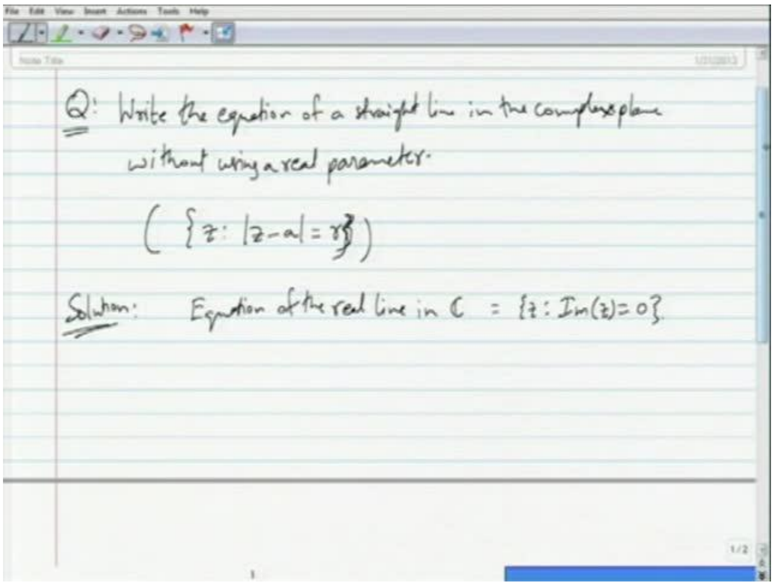

So, write the equation of a straight line in the complex plane without using a real parameter. What I mean by that is, recall we have written the equation of a circle as set of all z such that the modulus of z minus a is equal to r for some positive real number r. So, in the case of a straight line, it is possible to write the equation without using any real parameter. So, try to solve this question and I will provide the solution here.

So, what we can do is I will first give you the equation of the real line in the complex plane, the equation of real line in C is basically the set of all z such that the imaginary part of z is equal to 0. So, it is a real number. So, the imaginary part of z is equal to 0. So, this is what I am looking for. Now, let us try to see a general straight line in the complex plane.

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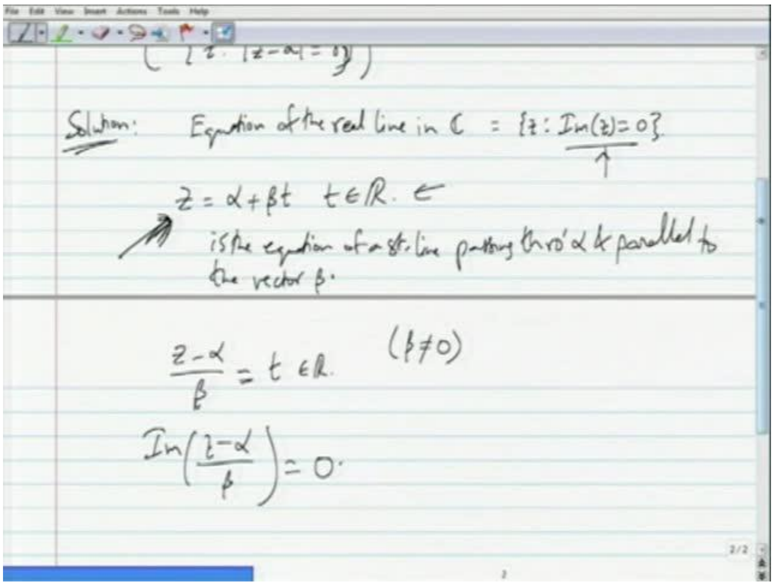

It looks like z equals, it is the locus of all point z such that z equals alpha plus beta t where t belongs to R. So, any straight line has to look like that. This is the equation of a straight line passing through alpha and parallel to the vector beta. So, the viewer is familiar with this equation from geometry in the, in the plane. So, now let us try to eliminate the real parameter t. So, we do not want a real parameter. So, what we can do is we can write z minus alpha by beta and look at, look at its value, we will assume that beta is non zeros, well if beta is 0 we do not have a line here.

Beta is non-zero, if beta is 0 we just have a point so it is degenerate line. So, we will allow a beta not to be 0 and then we have z minus alpha by beta is equal to t belongs to R. So of course, the equation of any, the set of all real numbers is imaginary part of z is equal to 0. So, the equation of a straight line of this particular straight line is precisely the imaginary part of z minus alpha divided by beta is equal to 0. So, this is what I am looking for.

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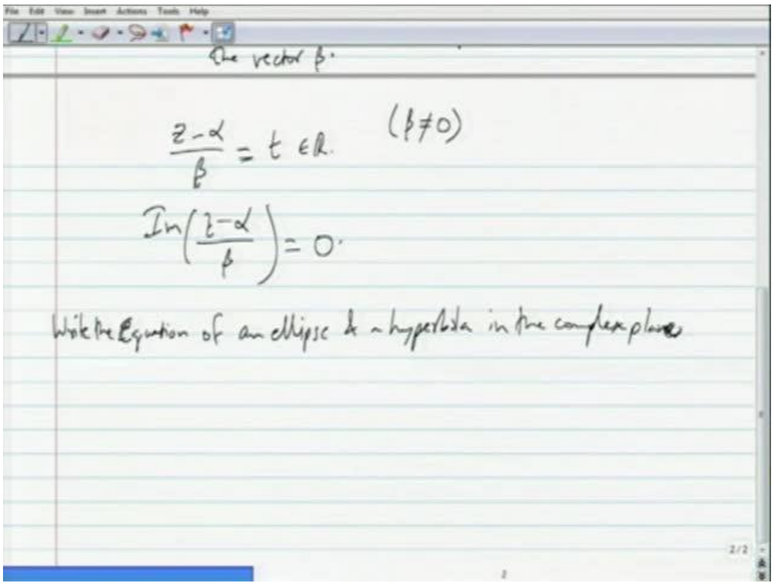

So, one can also try to write the equation of, so these are exercises. Write the equation of an ellipse and a hyperbola in the complex plane. So, once again try not, you should not use real parameter in the case of equation of a circle for example, there are no real parameters r is a constant, a is a constant the only variable z which is the complex number. So, likewise write the equation of ellipse and hyperbola in the complex plane without using real parameters. So, that is an exercise. The next question is simple minded.

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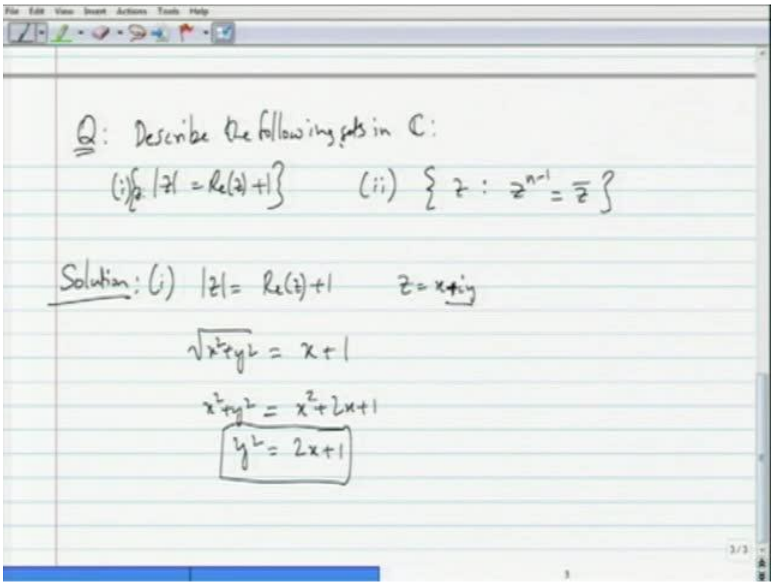

So, it is about describing sets geometrically. We have already seen such examples when we studied the geometry in the complex plane. So, let me do a couple of more simple minded example, examples here. So, the the question is as follows. Describe the following sets in the complex plane. So, the first example is real part of the modulus of z is equal to real part of z plus 1 and then the second set is the set of all z. So, I said sets so I will write it in the set format set of formats all z such that this happens and the set of all z such that z power n minus 1 is equal to the conjugate of z. So, try to describe these two sets geometrically in the complex plane.

So, thus I will present this solution here. So, the first set well we can do this problem analytically. What I mean by that is we will consider mod z. So, the mod z is equal to real part of z plus 1. So, allow me to write z as x plus i y and then I can use my knowledge of you know the plane to actually come up with the description of the set. So, modulus of z is square root of x square plus y square when I write z like that and the real part of z is x and then plus 1. So, this is what you are looking at. So, square both sides you get this. So, you get y square is 2 x plus 1. So, that is the description.

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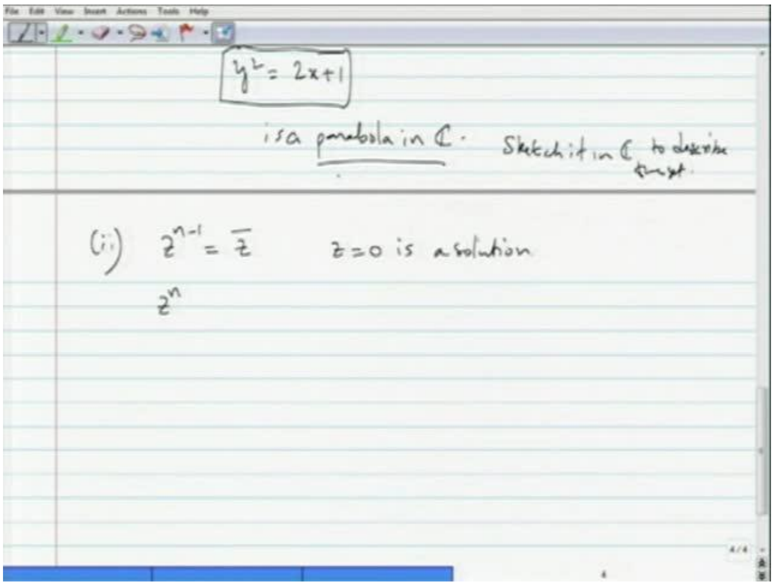

So, and we know that y square is equal to 2 x plus 1 is a parabola in C in the complex plane and you can sketch this parabola. So, this is the, is the parabola in C this is description I am looking for and sketch it in C to describe the set. So, the second set z power n minus 1. So, if you look at the set of all z such that z power n minus 1 is z conjugate. So, what kinds of point are in this set? So, then you notice that what we can do is we can multiply both sides with z so we get, firstly z is equal to 0 is definitely a solution n here I did not mention.

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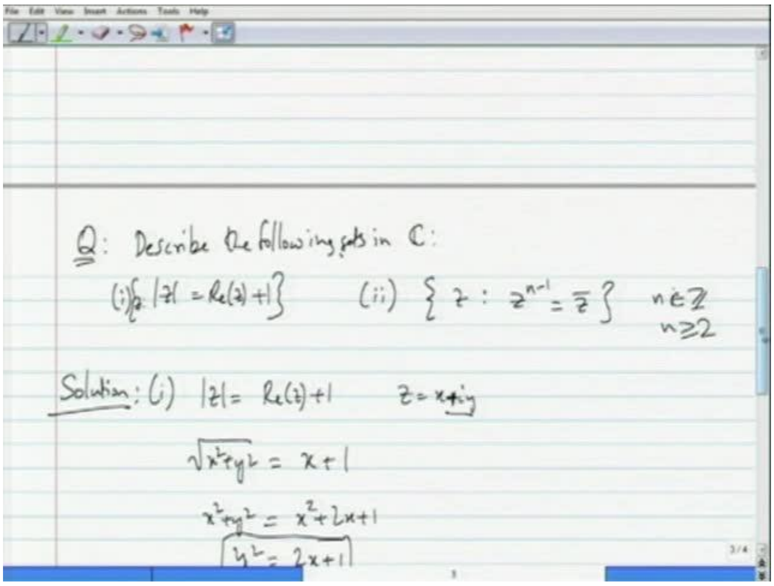

So, n here should be an integer. So, n belongs to z n greater than or equal to 2. So, n, n can be made to be a non positive non negative integer, but the cases 0 1 are trivial. So, I will start with n equals 2 so integers n equals 2. So, let us try to describe z power n minus 1 as z bar. So, z equals 0 is a solution. So, we will keep this in mind.

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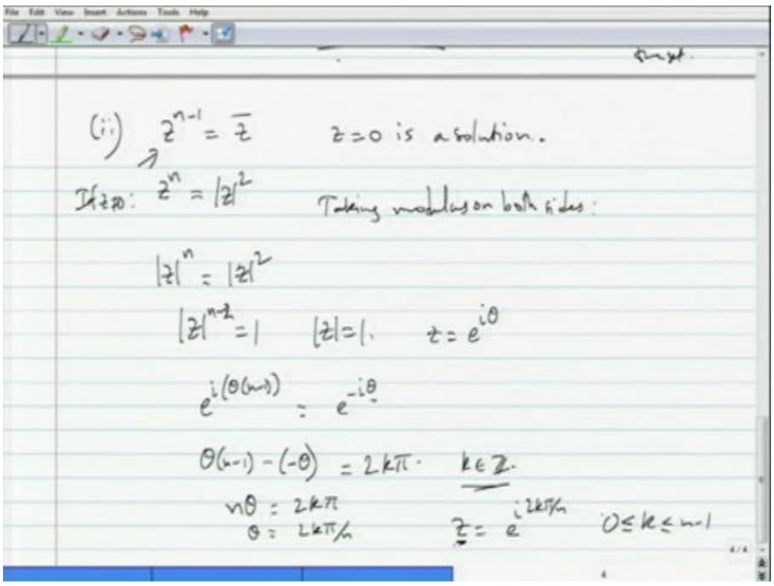

Apart from this if z is non zero, if z is not equal to 0 then what do we have? We have z power n we can multiply both sides with z. So, we get z power n is equal to the modulus of z square. So, when z power n is modulus of z square. So, you notice that taking modulus on the both sides. So, what what I am trying to do is that well if two complex numbers are equal there moduli have to be equal at least. So, we get modulus of z power n is equal to modulus of z square because the right hand side is already a real number, its modulus is itself. So, what we get is that the modulus of z power n minus 2 is equal to 1.

So, we conclude that the modulus of z is equal to 1. So, I can write z as e power i theta for some real number theta then let me look at the equation again I get e raise to i times theta times n minus 1 is equal to e raise to minus i theta. The conjugate of e power i theta is e power minus i theta. So, from this one can conclude that theta power n minus 1 and minus theta, minus theta here they are, they differ by a 2 k pi an integer multiple of 2 pi. So, they differ by 2 k pi this gives us theta times n, n theta is 2 k pi so theta is 2 k pi by n where k is any integer here.

So, theta is an nth root of, I mean theta is 2 k pi by n. So, z is an nth root of unity. It is i times 2 k pi by n k belongs to z. Well, k can be a restricted to 0 through n minus 1 and this gives all the values of, all the values of z. So, that is the solution to this. So, we notice that the solution set or the set, the description of this set is a discrete set. What I mean by that is it is a bunch of points like that based on n it is just a a bunch of points not more than n. So, of course, 0 is always included in this set of solutions like we observed. So, that is the description of 2. So, let us move to the next question.

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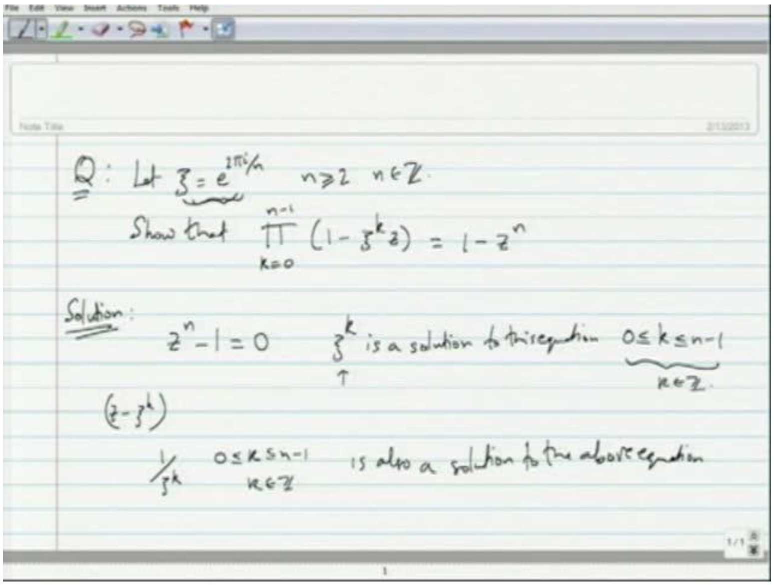

So, the next question is as follows. Let zeta equal the nth root of 1, a specific nth root of 1 where n here is greater than or equal to 2, n is an integer. Show that the product of from k equals 0 through n minus 1 of 1 minus zeta power k z. So, the product of these n factors 0 through n minus 1, 1 minus zeta power k times z these are n number of factors, product of this show that this is equal to 1 minus z power n. So, that is the question. Try to solve this question and I will provide the solution. So, the solution to this problem is as follows. So, there is a factor theorem for real polynomials recall that if a is a root of a polynomial then z minus or x minus a is the, is a factor of a polynomial f of x for real numbers. Similar factor theorem holds for complex numbers, complex polynomials as well although we did not prove it. If a is a root of a polynomial then z minus a is a factor of the polynomial f of z.

So, we will use that. So, if z power… So, you consider this equations z power n minus 1 is equal to 0. We know that the nth roots of unity are precisely the solutions or the roots of the polynomial z power n minus 1. So, zeta power k is a solution. So, not only zeta which is mentioned in the question, but zeta power k is a solution for, to this equation for 0 less than or equal to k, less than or equal to n minus 1.

So, you can you can put any other integer n, but it is going to be repetition of these roots for k belongs to z. So, it is going to be a repetition of one of these roots which are present here in particular k equals 0 corresponds to 1, the root 1 and then k equals 1 corresponds to the given root zeta itself etcetera. So, but this is not exactly what we want here, this would give us the factor z minus zeta power k, but this is not we wanted here. What we will do is we will observe the following. Notice, that 1 by zeta power k where k is in between 0 and n minus 1 k n integer is also a solution to the above equation.

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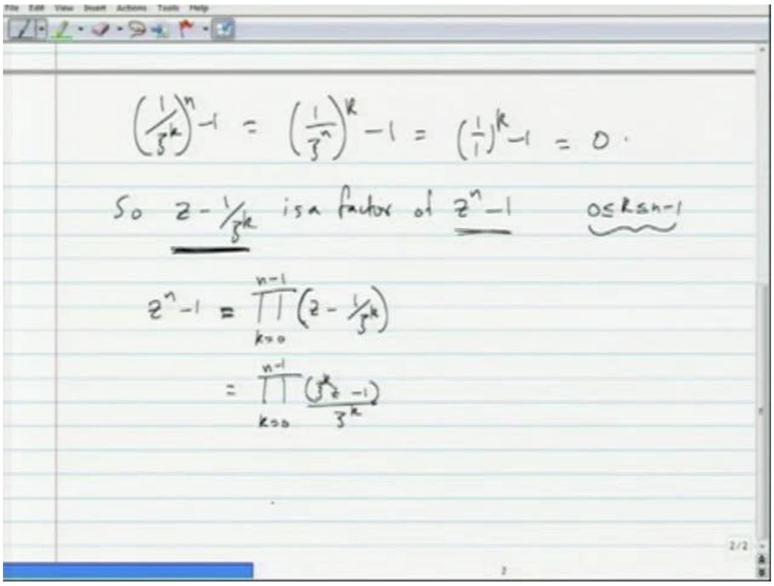

That is because well 1 by zeta power k raise to n minus 1 is equal to 1 by zeta power n raise to k minus 1 which is 1 by 1 power k minus 1 which is equal to 0. So, 1 by zeta power k is also a solution. So, what that gives us is that it gives a factor. So, z minus 1 by zeta power k is the factor of z power n minus 1 where k, I mean this is true for all k, k between 0 and n minus 1. And so we have got n factors for an nth degree polynomial z power n minus 1. What that means is we have a complete factorization of this polynomial. So, then z power n minus 1 is now equal to because of what I have said. This is now equal to the product from k equals 0 through n minus 1 of z minus 1 by zeta power k.

So, these are all distinct roots, that is also important. So, we have distinct factors and the product of these n factors. n factors gives us z power n minus 1. Now, I will multiply zeta power k or I will clear the denominator to get the product from k equals 0 through n minus 1 of zeta power k z minus 1 divided by zeta power k. Now, this looks almost like what I want, but I, but what I want here is 1 minus zeta power k times z equals 1 minus z power n. The product of them is 1 minus z power n.

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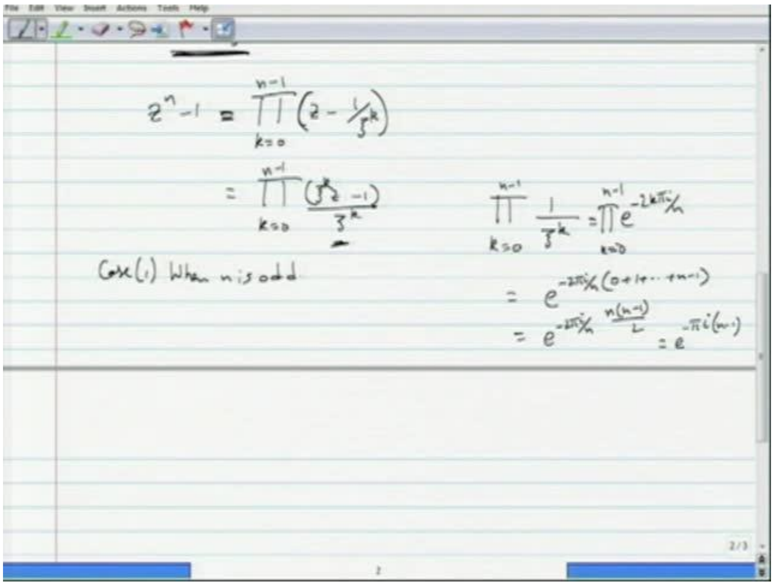

Product notice of k equals 0 through n minus 1 of 1 by zeta power k. The product of these denominators is equal to e power minus 2 k pi i by n, I am just substituting what zeta is. Zeta is 2 pi i by n. So, 1 by zeta power k will be e power minus 2 pi 2 k pi i by n product from k equals 0 through n minus 1. So, this product will give me e raise to minus 2 pi i by n times I will have to add all these.

So, I will get 0 plus 1 plus so on until n minus 1 because in the powers these case get added when I form the product. So, this gives me e raise to minus 2 pi i by n, the sum from n from 0 through n minus 1 is n times n minus 1 by 2. So, that gives me after cancellation that gives me e raise to minus pi i times n minus 1. So, now I will consider the two cases.

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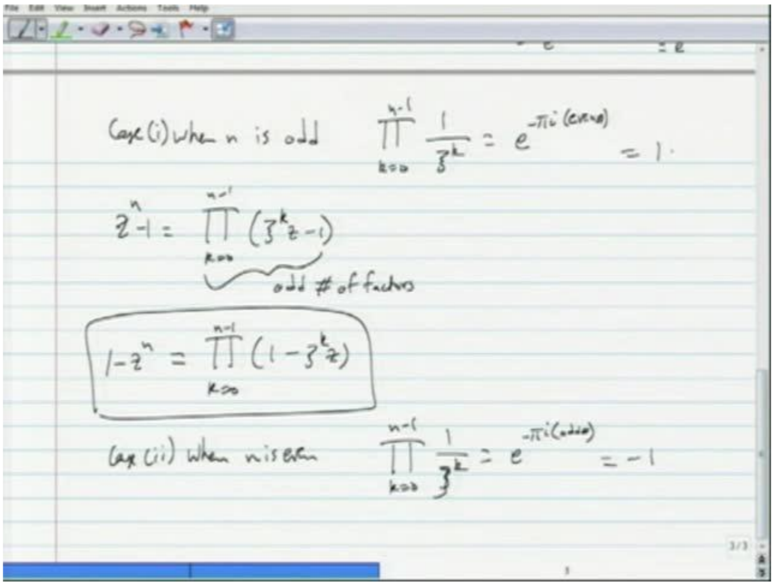

So, case one, when n is odd this product pi from k equals 0 through n minus 1 e raise to minus pi i times n minus 1 is or sorry when n is odd the product k equals 0 through n minus 1 of 1 by zeta power k is e raise to minus pi i times an even number. So, that gives me 1. So, this product z power n minus 1 is equal to the product from k equal to 0 through n minus 1 of zeta I apologize z for n minus 1, this is zeta power k z minus 1, these are odd number of factors.

So, when I switch each one of them to be 1 minus zeta power k z k equals 0 through n minus 1. I have odd number of these so I get a minus 1 outside. I have made odd number of switches so I will use the minus 1 to switch the left hand side to make it 1 minus z power n. So, that is easy. So, this shows what we want when n is odd. When n is even case two, when n is even the product in the denominator gives us pi k equals 0 through n minus 1 gives us 1 by zeta power k is e power minus pi times an odd number. So, this is minus 1.

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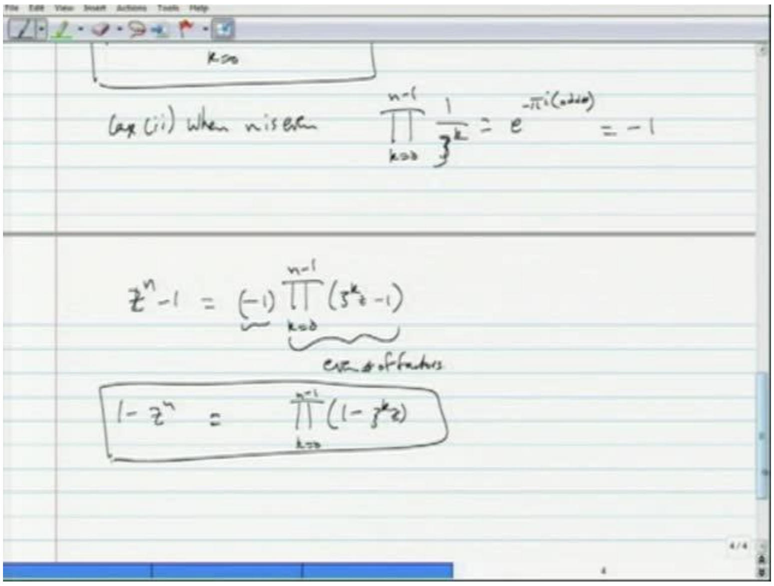

So, that leads us to 1 minus or z power n minus 1 equals minus 1 times the product from k equals 0 through n minus 1 zeta power k z minus 1. Now, these are even number of factors. So, when I switch them to 1 minus zeta power k z even number of switches will give me even number of minus 1 which cancels and given me 1. So, this minus 1 in you know ahead of all these I will use that to switch the left hand side and write this as 1 minus z power n which still gives me what I want. So, that shows this problem. Let us look at the following problem.

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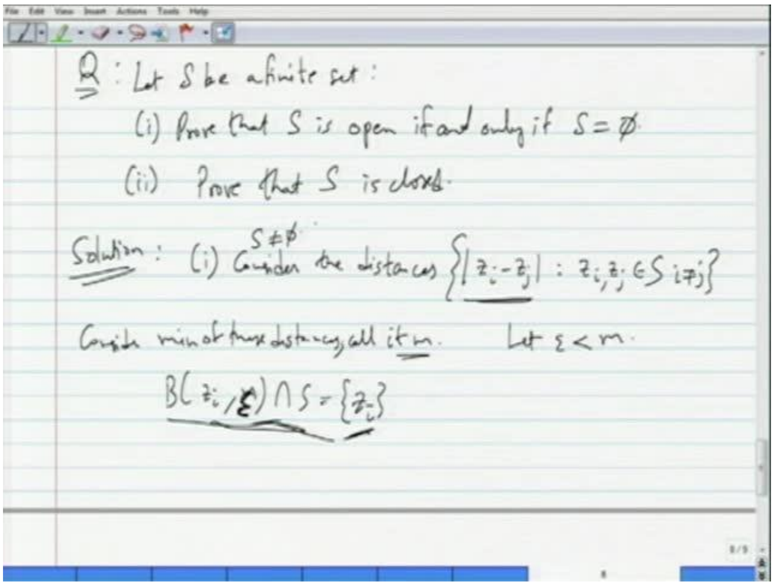

So, let S be a finite set. One, prove that S is open if and only if S is empty. Two, prove that S is closed. So, try to provide the solutions and then here is the solution. So, I will not really give the complete solution, but given intuition it is easy to see either of them. So, firstly S is a finite set, it could be empty. Consider the distances between all the points in S. So, suppose S is non empty so consider distances, modulus of z i minus z j such that z i comma z j belong to S. So, consider these distances i naught equal to j and there are only finitely many points in S.

So, we can consider these distances, consider the minimum of these distances. So, let let me call the minimum as little m, call it m, call it m, little m. Let epsilon be strictly less than m. Then if I consider ball of radius epsilon around z i then the intersection with S is just the point z i and so the set S cannot be open if it is non empty. And of course, if S is empty then it is definitely open so the opposite implication is the converse of this statement is clearly true.

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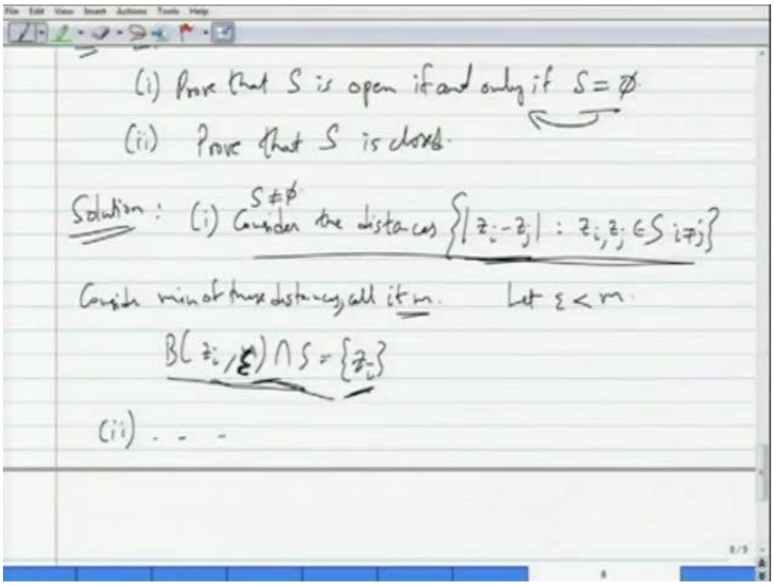

So, that is the proof of one and then two. Two is an easy exercise, show that, you know use the same principle, consider this minimum of distances and try to show that each point of S is an isolated point and so S union, the set of limit points is S itself because there are no limit points in S. So, each point is isolated so S becomes closed. So, two is an easy exercise.

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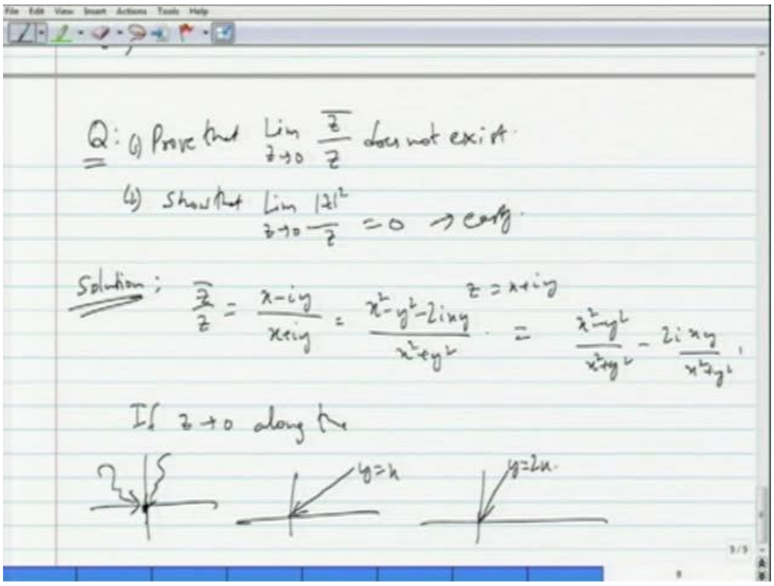

So, the next question is as follows. So, prove that the limit as z goes to 0 of z bar by z does not exist. And then that is one. Two, well show that the limit as z goes to 0 mod z square by z is equal to 0. This is easy, it is very clear. Try to show one and two. So, I will provide the solution here only to one, two is straight forward. So, z bar by z. Well, let us look at z bar by z that is x minus i y by x plus i y, if I take z equals x plus i y and by multiplying the conjugate in the denominator I get x square minus y square plus or may be minus 2 i x y divided by x square plus y square.

This is x square minus y square by x square plus y square minus 2 i x y by x square plus y square. So, if z approaches 0 along the line there are various directions in which z can approach 0. So, z could approach 0 in any fashion that it likes. So, in particular it can approach 0 along the line y equals x, it can approach 0 along the line y equals 2 x for example. So, let us consider these two lines. If z approaches 0 along the line y equals x then set y equals x in this expression.

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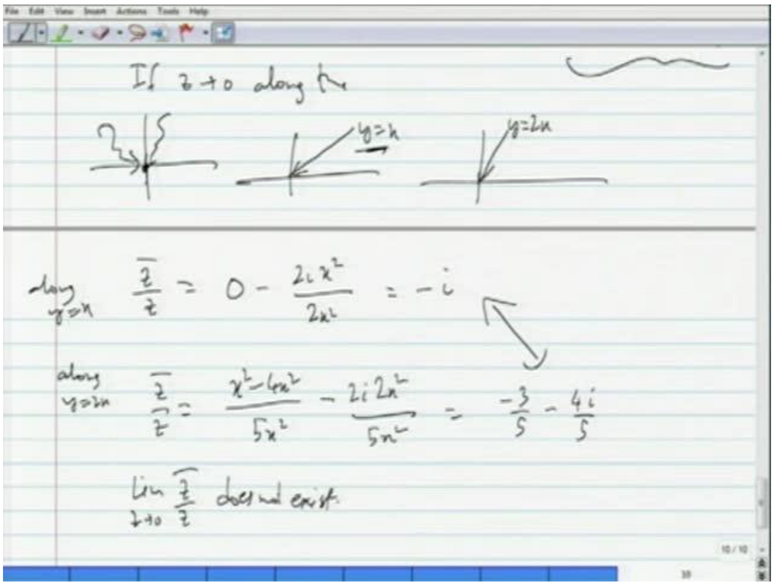

So, you get z bar by z along y equals x is equal to x square minus x square 0 so the real part is 0 and the imaginary part is x square divided by 2 x square. So, this is minus i and whereas, along y equals 2 x z bar by z is equal to… if I put y equals to x, I get x square minus 4 x square divided by x square plus 4 x square which is 5 x square minus 2 i times 2 x square divided by 5 x square which gives me minus 3 by 5 minus 4 by 4 i by 5 and these two are clearly not equal. So, the limit of z goes to 0 of z bar by z is does not exist. So, that is an easy example.

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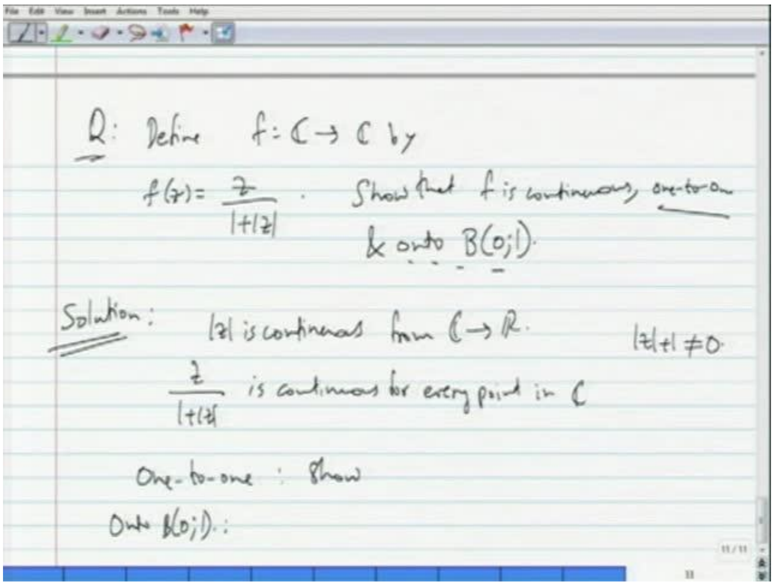

So, the next question is about continuous functions. So, define f from C to C by f of z is equal to z by 1 plus mod z. Show that f is continuous, one to one and onto d B 0 1. So, this is recall this is the ball of radius 1 centred at 0, open ball. Show that it is onto B 0 1. So, the solution is as follows. Well I will show that f is continuous, it is easy. So, 1 plus mod z notice the modulus function is continuous from C to R.

So, mod z is continuous from C to R. In particular you can consider it to be a function from C to C where R is identified with the real line in the complex plane. So, mod z is continuous and z is clearly continuous. So, z by 1 plus mod z is continuous by the rules for continuous functions. So, it is continuous for every point in C. Notice, mod z plus 1 is never 0, mod z is greater than or equal to 0 and so mod z plus 1 can is is greater than or equal to 1. So, it is never 0. So, this is continuous. That is easy. So, one to one is an exercise. That is an interesting exercise, show let us see our self and then it is on to B 0 1. So, why is this on to B 0 1? Firstly, notice that the image lies in B 0 1.

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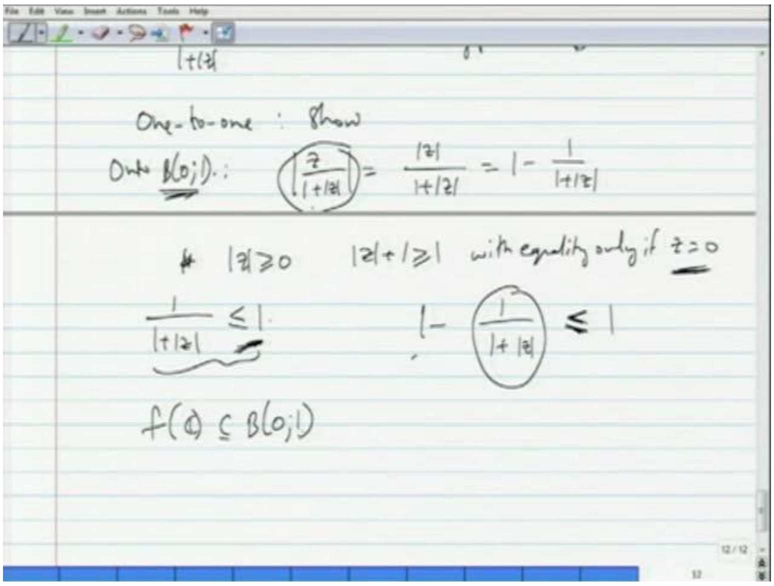

Because the modulus of z by 1 plus mod z, this is equal to mod z by 1 plus mod z which is 1 minus 1 by 1 plus mod z and mod z like I remarked earlier mod z greater than or equal to 0 so mod z plus 1 is greater than or equal to 1 with equality only if z is equal to 0. For any other z the the modulus of z plus 1 is strictly greater than 1. So, 1 by 1 plus mod z is less than or equal to 1 with equality only if z is equal to 0. So, 1 minus 1 by 1 plus mod z is less than or equal to… So, this is less than or equal to 1.

You are removing a quantity which is strictly less than or less than or equal to 1, let us say strictly less than 1 from 1 that will be strictly less than 1 and when z is equal to 0 equality holds here and then clearly this number 0. So, that is also in B 0 1. So, all in all the image of this function lies in B 0 1. So, it is, it is definitely contained in B 0 1. I mean f of C is contained in B 0 1, but is it onto so if w is a point in the unit ball.

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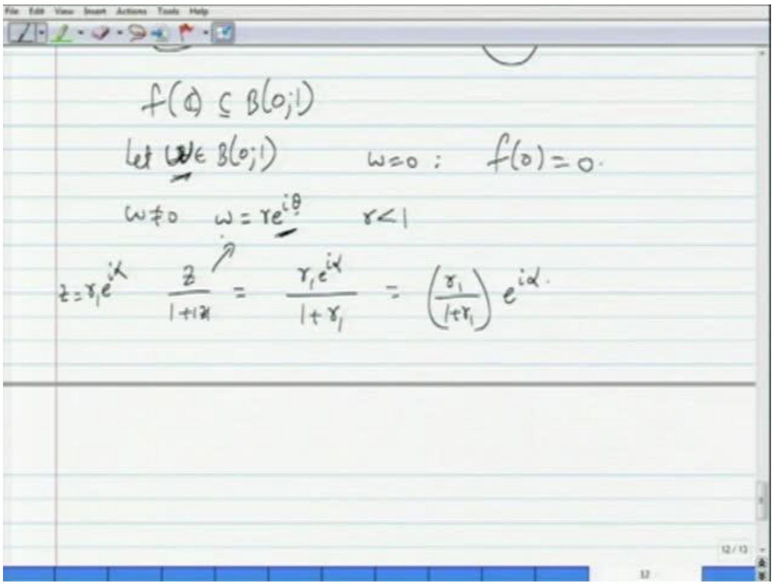

So, w can be 0 in which case set I mean f of 0 we know is 0. So, w is in the image of f. If w is non zero it can be written as w is r r e power i theta where we know that r is strictly less than 1 and we also know that theta well, theta is any real number. So, theta is a particular real number corresponding to the argument of w.

So, what you can do is z by 1 plus mod z, if you write z as r 1 e power i alpha then we want the, this to be equal to r e power i theta, this quantity to be equal to r e power i theta. This is nothing but r 1 e power i alpha divided by 1 plus r 1. So, this can be written as r 1 by 1 plus r 1 times e power i alpha. So, definitely you can find a number alpha.

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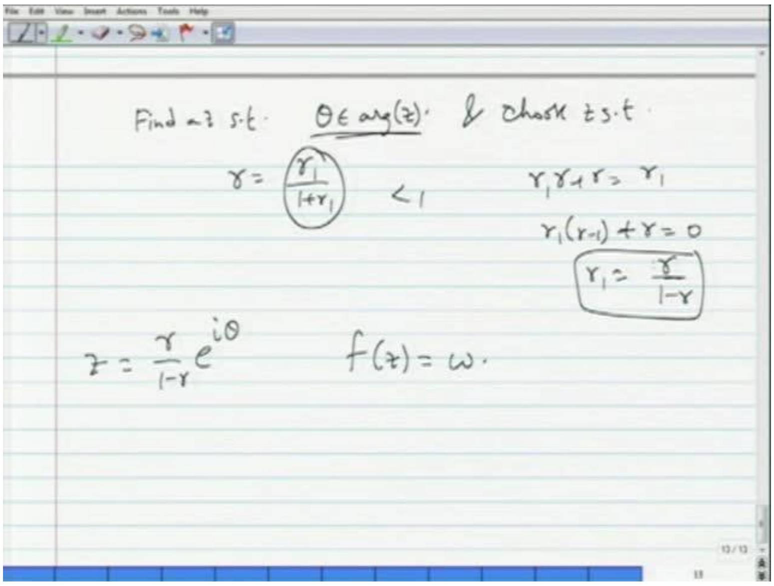

So, you find a find a z such that argument of z there is I mean theta belongs to argument of z, theta belongs to argument of z. That is easy, there are many points to pick from and then you choose, you know choose r, choose and choose z such that r is equal to r 1 by 1 plus r 1. Notice, r 1 by 1 plus r 1 is strictly less than 1. So, you can solve for r 1. So, r 1 times r 1 times r plus r is equal to r 1.

So, r 1 times r minus 1 is equal to plus r is equal to 0 r by 1 minus r. So, if you choose r 1 is equal to r by 1 minus r and theta belongs to r z that that z so r by 1 minus r e power i theta is a complex number z whose image is equal to the given w. So, definitely this map is onto, so that shows this exercise. The next problem is as follows.

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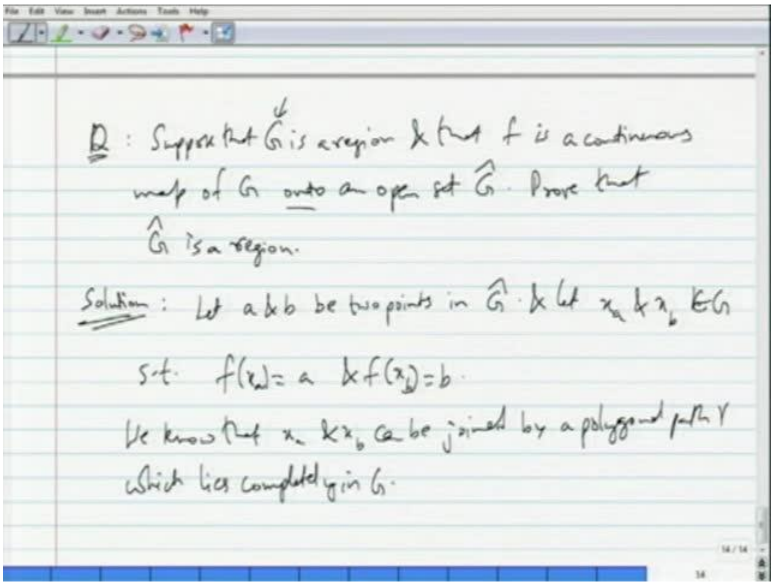

Suppose, that G is a region and that f is a continuous map of G onto, it is important, onto an open set G hat. Prove that G hat is a region; it is given to be an open set. So, show that it is actually region which means you have to show that it is connected. It is non empty; it is clear because it is a image of a function of a image of a non empty set under a function.

So, it is definitely non empty, but try to show that it is connected. So, this is the problem and I will provide the solution here. So, let a and b be two points in G hat. So, recall what we have to show? In order to show that G hat is the region we have to show that these points a and b are connected via a polygonal path. So, we need to demonstrate this path. All we know is that the pre images of a and b in G can be joined by polygonal path because after all G is given to be connected.

So, let a and b be two points in G hat and let x a and x b belong to G such that f of x a is equal to a and f of x b is equal to b. There can be many choices, you choose one x a and x b and we know that x a and x b can be joined by a polygonal path, polygonal path gamma which lies completely in G.

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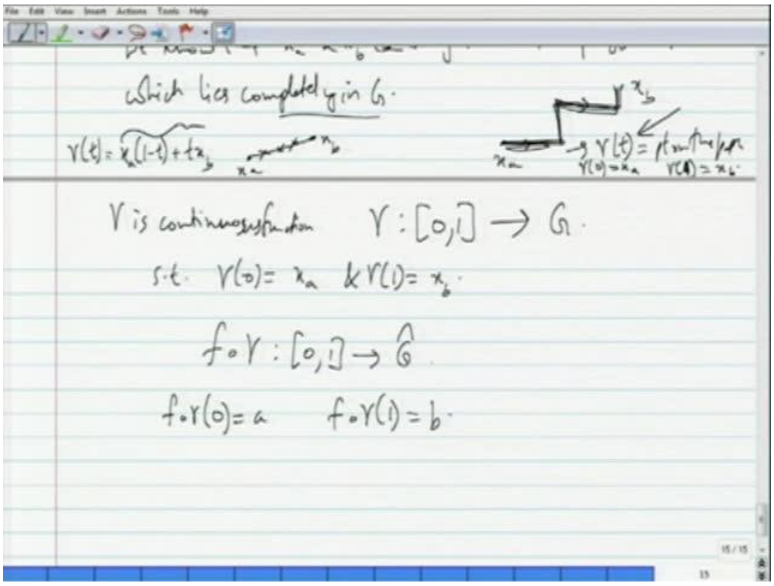

So, if you have a polygonal path which starts at x a and ends at x b you can always parameterise it, what I mean by that is you can write equations so that gamma of t is equal to any point on this path, point on the path with gamma of 0 is x a and gamma of 1 is x b for example, if you have a straight line segment joining x a and x b, gamma of t can be x a times 1 minus t plus t times x b, t equals 0 gives us x a t equals 1 gives us x b and any point here is on the straight line segment connecting x a and x b.

So, you can extend this to these kinds of a polygonal paths which are completely contained in G. So, the property of such a gamma is that you can ensure that gamma is continuous function. So, what kind of entity is this gamma? Gamma is actually a function from 0 1 to the complex plane or in particular to G because the image of gamma completely lies in G, the polygonal path lies in G.

So, gamma is a continuous function, continuous function gamma from 0 1 to G such that gamma of 0 is equal to x a and gamma of 1 is equal to x b. Then what you can do is you compose f with gamma. So, f circle gamma, f is given to be continuous from 0 1. Now, the domain of gamma is 0 1, this is going to G and then G, f takes G to G hat. So, f circle gamma is a function from 0 1 to G hat, it has the property that f circle gamma of 0 is a and f circle gamma of 1 is equal to b.

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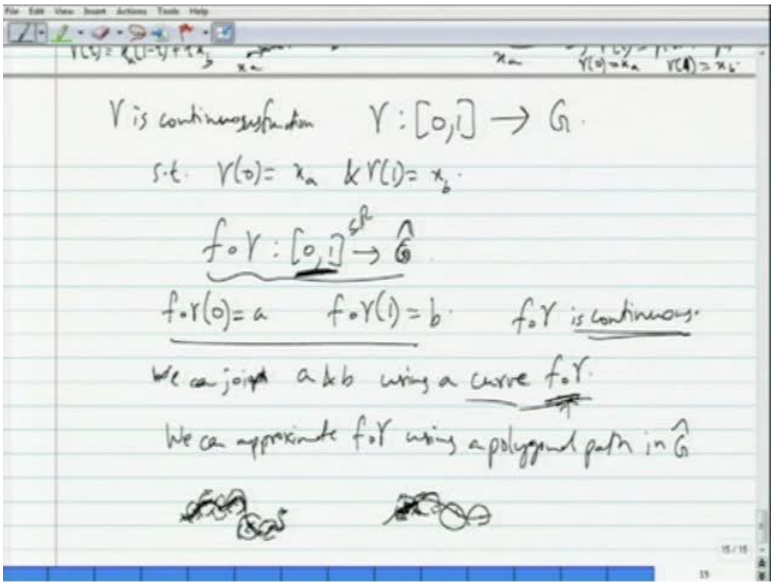

And f circle gamma is continuous because it is the composition of continuous functions. So, so using this f circle gamma what you can do is you can… so, we can join, we can join a and b using a path, a curve, let us call this a curve, a little informally a curve f circle gamma. Now, f circle gamma need not be a a path, a polygonal path, but once we have a curve joining f one joining a and b, we can correct it, we can sort of linearize it, approximate it using using a… So, we can approximate f circle gamma using a polygonal path in G hat. The approximation comes from the fact that G hat is open.

So, once you have two points and then there is some curve joining these two, around each point you can, this is a compact set because it is the image of a compact set, 0 1 is compact in real numbers and then the the image of a, continuous image of a compact set is compact in G hat. So, when you have a compact set you collect all the balls of a certain radius and that forms an open cover of this path and so you can by compactness you can choose finitely many balls and then those epsilon balls will cover the path, those finitely many balls will cover the path, the the curve f circle gamma.

And within each of these balls what you can do is, however curvy this is, however curvy f circle gamma is you can correct it, you can correct it sort of to be a polygonal path. You just pick a point here, a point here and then take the straight line for example, connecting these two points and then now this point, the next point will lie in the overlap of then, of this ball and the next ball and you can keep repeating this process. So, that is polygonal path in G hat and that completes this problem. So, we will see more concretely paths and curves in next module, but for now that that is an explanation for this problem. So, next we have the following question.

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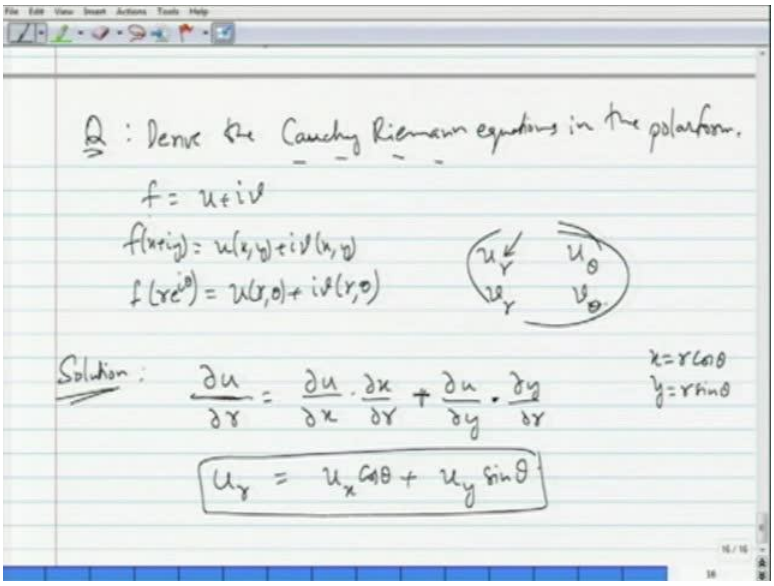

So, derive the Cauchy Riemann equations in the polar form. What I mean by that is use the polar coordinates to get equivalent of Cauchy Riemann equations. So, your equations should finally, look like the following. So, if f is equal to u plus i v, f is a function of z, f is a function of x plus i y is u of x comma y plus i times v of x comma y. So, you can also consider if x plus i y is non zero you can also consider f to be a function of r e power i theta then you have u of r comma theta plus i times v of r comma theta. So, there are partial derivatives u r and u theta and v r and v theta.

So, use the already known Cauchy Riemann equations to establish a relationship among these partial derivatives. So, that is the problem. Try to do this exercise and I will present the solution here. So, we know that well dou u by dou r so, I will use chain rule on partial derivatives, dou u by dou r which is u r here is equal to dou u by dou x times dou x by dou r. Use the function of both x and y so this is also plus dou u by dou y times dou y by dou r. So, this is a simple exercise in chain rule for partial derivatives. So, this is u x and then x recall is r cosine theta and y is r sin theta. So, the derivative of x with respect to r is simply cosine theta and then this is u y times the derivative of y with respect to r is a sin theta. So, I will write this as u r for short. We will preserve this.

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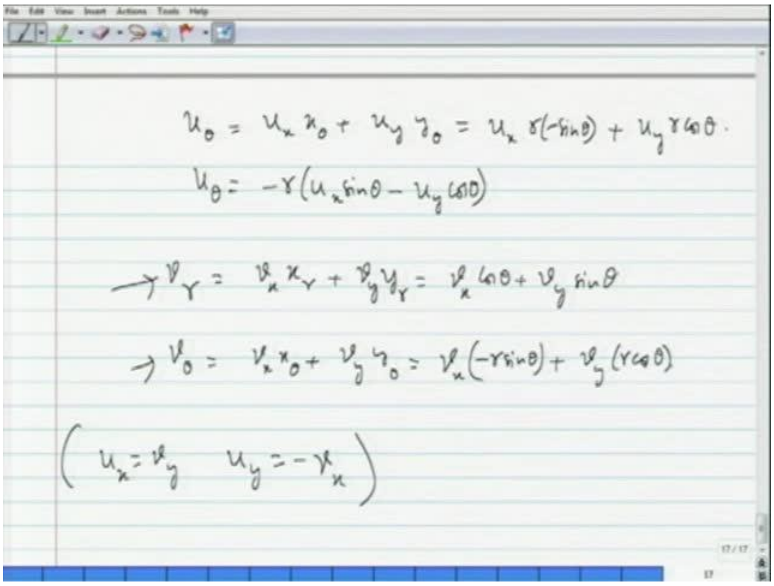

And likewise u theta is u x x theta plus u y y theta. So, the derivative of u with respect to x times x theta, the derivative of x with respect to theta is r times minus sin theta plus u y times the derivative of y with respect to theta is r cosine theta. So, u theta is minus r times u x sin theta minus u y cosine theta. Likewise v x one can calculate is sorry v r one can calculate is v x x r plus v y y r which is going to be v x cosine theta plus v y sin theta. And v theta is going to be v x x theta plus v y y theta which is v x minus r sin theta plus v y r cosine theta and we will use the fact that, I mean we will use the Cauchy Riemann equations u x equals v y and u y is equal to v x or minus v x sorry minus v x in these two derivatives.

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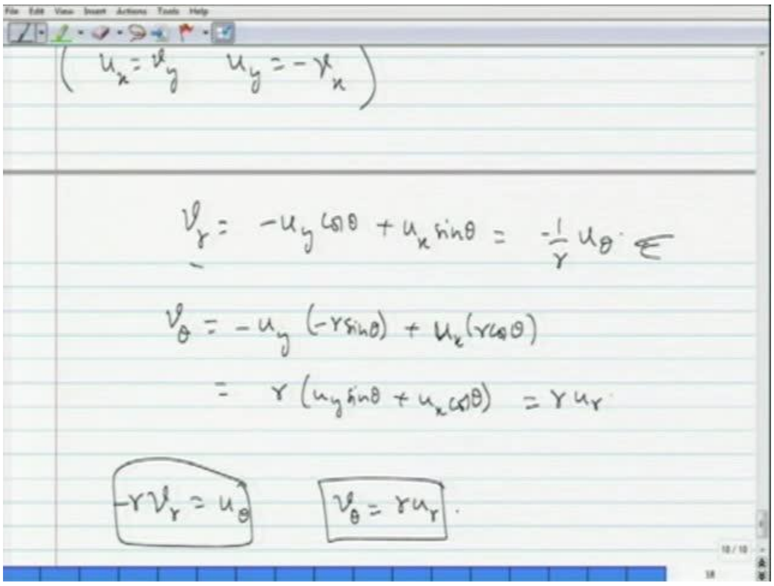

So, using those what we get is v r is equal to v x is minus u y cosine theta plus v y is u x sin theta and notice from one of the preserved expressions, this is minus u y cosine theta plus u x sin theta appears here. So, this is minus 1 by r times u theta. So, this is minus 1 by r times u theta, and then likewise v theta is v x, which is now minus u y times minus r sin theta plus v y, which is now u x times r cosine theta is going to be r times u y sin theta plus u x cosine theta and u y sin theta plus u x cosine theta is u r.

So, this is r times u r. So, the… We have new Cauchy Riemann equations in polar coordinates, it says v r or minus r v r is u theta, this is from here and then v theta is r u r. So, these are the Cauchy Riemann equations in the polar form.