Complex Analysis
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Module - 2
Complex Functions: Limits, Continuity and Differentiation
Lecture - 7
Branches of Multifunctions; Hyperbolic Functions

Hello viewers, we saw that the argument of a complex number is not a function surely, because there is no unique choice of argument. So, that happens, because the functions the real cosine and sin functions are 2 pi periodic. So, that has other repercussions as we will find out today.

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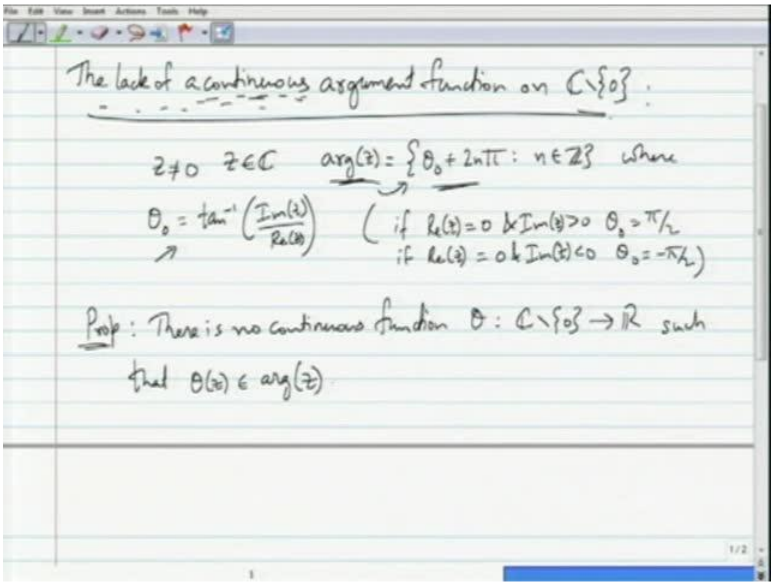

So, let us start with the lack of a continuous argument function on C minus 0. So, of course we do not define the argument of the complex number 0, but the emphasis here is one continuous. There is no continuous a argument function on C minus 0. So, what I mean by this is, so given a complex number z non zero, so we define the argument of z to be the set of all theta naught plus 2 n pi, such that n belongs to integers, where theta naught is equal to recall the tan inverse of y or I will say the imaginary part of z divided by real part of z.

With the abuse interpretation that if real part of z is equal to 0, and imaginary part of z is strictly positive, we say theta naught is pi by 2. If real part of 0 z is 0 and imaginary part of z is strictly less than 0, then we take theta naught to the minus pi by 2 or you can take that to be 3 pi by 2. So, with these, so with these modifications theta naught given by this formula gives us a set argument of z is equal to this set here, all right? So, the point of this statement is that or the point of this contention here is that there is no continuous way in which we can pick one argument for each z in C minus 0.

So, here is a preposition which states this concretely. So, preposition, there is no continuous function theta from C minus 0 to R, such that theta of z belongs to the argument of… So, we cannot pick arguments in in a coherent way such that, you you have a continuous function and for each z you actually extract the argument from that set theta naught plus 2 n pi. So, here is a proof.

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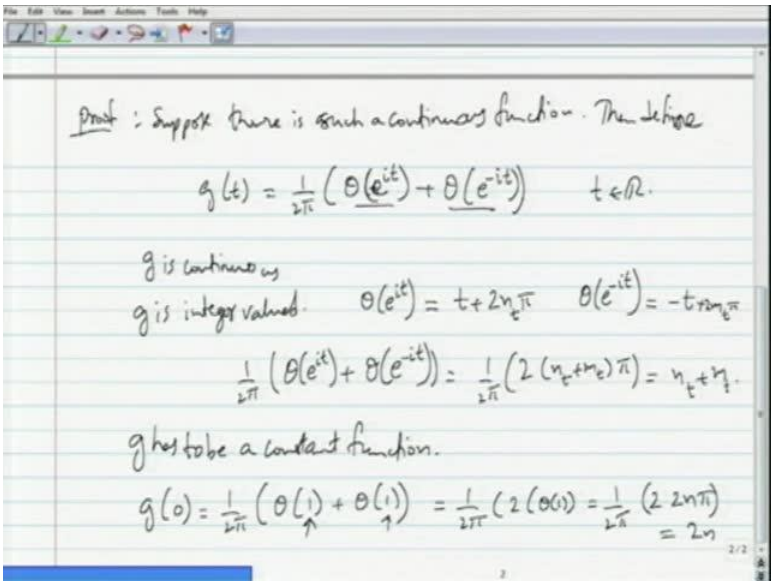

Suppose, suppose there is such a continuous function. So, define then define g from well, g of t is equal to 1 by 2 pi of 1 by 2 pi times theta of e power i t, plus theta of e power minus i t. So, where g is a function, where t is a real number. So, then g of g is continuous firstly. The addition of two continuous functions divided by 2 pi. So, g is continuous that is clear and g is integer value because theta of e power i t, that has to look like t plus 2 n t pi for some integer n and theta of e power minus i t has to look like minus t.

That is one argument of e power minus i t plus 2 m t pi, for some other integer m t possibly. So, then theta of e power i t plus theta of e power minus i t 1 by 2 pi of this is 1 by 2 pi times minus e and t cancels. So, you have 2 times n t plus m t times pi, which is n t plus m t, which is definitely an integer. So, g is integer value and it is a continuous function. The only continuous integer value functions from real numbers 2 R, R constant.

So, g has to be a constant, so g has to be a constant function, but notice that g of 0 is one by 2 pi times theta of e power 0, which is 1 plus theta of 1, which is going to be 1 by 2 pi time spell. You have to choose an argument for 1 and that same thing applies to this. So, that has to be 2 times theta of 1. So, this is 1 by 2 pi times, what is an argument for theta of 1? It has to look like 2 n pi. So, this is going to be 4 2 n, which is an even number, even integer, this is an even integer.

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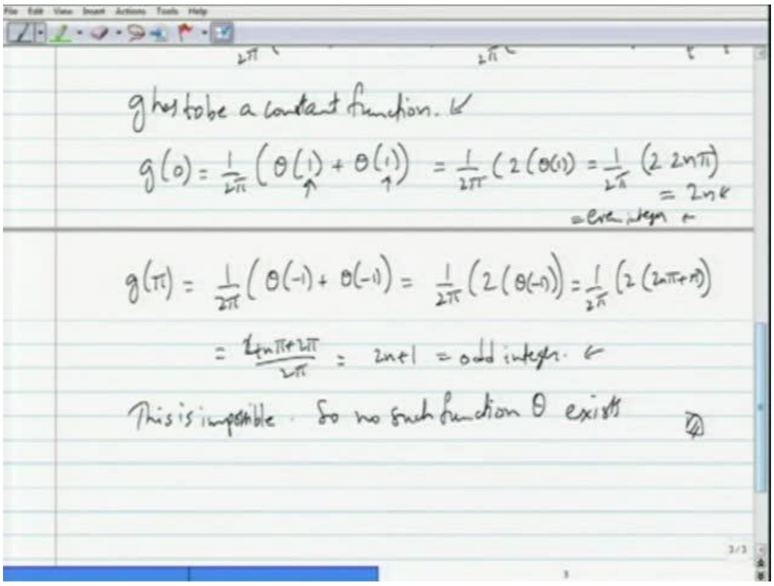

Whereas g of pi observe is equal to 1 by 2 pi times theta of e power i pi, which is minus 1 plus theta of minus 1, which is 1 by 2 pi times 2 times theta of minus 1, well an argument for theta of minus 1 has to look like 2 n pi plus pi for some integer n. So, that gives us two times. So, 4 pi plus 2 pi by 2 pi, which is 2 n plus 1, which is an odd integer. So, this n and this n need not be same, so that could I I could have taken an m here, but none of less this is an odd integer. So, we have a constant function, which takes even value at 0 and odd integer at pi, this is impossible, so this is impossible. So, no such function theta exists. That shows the impossibility of a continuous argument function on the complex plane minus 0.

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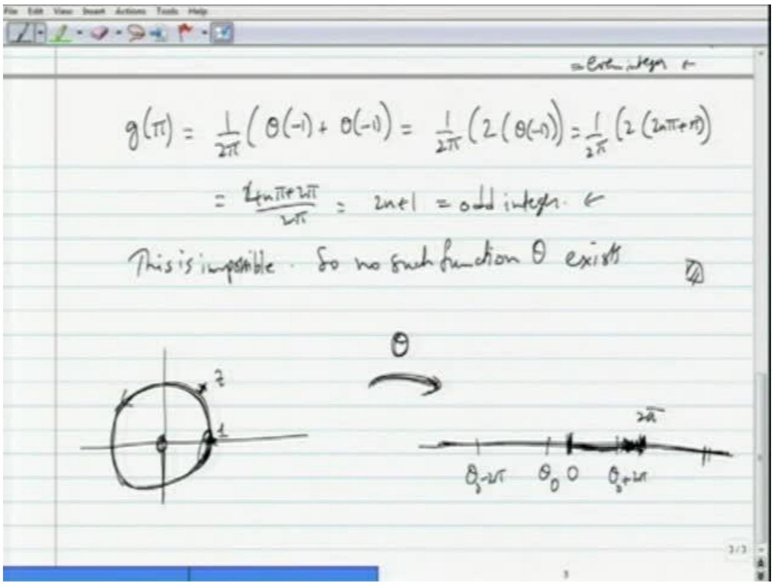

So, if we look at the visualization of this argument of this argument in the complex plane. So, if we look at C minus 0 and then this argument is what call as an multifunction because for each z, we have infinitely many values of the argument. So, this is a multifunction theta. So, if we look at any z and if we pick one value theta naught, which is an argument for this z, which belongs to arguments of z, then theta naught plus 2 pi theta naught minus 2 pi etcetera on the real line.

They are all us you know, they are all values of this theta for this particular z. So, in particular when we consider a circle like that, let us pretend that is the unit circle. As we traverse along the circle in this direction, so we are starting at a particular argument for 1 let say. So, we are starting at let us pretend 0, we start at 0, which is an argument, n argument for 1. As we go like this in choosing a continuous theta naught, we end up you know closer to 2 pi, by the time we reach here.

So, we end up at 2 pi by the time we reach here. So, a points here, near here which should be closer to 0 r naught and hence there is no continuous. So, in the domain this points definitely are closer to 1, but in the co domain for this, suppose multifunction the points are closer to 2 pi they are not closer to 0. So, this vaguely explains why the above preposition works. So, visually explains why the above preposition works. So, we are closer to 2 pi and not to 0.

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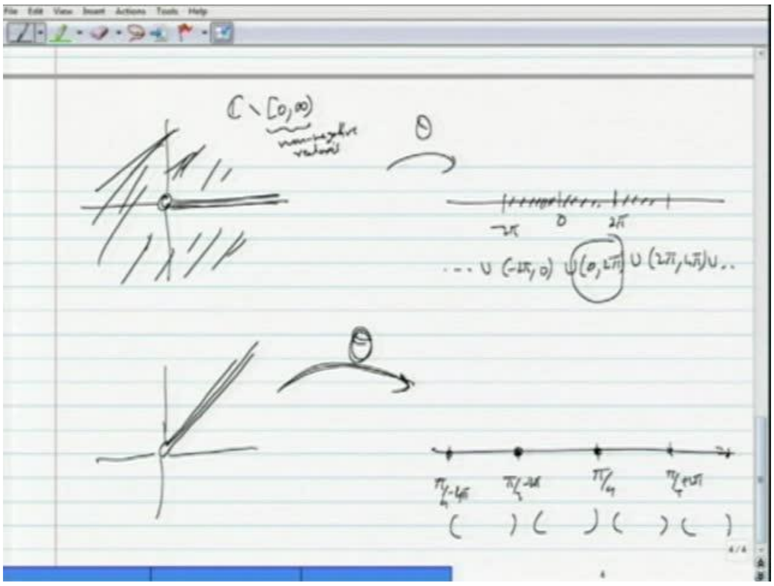

So, what we can do to remedies a situation is we can actually delete the non negative real accesses from the complex plane. And if we consider this particular domain, so this is C minus 0 infinity, this stands for the non negative real access. So, then there is a multifunction theta, what it does is. It takes it takes C minus 0 infinity to pieces of intervals between multiples integer multiples of 2 pi. So, it maps this whole thing to give pieces of intervals like that minus 2 pi comma 0 union 0 comma 2 pi union 2 pi comma 4 pi etcetera.

So, in this direction and this direction as well. So, you have several pieces of this intervals, open intervals. You have several pieces and then they are of length 2 pi. So, what we can do to construct a continuous function is we pick, we choose on particular interval like that. Let us say 0 to 2 pi, if you pick the piece 0 to 2 pi then the function theta the the argument function, we can construct a continuous argument function from C minus 0 infinity to this interval. So, that is the idea. So, we can remedy the situation by deleting a particular half ray including 0 like that, starting at 0 like that.

So, there is no reason why we should pick the real axis the non negative real axis alone. What we can do is, we can pick any half ray any half ray, let us say the 45 degree line like that, starting at 0. So, if you delete the 45 degree line, then also we can make a choice of theta, we can make a continuous function out of this multifunction theta. I am calling it multifunction because the for any given complex number z in C minus 0, there are infinite values remaining on the in the co domain. So, it is not a really a function, it is a multifunction.

So, so if you pick this 45 degree line we have, we have now or if you delete that 45 degree line we have now pi by 4 and pie by 4 minus 2 pi pi by 4 minus 4 pi in that direction and pi by 4 plus 2 pi pi by 4 plus 4 pi etcetera. So, actually you are deleting these these points from the real line y R theta. So, you have these intervals this open intervals whose union is in the range of this multifunction theta. So, and also given any complex number z, which is not on this half ray it has a unique value in any of this given intervals.

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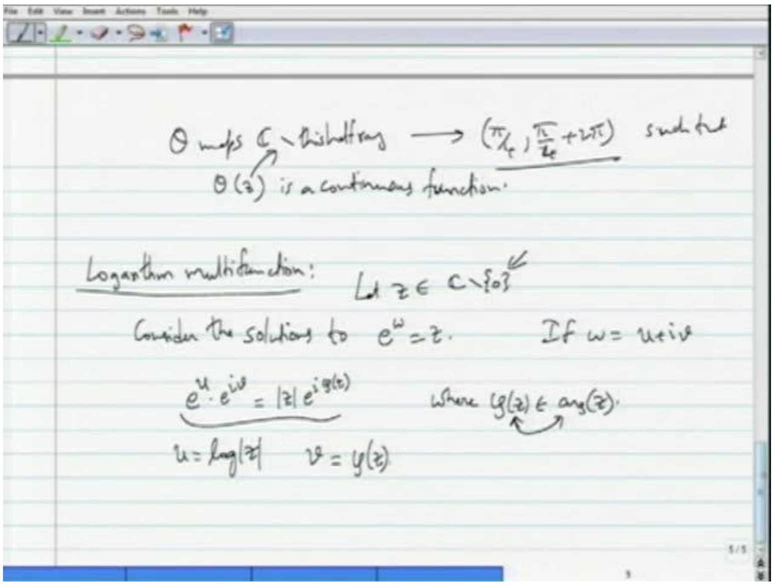

So, we can pick any one of the intervals and so theta theta maps C minus this half ray, half ray to pi by 4. Let say I will I will make a choice, I will pick one of them pi by 4 comma pi by 4 plus 2 pi, such that theta of z belonging to this C minus this half ray is a continuous function. It is a function firstly and then it is a continuous function. So, there are two components which need observation here. So, there is a choice of this half ray first, which divides the real line into pieces and then there is a second choice that is made, you pick one of this pieces.

The real line follows apart into pieces, you pick one of these. So, there are two choices that we make to make theta a continuous function. So, that is one remedy add to this situation. So, we will see that this has other repercussions. So, the impossibility of a continuous argument function C minus 0 has other repercussions. So, here is the logarithm multifunction. So, we have already seen the exponential function. So, consider the solutions to… So, let us first, let z belong to C minus 0, so consider the solutions to z equals e power w or e power w equals z.

So, we know that e power w never take the value 0, that is why we omitted 0. So, we are interested in all such w, which give us e power w equals z. So, if w equals u plus i v, you immediately observe that, you can write e power u times e power i v is equal to modulus of z times e power i times a phi of z, where phi of z belongs to argument of z. So, it is a choice of an argument of z. So, any choice of an argument of z gives this equation.

So, u equals setting u equals logarithm, the real logarithm of the modulus of z. Modulus of z is always strictly positive for z naught equal to 0. So, we the logarithm of that is defined. Of course, all these are to the base e. So, I will ignore the base e because I will always consider the base e in this context. Then you notice that v can be taken to b phi of z. So, any values here can be entertained to be phi of z and then v with that will satisfy this equation.

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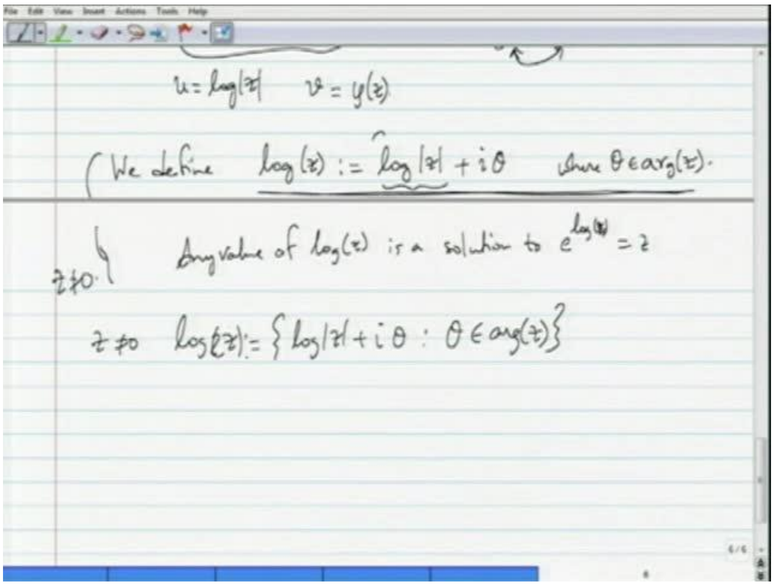

So, we define the logarithm of a complex number z, non zero complex number z to be the logarithm of the modulus of z plus. So, this is the real logarithm, note plus I times theta, where theta belongs to the argument of z. So, the motivation is that e this is a solution to…

So any value of log z is a solution to the above to e power log z is equal to z. z is non zero in all of this discussion z is non zero, so the log as it stands is a set because the theta belongs to argument of z. So, this is really a set, so log z can should actually be written as log mod z plus i times theta, such that theta belongs to argument of z. z naught equal to 0, so z naught equal to 0. So, once again log z because argument is a multifunction log z is a multifunction and it takes many values that is the idea and once again.

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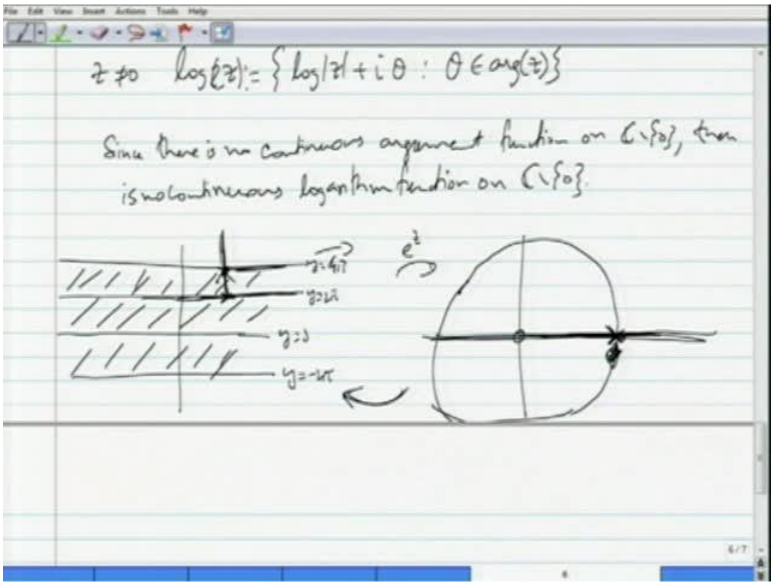

Since, there is no continuous argument function on C minus 0, there is no continuous log function logarithm function on C minus 0. So, individually finding a logarithm of an non zero complex number is not a problem. You just make a choice from this defined set, that works or you can say that for a given non zero z, you know these are all the possible logarithms. But the problem you know comes from varying z, all across the the range of e power z, namely complex plane minus 0. If you start varying z in C minus 0, we cannot, unfortunately we cannot find continuous logarithm function.

So, like we have done for the argument function, what we can do to the remedies a situation is, we can delete a particular half ray. We know the, then there is a continuous argument function and hence there will be a continuous logarithm function. So, let us look at the visualization of the exponential function, which we saw earlier and try to see what happens when we try to get an inverse function to the exponential function. After all that is what this logarithm function is we are trying seek solutions to e power w, w equals to z.

So, we saw that the exponential function takes this stags of horizontal strips. So, these are y equals 0 y equals 2 pi y equals 4 pi etcetera y equals minus 2 pi. So, it it takes these stags of vertical a horizontal strips and maps each of the stags to all of the complex plane minus 0. So, if you start at any particular y equals to n pi line. So, then you are beginning on this half ray as you proceed along a vertical line like that. So, if you start at any point here, you will start at some point here and as you proceed along a vertical line, you go along a circle like that.

We saw this during mapping properties of e power z and when you approach that next to n plus 1 pi. So, in this case 4 pi you are closer to that starting point, where you started off. So, and hence you know there is no continuous choice, when you want to come back. So, when you come from here to here these points are all closer to this point, but these points are not closer to the starting point. So, we remedied the situation by actually deleting a half ray herein, the co domain of e power z in in the case of the argument function multifunction, we had the real line.

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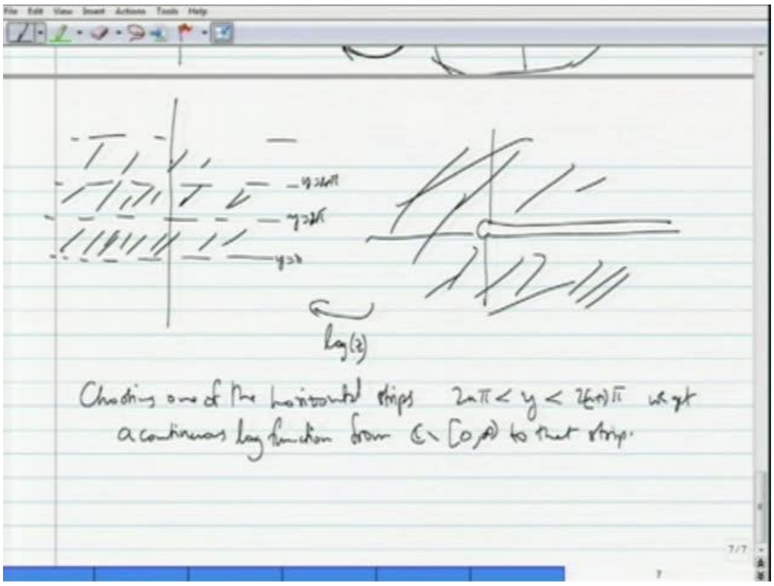

Here for e power z, what we will do is we will delete a half ray. So, that the image of this under this multifunction, now is bunch of horizontal strips with that line missing. So, these are lines y equals 0 y equal 2 pi y equals 4 pi etcetera. So, you can pick one of these horizontal strips and then the log z, where z belongs to this domain here is a continuous function. So, choosing one of the horizontal strips 2 n pi less than y strictly less than 2 n plus 1 pi, we get a continuous a log a function from C minus 0 infinity to that strip.

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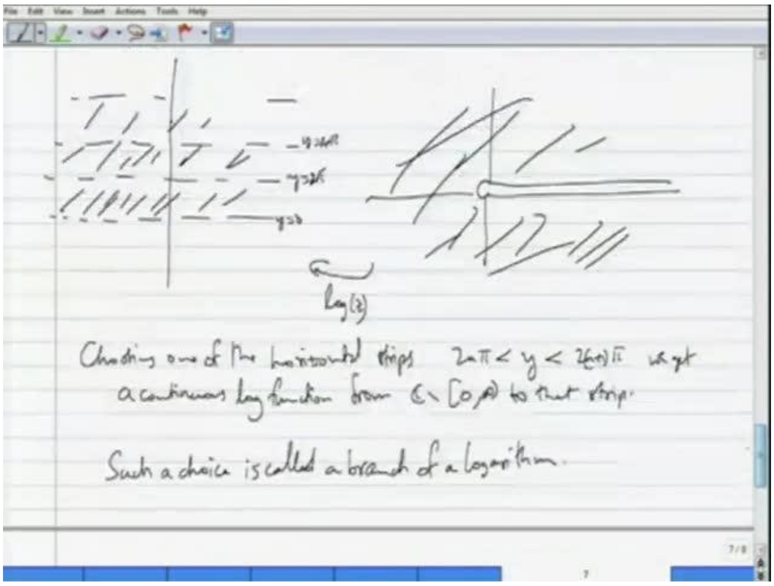

So, then such a choice is called a branch of a logarithm so and such a choice is called a branch of a logarithm. So, notice that this exactly corresponds to what we have done with the argument function argument multifunction, after all these angles correspond to the numbers on the y axis.

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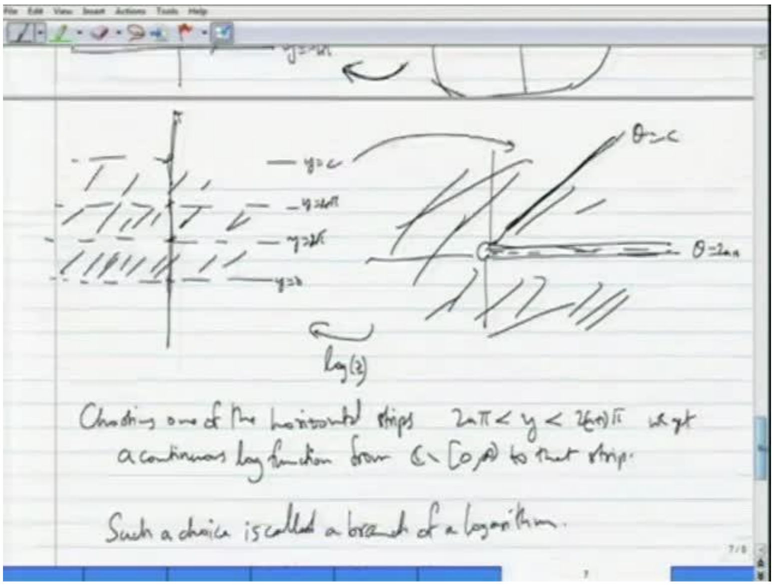

A constant y equals C, where are the exponential map is mapped on to theta equals C in the complex plain. So, y equals C goes theta equal C, we saw while, we saw the visualization of exponential function. So, y equals 4 pi or y equals 2 pi etcetera. They go to theta equals to n pi, which exactly corresponds to the real line. So, if you drop these exact points like we did for the argument function, we get these horizontal strips and and hence we can construct, what is called a branch of a logarithm. So, once again there are two choices involved, first is the selection of this half ray.

I could have chosen the 45 degree line or any theta equals C half ray. I mean so any theta equals C half ray and that would have given me a corresponding cutting of the complex plane. Here in the in the domain of e power z in to stags of horizontal strips. Then the second choice is of one of these horizontal strips to construct a continuous logarithm function from from the domain the complex plane minus half ray. So, that is the idea and then we will define a branch of a logarithm. So a function...

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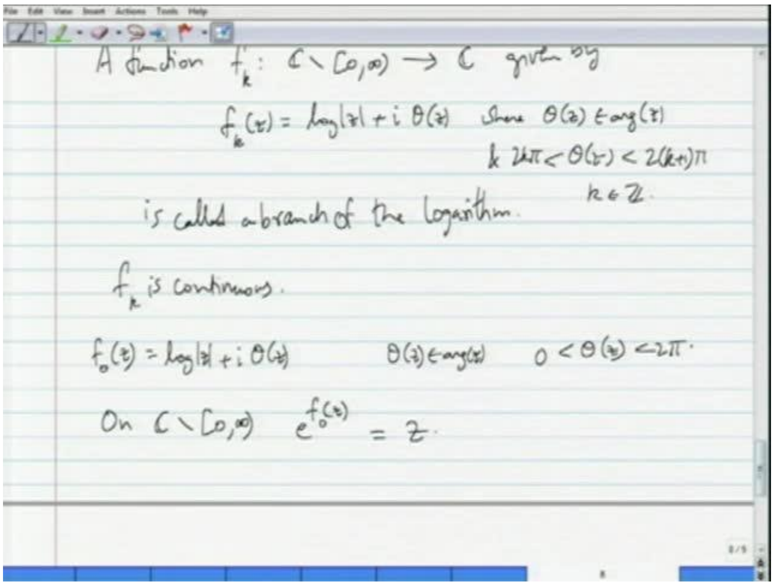

So, for one choice of these half ray namely a function f k from C minus 0 infinity to C given by f k of z is equal to log mod z plus I times theta of z, where theta of z belongs to argument of z and theta of z lies between at 2 k pi and 2 k plus 1 by is called a branch of logarithm, the logarithm. So, read that as branch of the logarithm multi function, so what we will show is. So, f k is continuous because the real and imaginary parts of f k are continuous functions on C min on that on the domain.

So, now we will use this to show that so k, I did not mention what k is? What k should be? k is any particular integer, k is the fixed integer. So, for any value of k we will show that f k is actually analytic not only continuous, but also analytic. So, here is the proof, so let me choose a particular k and that will work for any general k. So, let us look at f 0 of z, so which is log mod z plus i times theta of z, where theta of z lies between belongs to argument of z and theta of z lies between 0 and 2 pi so on. C minus 0 infinity, it is also true well by construction e power f naught of z is equal to z.

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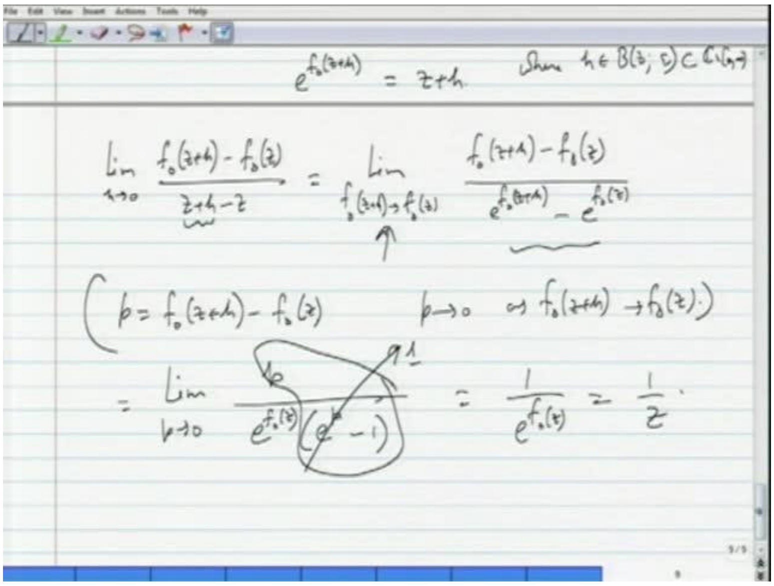

So, e power f naught of z plus h is equal to z plus h, where h belongs to such a ball around z, such that this is completely contained in the domain of f naught contained in C minus. So, that is true and then so the limit as h goes to 0 of f naught of z plus h minus f naught of z by z plus h minus z, which is h. This can be written as limit as f naught of z plus h goes to f naught of z. Now, that true because f is continuous.

So, we are using the continuity of f naught here and then the numerators f naught of z plus h minus f naught of z divided by… I will write z plus h here as e power f naught of z plus h minus and then z as e power f naught of z. Then what I can do is, I will write this further as limit as f naught of z plus h minus f naught of z. So, let me take p equals f naught of z plus h minus f naught of z and p approaches 0 as f naught of z plus h tend to f naught of z. So, this limit here can be written as this is equal to a limit as p goes to 0 of the numerator.

Now, becomes p divided by e raise to well I can say f naught of z I can take f naught of z outside. Then I have e raise to p, so e power p times e power f naught of z give me p plus e raise to p plus f naught of z, which is f naught of z plus h. So, this is minus 1. Then we can identify the limit of this quantity as p goes to 0 is 1. So, this gives us limit as well that is that quantity goes to 1, so this is 1 by e power f naught of z and all of the domain of this function f naught, this is nothing but z. so indeed f naught is analytic f naught is analytic.

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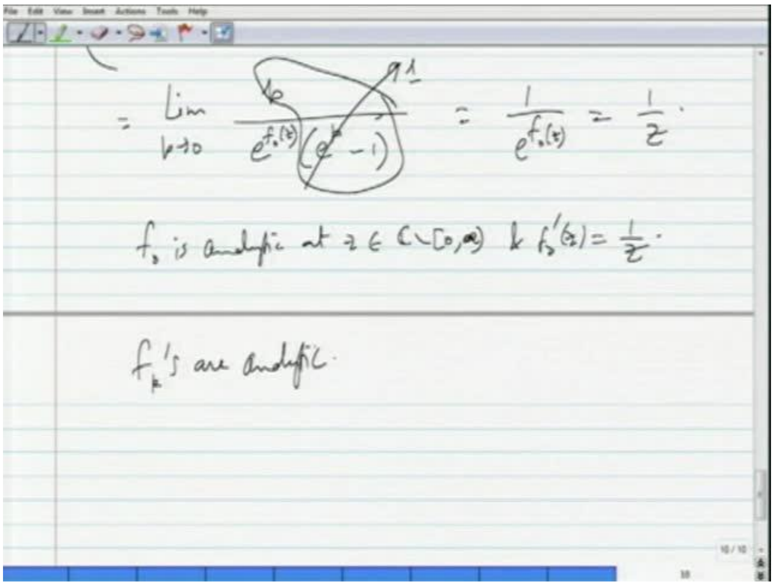

At z belongs to C minus 0 infinity at any C minus 0 infinity and f naught prime of z is equal to 1 by z. So, it matches up with the derivative of log x being 1 by x on the positive real axis. So, a fact we know from real analysis, real logarithm. So, these f ks in general this is true for any f k, so f ks are analytic. And they are often called the analytic branches of a logarithm. For each each integer k, there is an analytic branch of logarithm. During this course we will put these multifunctions and analytical branches of logarithms to use.

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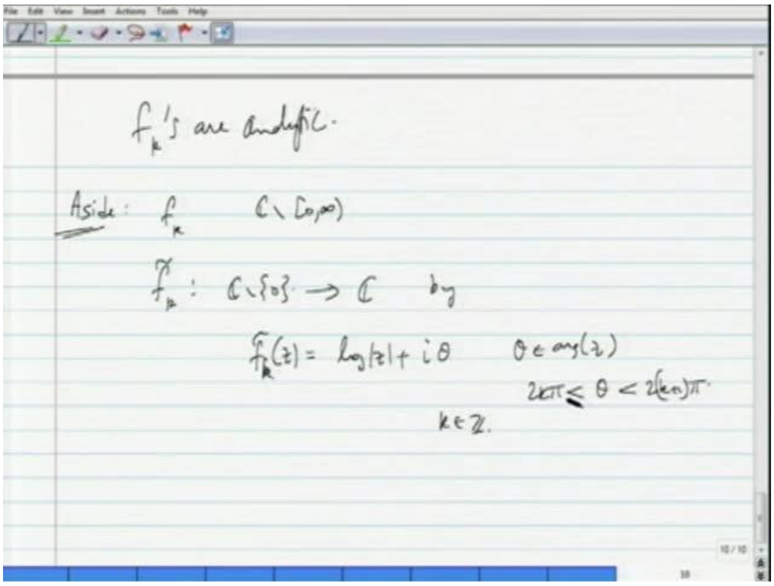

So, as a side though, what we will see is that these f ks have a certain nice extension. So, f ks the domain notice is C minus 0 infinity. So, we saw one way to resolve the problem of log z being a multi function by cutting a half ray on the complex plane and choosing a particular horizontal strip. So, what what can also be done is that we will take many copies of this C many copies of the complex plane minus 0. Then we that f k can be extended to an s k tilde on all of C minus 0 in the following fashion.

So, define f k tilde from C minus 0 to C by f k tilde of z is equal to log mod z plus i times theta, where theta belongs to argument of z with 0. In this case 2 k pi strictly less than or 2 k pi or less than or equal to theta less than 2 k plus 1 pi k belongs to z k is a fixed integer. So, what we can do is by putting an equality here, we can extend f k to f k tilde. It has the following property from the geometry.

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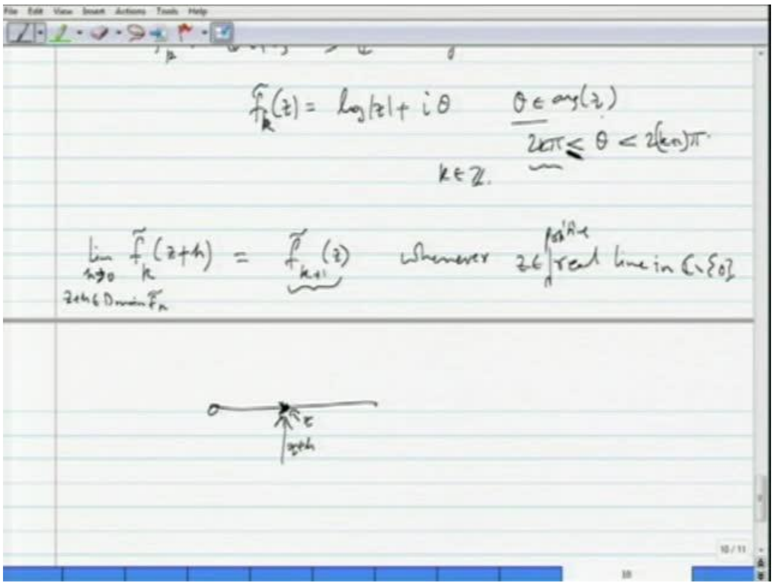

We explained earlier f k tilde of z minus h, the limit as h goes to 0 h having, so and z plus h belonging to domain of f k tilde. So, this z plus h here is equal to f k plus 1 tilde of z, whenever z belongs to the real line in C minus 0. So, what I want to say here is that when you have a point on the real line, which is non zero on the positive real line. So, this should be positive real line. As z plus h approaches this z and then the value of f k tilde of z plus h, actually approaches the value of f k plus 1 of z. That is due to the property of argument.

So, this f k tilde trans trances into f k plus 1 like that. So, in order to remedy the multi function situation, what you can do is you can take a stack of C minus 0 and you you consider a race. So, each C minus 0 should be viewed as half rays being traced from 2 k pi to 2 k plus 1 pi and then as you move 2 k pi to 2 k plus 1 pi when you reach 2 k plus 1 pi, you trance it via this limit from f k tilde to f k plus 1 tilde. So, you have this kind of a spiral C minus zeros sued like that and then there are also in stacks like that and then you have a function from that particular complex construct, which is now is stack of C minus zeros to the complex plane. Then the logarithm function from this domain to the complex plane exists.

Now, logarithm becomes a function, because for each z you have take multiple copies of z by this complex construct. This kind of construct was given by Riemann and this general surface, let us call it as surface is called Riemann surface for the multifunction log z. So, by constructing a multi, by constructing a Riemann surface for this multifunction, we have made the logarithm function now to be genuine function to the complex plane.

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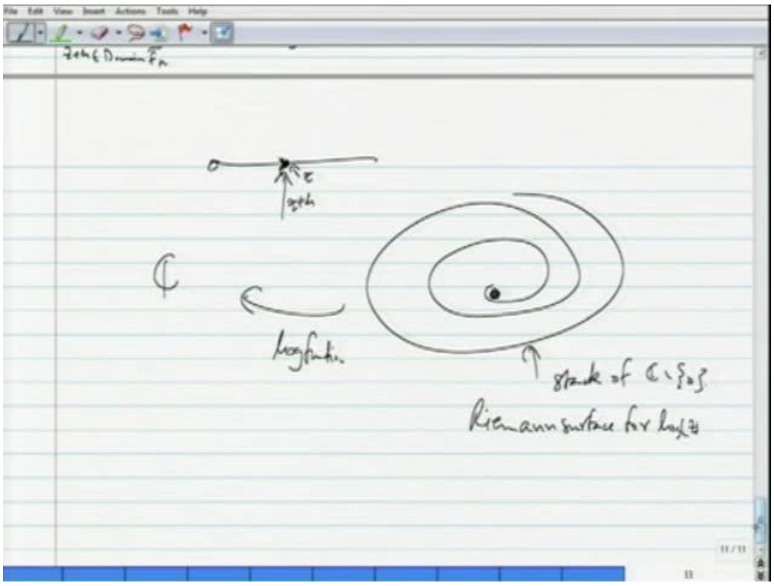

So, from that stack of planes from, so it looks like a stack of C minus zeros to the complex plane you have. Of course, 0 is such a point which is a removed all throughout. Then you have a logarithm function and this stack of C minus 0, such that you glue the real accesses of one sheet of C minus 0 to the to the sheet, which are occurs prior to that.

So, this stack of C minus zeros is called a Riemann surface for log z. So, that is about logarithm, log logarithm multifunction.

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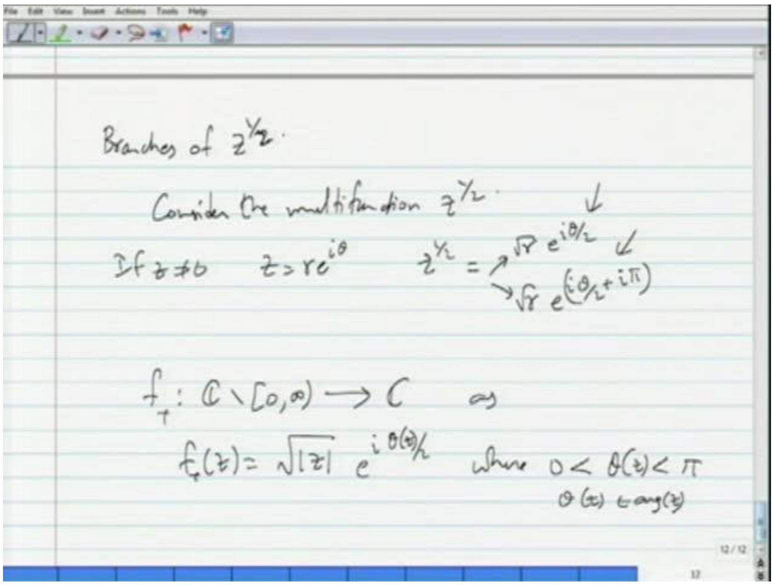

So, next let us consider owing to the logarithm function. Let us also consider branches of z power half. So, we have a multifunction z power half, so consider the multifunction z power half, I am calling it multifunction because if z is not equal to 0 and z is r i e power theta, z power half can be either square root r e power i theta by 2 or it can be square root r e power i theta by 2 plus i pi, which is i 2 pi by 2. So, we have two possible values of z power half.

So, it is a multi function, so what we can do is we can define f plus from C minus 0 infinity. The strategies once again the same, you remove a half ray and once again the problem occurs due to argument function. Then you remove a half ray and then define f plus from C minus 0 infinity to C as f plus of z is equal to square root of the modulus of z times e power i times the argument of or I will say theta of z by 2, where 0 strictly less than theta of z is strictly less than pi theta of z, belongs to the argument of z.

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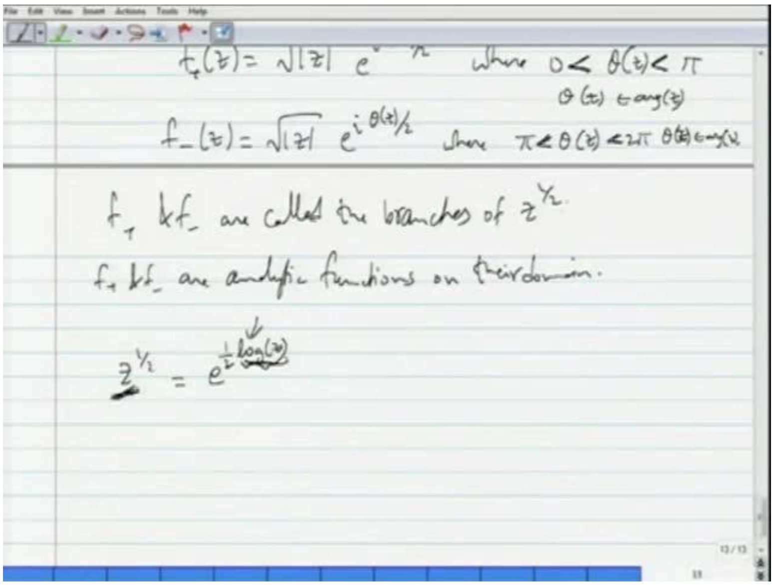

You can define a second function f minus of z likewise, which is the square root of modulus of z e power i theta of z by 2, where you restrict theta now to be between pi and 2 pi and theta outside belongs to argument of z. f plus and f minus are called the branches of z power half and by eliminating 0 and pi and pi and 2 pi here in the definitions, we have actually made it a continuous function. So, f plus and f minus are actually analytic. One can show that they are analytic; one can use the Cauchy Riemann equations.

For example, to calculate the real and imaginary parts of f plus n, show that f plus and f minus are analytic, are analytic functions on on there domain. So, these are called branches of z power half. So, owing to this so we can actually view z power half is equal to e raise to half log of z, where we use the logarithm with multifunction and by choosing a branch of logarithm, we can choose a branch of z power half. So, if you are wondering that well log z has infinitely many branches like I showed corresponding each integer, there is a branch of logarithm, but z power half has only two branches.

So, I will ask the viewer to actually realize that if you take the branches of logarithm, which are 4 pi away are which correspond to even integers or which correspond to odd integers respectively. They fall into two classes and all the even integers give you one branch of z power half and all the branches corresponding to odd integers give you another branch of z power half.

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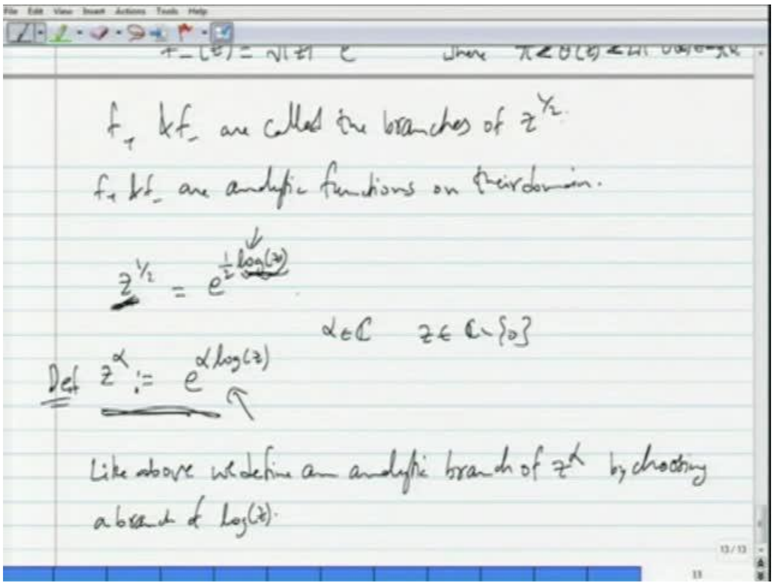

Now, owing to this we will make a definition. So, define z power alpha is equal to e raise to alpha log z. So, we are defining z power alpha, where alpha belongs to complex numbers and z belongs to C minus 0. So, we make a definition like this and this is a multifunction for the same reason, like we saw except when alpha is an integer positive integer positive or negative integer. So, then this becomes a multifunction and we define, so like above like we did for z power half, we define an analytic branch of z power alpha by choosing a branch of logarithm of z. So, that concludes a discussion about analytic branches of a logarithms and and a z power alpha etcetera. So, next we will look at the hyperbolic functions.

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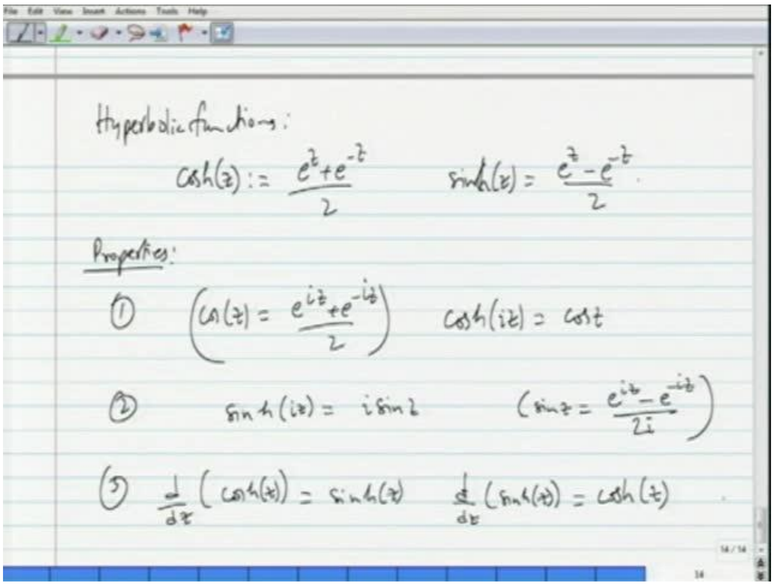

Owing to the definition of the real cosine hyperbolic and real sin hyperbolic functions, we define the complex hyperbolic function to be e power z plus e power minus z by 2 and the sin hyperbolic of z function of z is e power z minus e power minus z by 2. So, since we already know that the cosine theta a cosine of z is defined to be e power i z plus e power e power minus i z by 2. So, since this is true we have cosine h of i z is equal to cosine z, so in complex numbers we see the connection between the hyperbolic functions. There corresponding trigonometric functions. Likewise a sine h of i z is equal to i sine z. Notice that i in front, so that stands because of the fact that sine z is e power i z minus e power minus i z divided by i 2 i. Then the third property is that the derivative one can verify directly, that the derivative of cosine h z is equal to sine h z and likewise the derivative of the sine hyperbolic z is equal to the cosine hyperbolic cosine.

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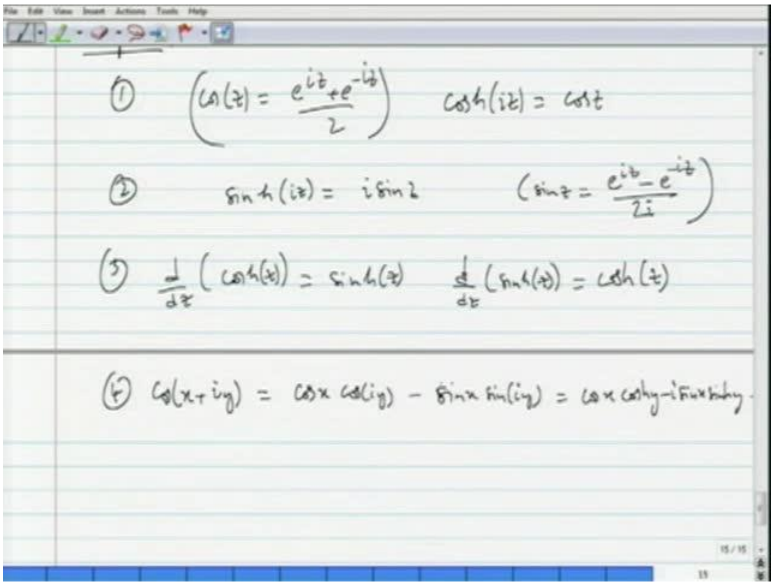

We can see that in the addition formula for cosine, the hyperbolic cosine and hyperbolic sine figure. So, it was an exercise when we introduced a complex cosine and sine, that the the addition rule holds for the addition rule we know from a standard trigonometric holds for complex numbers. So, this gives us cosine x cosine i y minus i minus sine x sine i y, which now turns out to be cosine x cosine h y and minus i sine x sine h y. So, these are the some of the properties of hyperbolic functions and so with these we conclude our collection of holomorphic functions or analytic functions. With this we conclude this module. In the next module, we will use integration as a tool to study properties of these analytic functions.