Complex Analysis
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Module - 2
Complex Functions: Limits, Continuity and Differentiation.
Lecture - 6
Sine Cosine and Harmonic functions

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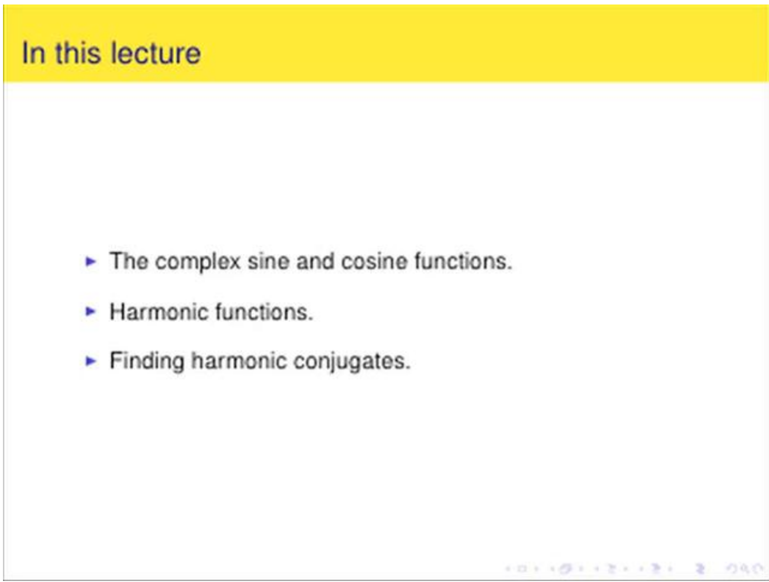

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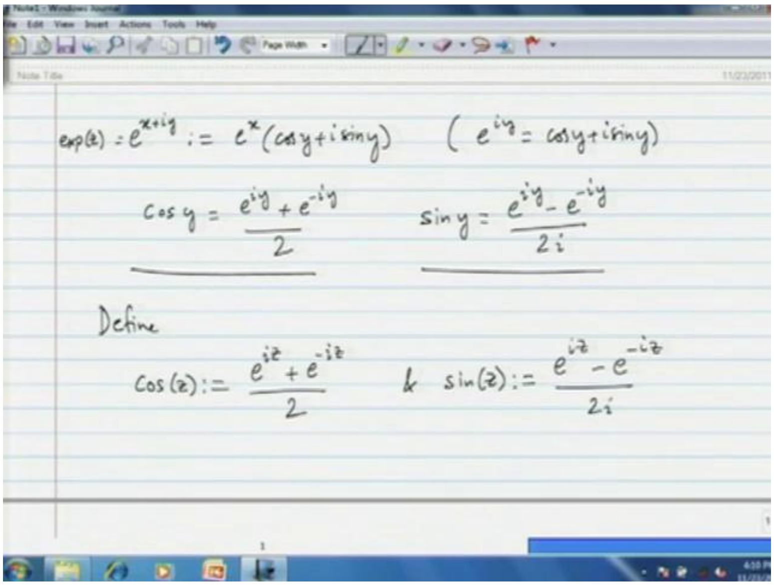

Last time we have defined the function e power or the exponential of the complex number x plus i y as e raise to x. Now this is the usual real exponential function e power x times cosine y plus i sin y, that was our definition of the complex exponential function. So, I will write exp of z, that was our definition for any, for any complex number z. So, from this, we see that we can quickly see that cosine of y is actually e power i y plus e power minus i y by 2. So, just to put this really clearly, if you take x equals 0, you get e power i y equals cosine y plus i sin y. Okay fine, from here we can clearly see that cosine in y can be recaptured by taking e power i y plus e power minus i y divided by 2.

So, and likewise sin y can be recaptured by taking on the right hand side e power i y minus e power minus i y divided by 2 i though. So, taking notice that the left hand side of either of these equations are the real cosine and sin functions, they are functions of real numbers y. So, owing to these two, these two equations, we define the cosine and sin functions for a complex number z as follows. So, define cosine of a complex number z as, so the colon means that the left hand side has being defined cosine z is defined as the exponential function of i z plus e power minus i z by 2, and sin of z, the sin of the complex number z is defined as e power i z minus e power minus i z by 2 i.

So, at least this definition makes sure that when we substitute a real number in place of the complex number z, then the older definitions of the real, and the the real cosine and sine are recaptured. So, when z is a real number, you get back your sin and cosine which you already know from real analyses. So, that is your definition of the complex cosine and sin functions, and owe, owing to this you can actually define all the other trigonometric functions, so for example, tan z is defined as, the tangent z is defined as sin z by cosine z. So, let us now at least look at these functions cosine z and sine z and capture some of the properties.

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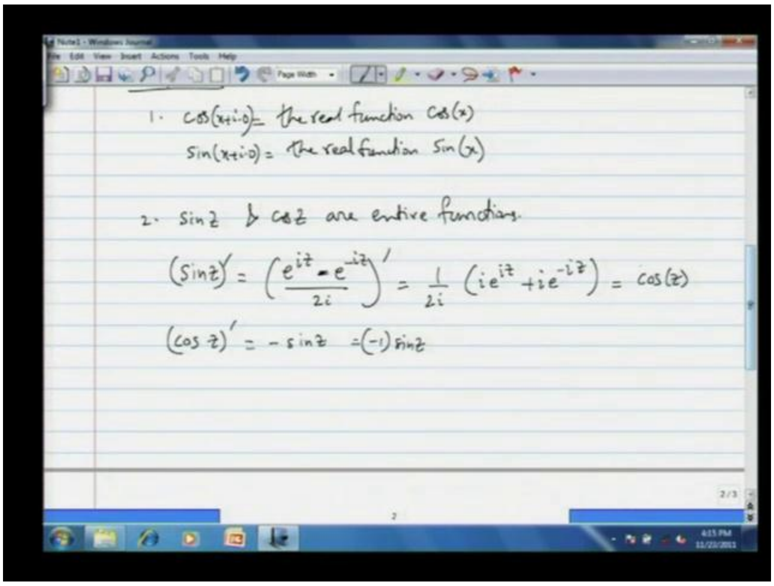

So, properties which follow from this definition of cosine z and sin z are as follows. One cosine of x is equal to the real function cosine x. So, the left hand side should really say cosine x plus i times 0. So, the cosine of a complex number x plus i times 0, tallies with the real function cosine x, this I have said verbally already, and likewise sin of x plus i times 0 is the real function. So, by the real function I mean that it is the function of the real variable x which you, which you are already familiar with, and further properties are as follows 2 sin z and cosine z are entire functions.

So, sin z and cosine z are entire functions, because let us look at the definition, they are combinations of entire functions the exponential functions e power i z and e power i minus i z. So, so when you add or subtract entire functions, you get entire functions. Dividing by 2 will not affect the analyticity of a function at a point. So, or dividing by 2 i likewise, so these are entire functions, and the second or the third property is that, well let me, let me give you the derivative, then the derivative of sin z which we expected to be cosine z is indeed cosine z, because the derivative of sin z is the derivative of e power i z minus e power minus i z by 2 i .

So, since 2 i is a constant, I will just pull it out, and the derivative of e power i z is i e power i z, and the derivative of e power minus i z is plus i e power minus i z. So, the derivative of e power minus i z is minus i e power minus i z, then I multiply i in the minus in the front here. So, I will get a plus e power plus i e power i z, and then one cancels the i to notice that what you have is cosine z. Likewise one can compute that the derivative of cosine z is a minus sin z. So, it is actually the complex number minus one times sin z to be more précised. Well, so those are the derivatives.

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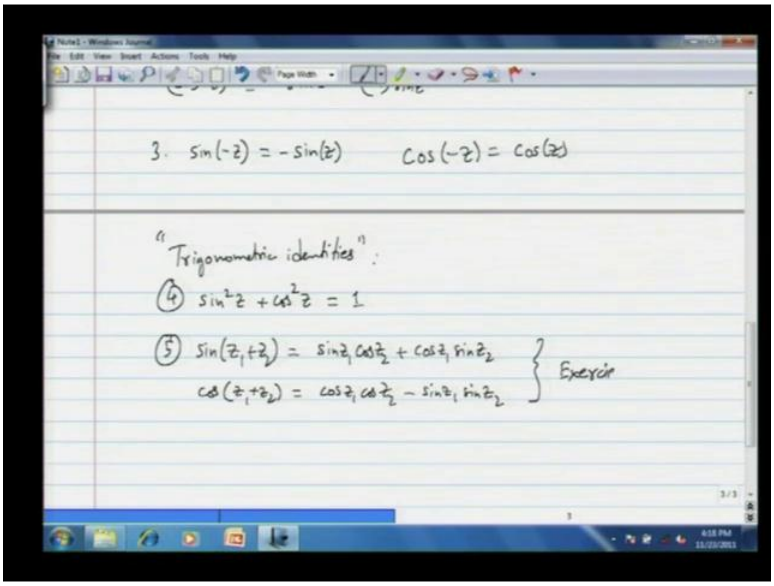

Then the third of the properties is that the sin of minus z is minus sin z, . So, many of the properties we are familiar with of the sin and cosine hold also for complex numbers. So, sin of the complex number minus z, which is minus x minus i y when a z is written as x plus i y is the, is minus one times sin of x plus i y.

And likewise, cosine of minus z will be cosine z, so these tally with these properties we are familiar with for functions of real numbers, the the sin and cosine functions of real numbers. And then, usual trigonometric identities we are familiar with from trigonometry also hold for complex sin and cosine functions. So, for example, so I will put that in codes really, so I will say sin squared z plus cosine squared z is equal to 1. So, here we are defining the sin function the cosine function in terms of the exponential function, so one can directly compute the left hand side using those functions, using the exponential function and one can arrive at this equation four. And likewise I will say that sin of z 1 plus z 2, the complex number z 1 plus z 2 is sin z 1 cosine z 2 plus cosine z 1 sin z 2.

So, this is the identity familiar from trigonometry, and it holds for complex numbers as well. So, cosine z 1 plus z 2 now is cosine z 1 cosine z 2 minus sin z 1 sin z 2. So, the the proves are the the verification of these identities are exercises for the viewer. So, please verify any of these properties that you suspect, and a careful student is suspicious, so you might want to suspect all these and verify them for yourself.

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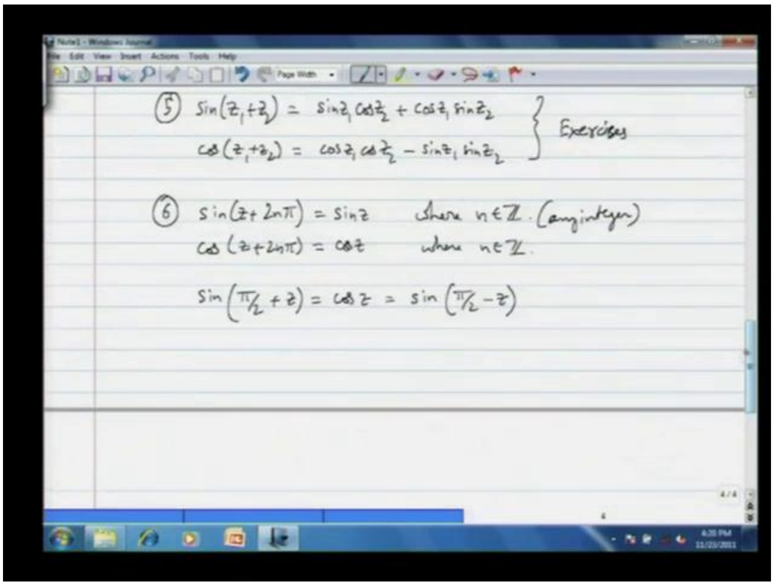

Then, the other properties are as follows, sin of z plus 2 n pi is indeed sin z where n is any integer, n is any integer. So, indeed the, the, the complex sin and cosine functions are also 2 pi periodic. So cosine, I will write that cosine z plus 2 n pi likewise is equal to cosine z, for all z and, and n is an integer.

So, these are some properties, and let me note couple others, so sin of pi by 2 plus z is going to give you cosine z, and this is also equal to sin pi by 2 minus z. So, that is the relation between the sin and the cosine, complex sin and cosine functions. So, the viewer once again is welcome to verify any of these identities. So, we have studied the mapping properties of the complex exponential function, and likewise the viewer is encouraged to chalk out the mapping properties of the sin and cosine functions of a complex number. So, we already know the graph of a sin function, let me go back to the real case the graph of a sin function has a y value at most 1 and at least minus 1.

So, by that I mean, sin x the function sin x is bounded by minus 1 and 1. So, but, you will notice that when we go to the complex case, the sin function or the cosine function are unbounded. By that I mean any complex number is really in the image of the complex sin function, or the complex cosine function. It is really interesting to see what the images look like for a, for the sin function. So, start with some contours and try to see if you can find the images like we have done for the exponential function.

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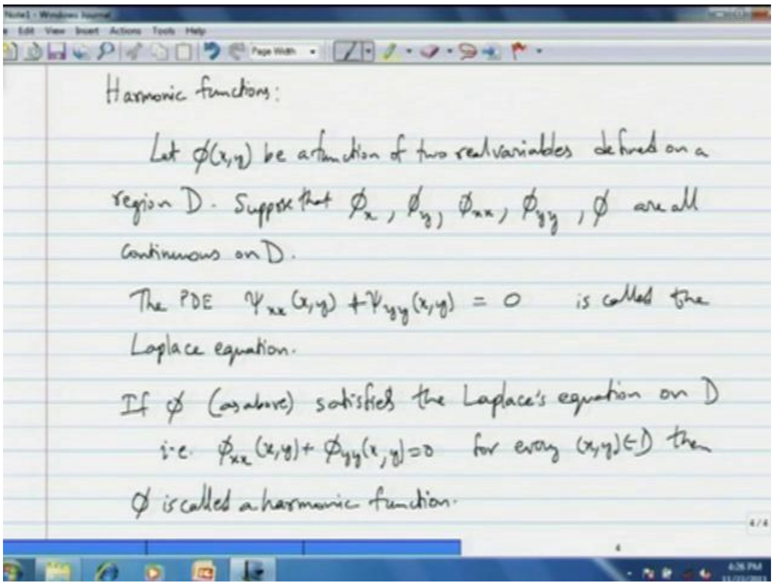

So, next I want to, I want to talk about harmonic functions. So, this is a slight detour from the main stream sort of complex analyses, but it has lot of, lot of material which comes from complex analyses and it is an important application. So, I am going to talk about harmonic functions here which appear really in, in the, in partial differential equations. Let phi of x y b a function of two real variables defined on a region D. Suppose, that the partial derivative of phi with respect to x, the partial derivative of phi with respect to y, the partial derivative of phi with respect to, or the partial of phi, the second partial oh phi with respect to x, and then again with respect to x and likewise the second partial of phi with respect to y, and then again with respect to y, and of course,, phi itself are all continuous on D. So, let us make these assumptions, the p d e phi x x of x y plus phi y y of x y, so the partial differential equation this equals 0 is called the Laplace equation ok.

So, this p d e is called a Laplace equation, and, so actually I should have taken some other name because I have used phi may be I will say this is si plus si y y of x y is equal to 0, so a function if, if phi as above satisfies the Laplace’s equation on the domain D. So for every point suppose that phi x x x, so i e phi x x rather of x y plus phi y y of x y is equal to 0, for every x y belongs to D then phi is called a harmonic function. So, such a function which satisfies this Laplace's equation is called a harmonic function. Now, if a function, if, if we take a, the real part of a complex analytic function it is true that due to the Cauchy Riemann's equations it already satisfies the Laplace's equation.

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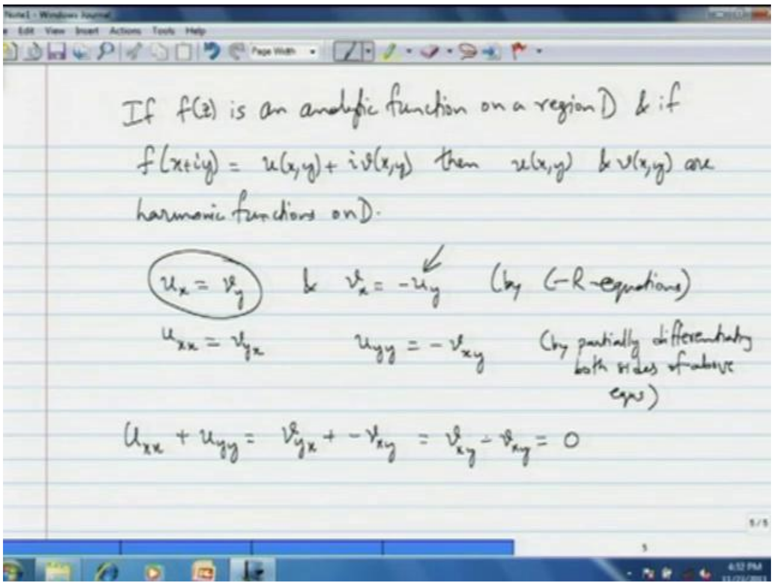

So, let us say that in the following words, So if, f of z is an analytic function on a region D, defined on a region D, and analytic on a region D, and if f of x plus i y, so allow me to write z as x plus i y this is equal to u of x y plus i times v of x y. So, I am capturing the real and the imaginary parts of this function, then u of x y and v of x y are harmonic functions on D. So, let us say y, this is due to the Cauchy Riemann equations. So, f is differentiable, so f is of course, continuous, so it follows that u and v are continuous, and the partial of u with respect to x is equal to, so I will, I will somehow suppress this of x y and say u x is v y and v x is minus u y. So, u, so by C R equations and C R equations say that the partial of u with respect to x is equal to the partial of v with respect to y on all of D, on all of the domain D.

Likewise is the other equation, so I will differentiate partially with respect to x on both sides of this equation, and get u x x is v y x, and for here I get u y y I will differentiate partially with respect to y on the right hand side and I get minus v x y. So, by partially differentiating both sides of above equations one with respect to x and the other with respect to y, I am not going to write that, I will just say that verbally and then what we get is that and this and this holds on all of D, so when we add u x x plus u y y we get v y x plus minus v x y, and since v y x is v x y, we get v x y minus v x y which gives us 0. So, u satisfies the, the Laplace's equation. So, if your wondering why v is differentiable? Why at all v or u is differentiable again with respect to x? How do I know that the partial derivatives exist? Or if you are wondering why the mixed partial v y x is equal to v x y? Let me state a fact which we are going to prove later that an analytic function f, is differentiable any number of times.

So, in particular if the take the real or the imaginary part of an analytic function, then all the partial, partial derivatives of all orders of u and v exist. So, in particular the second partial derivatives exist and are continuous which is enough to say that the mixed partials v y x and v x y are equal. Likewise it is thus justified that the mixed partials or rather the the second order partials exist at all. That is due to the fact that an analytic function is differentiable any number of times. And, this for now please accept that as a fact, we will prove that later when we study the integration theory. So, now owing to that fact we have that u satisfies the, the Laplace's equation.

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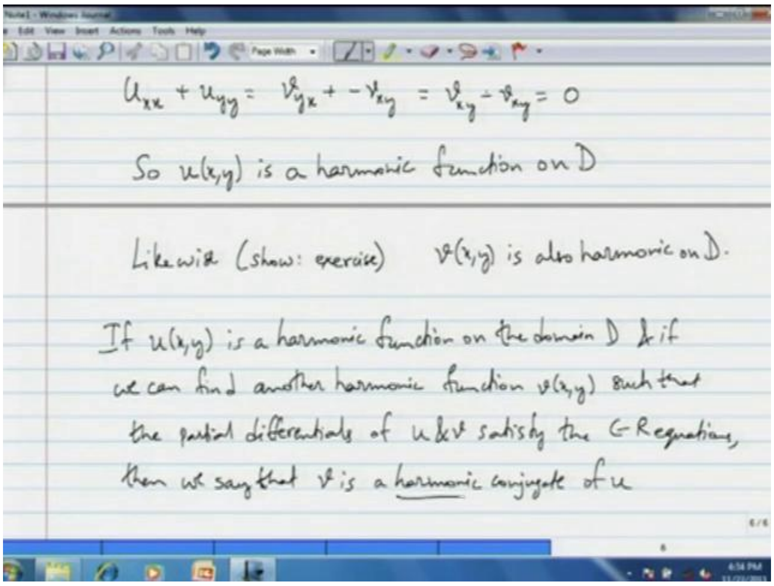

So, u of x y is a harmonic function on D. Likewise, one can show it is an exercise for the viewer, try to show v of x y is also harmonic on D. So, let us see the following, so if u of x y is a harmonic function on D, on the domain D, and if we can find another harmonic function v of x y such that the partial differentials of u and v satisfy the Cauchy Riemann equations, then we say that v is a harmonic conjugate of u. So, if you start with a harmonic function, and if your able to find a yet another harmonic function such that u and v satisfy the Cauchy Riemann equations then we say that v is a harmonic conjugate of u. So, notice that there is a slight asymmetry in, in this, in this definition.

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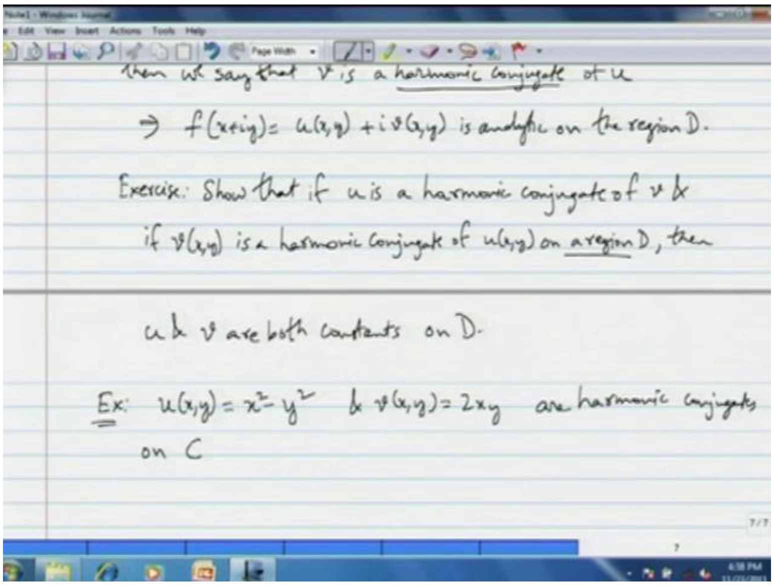

So, the following exercise which I am going to assign to the viewer will bring out the asymmetry. So, actually before the exercise let me say that this implies that, this definition implies that the function f of x plus i y defined as u of x y plus i times v of x y is analytic on the region D. So, if we take a f equals u plus i v then, since we are assuming that the partial second order partials of u and v exist, so the first order partials are continuous, and since u and v satisfy the Cauchy Riemann equations f equals u plus i v is indeed analytic on the region D, so on this region D where u and v are a harmonic. So, let me give an exercise which brings out the asymmetry in the statement, so the exercise is as follows show that if u is a harmonic conjugate of v, so here I mean u of x y and v of x y, I am suppressing that, of x y notation.

And, if v of x y is a harmonic conjugate of u x y on, on sub domain D, on a region D, then u and v then show that essentially u and v are both constants. So, constant functions on, on all of D, so I assume D is a region it is an open connected set, so if u, u is, I mean if the relation is mutual that use a harmonic conjugate of v, and v is a harmonic conjugate of u, then it has to be that both of them are constants. So, this brings out some asymmetry, so you have to be careful about the definition. So, having said that lets do an example here; so u of x y let me give you this polynomial into variables, u of x y is x squared minus y squared, and v of x y is 2 x y are harmonic conjugates on the entire complex plane.

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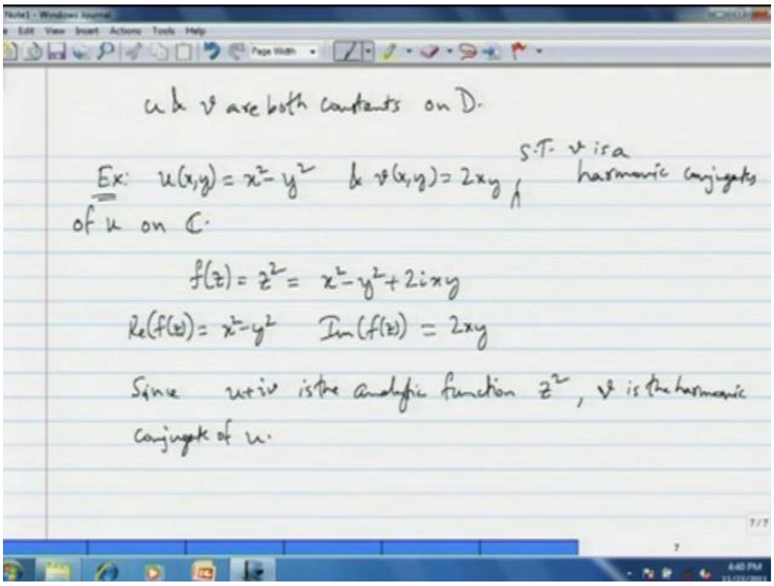

So, well that is fairly clear because f of z if you take that to be z squared which is in terms of x and y, x squared minus y squared plus 2 i x y. So, the real part of f of z is x squared minus y squared, and the imaginary part of f of z is 2 x y. So, since u and v are the real and imaginary parts respectively of this analytic function f of z equals z squared and entire function, so they satisfy the Laplace equation, so both of them are harmonic and, not only that, u, so v is a harmonic conjugate of u, so because u plus i v is an analytic function. So, since u plus i v is the analytic function, z squared u, sorry v is the harmonic conjugate of u. So, v is the harmonic conjugate of u I should have said, so I should have said in the exercise show that v is a harmonic conjugate of u on C, sorry about that. So, the question should say show that v is a harmonic conjugate of u on C. So let us see another example.

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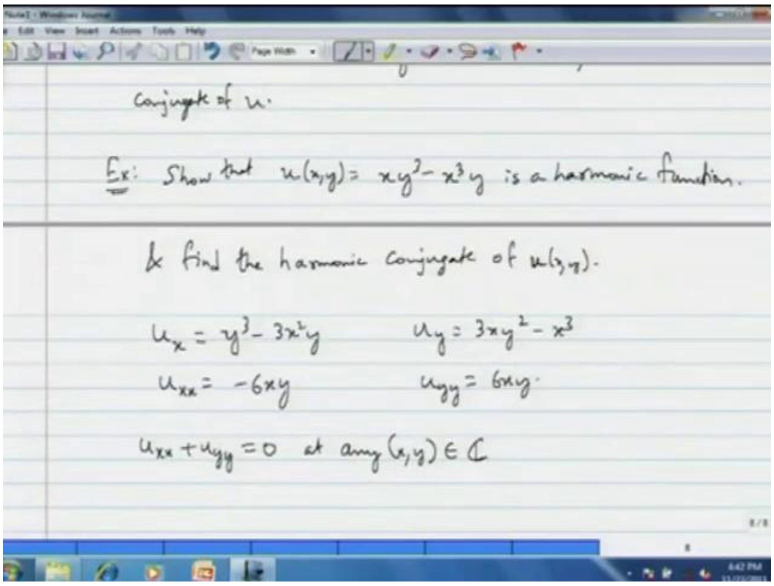

Show that u of x y equals x y cube minis x cube y is a harmonic function, that is easy, and find the harmonic conjugate of u of x y. So, firstly it is easy to verify that u is a harmonic function you just calculate u x which is y cube minus 3 x squared y, so you x x the second order partial is minus 6 x y u y the partial of u with respect to y is 3 x y squared minus x cube, and, so the second order partial of u with respect to y, and again with respect to y is 6 x y. So, it is clear that u x x plus u y y is 0 at every point x y, at any x y belonging to C. So, at any, so it is some abuse of notation I will just say belongs to C, so that should actually be x plus i y belongs to C.

So, now you have to find a harmonic conjugate of u, so a harmonic conjugate is a function which along with u satisfies the Cauchy Riemann equations.

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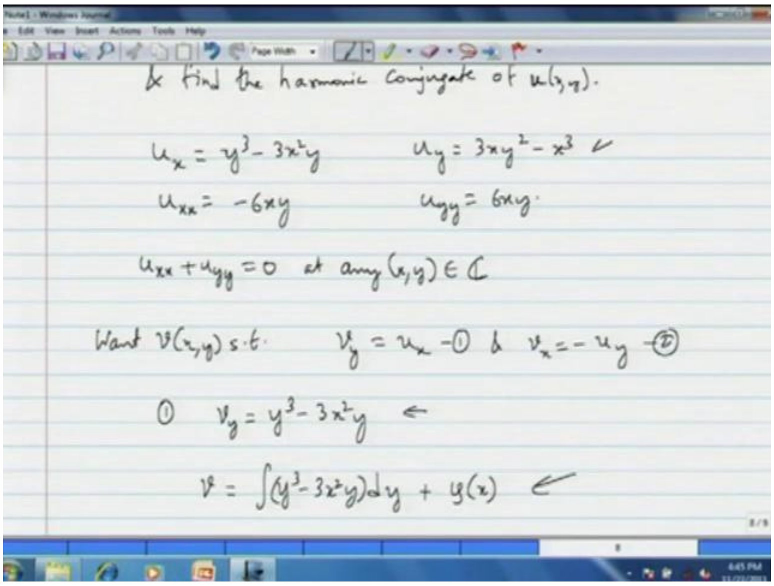

So, you want a v such that, so want v of x y such that the partial of with respect to y is equal to the partial of u with respect to x, and the partial of v with respect to x is the negative of the partial of u with respect to y. So, let us look at, so this is equation one this is equation two, these are the Cauchy Riemann equations. So, let us look at one, one implies that v y should look like v y should look like y cube minus 3 x squared y; and assuming that your working with an open connected region, so v then in that case will be the partial integration of this function with respect to y is an integration of this function with respect to y, and then there is possibly a function of x alone. So, phi of x let me call that function phi of x, so that when you differentiate this equation partially with respect to y you get back your v y in the, in that form.

So, so then v looks like y power 4 over 4 minus 3 x squared y squared over 2 plus, possibly some function of f x which we have to determine and to determine phi we will actually use the equation two. So, from here, from this equation we can calculate the partial of v with respect to x. So, we get minus 3 x y squared plus phi prime of x, this is your partial of v with respect to x from this equation. But, from the Cauchy Riemann equations or from two, we know that this should be equal to minus u y which we have already calculated or ya, u y is right here, so I am going to substitute minus u y will be minus 3 x y squared plus x cube, from two.

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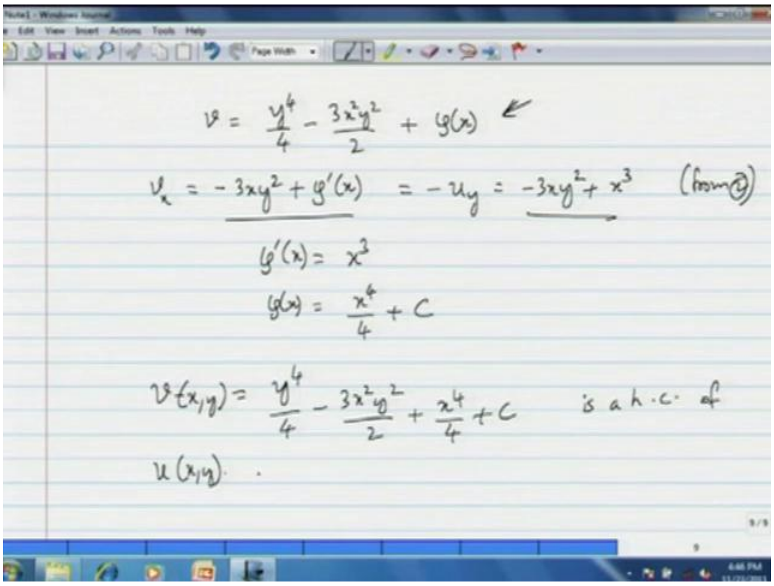

So, using the Cauchy Riemann equations, so then equating these two expressions I get phi prime of x should look like x cube. So, and then phi of x, candidate for phi of x is x raised to 4 by 4 plus a constant C, constant of integration C. So, your v of x y equals y power 4 by 4 minus 3 x squared y squared by 2 plus x power 4 by 4 plus a constant is a harmonic conjugate of the given u of x y. So, we have used the Cauchy Riemann equations to arrive at the harmonic conjugate. On, and this works on all of, so let us do another example of this sort.

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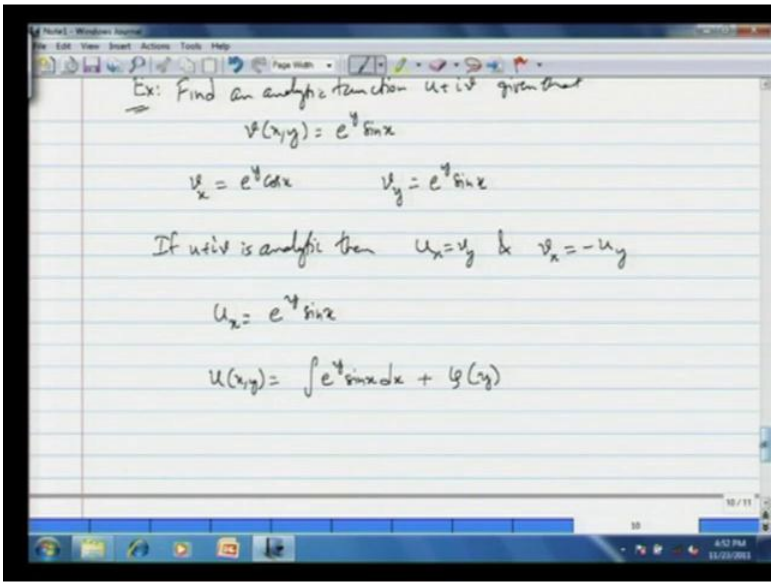

So, here is another example. Find an analytic function u plus i v, given that v of x y is e power y sin x. So, in this case we have been given the imaginary part of the analytic function, you have to find the real part, the procedure is pretty much similarly, what we do is we consider the partial of v with respect to x and that is going to give us e power y cosine x, and consider the partial of v with respect to y, so that gives us e power y sin x back again.

We will use the Cauchy Riemann equations, so if u plus i v is analytic, then v x is, then I should say u x is v y, and v x is minus u y. So, using the Cauchy Riemann equations, u x is e power y sin x, so partially integrating with respect to x, u of x y is showed, now look like the integration of e power y sin x, with respect to x plus possibly a function of y, a function of y alone. So, this works on all of the complex plane, and since the complex plane is open connected I can, I can do this, low from here to here.

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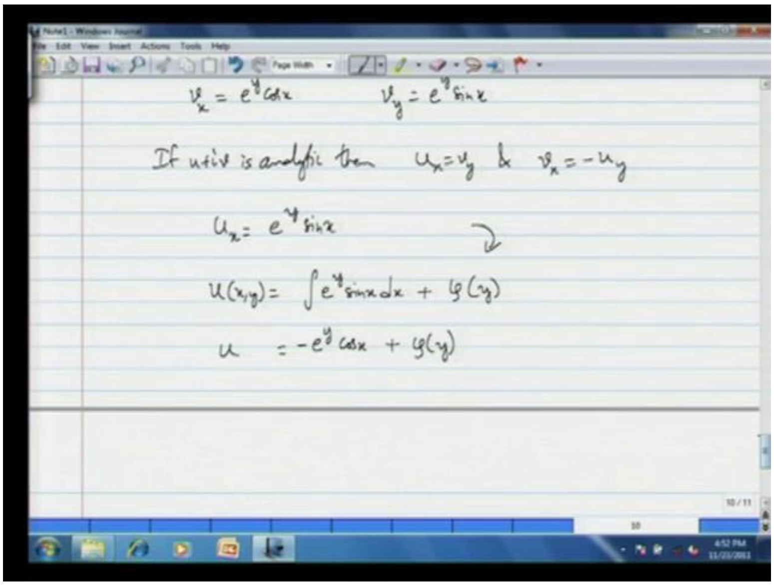

So, this gives me an expression for u as e power y negative cosine x, because the integration of sin x is minus cosine x, plus possibly a function of y.

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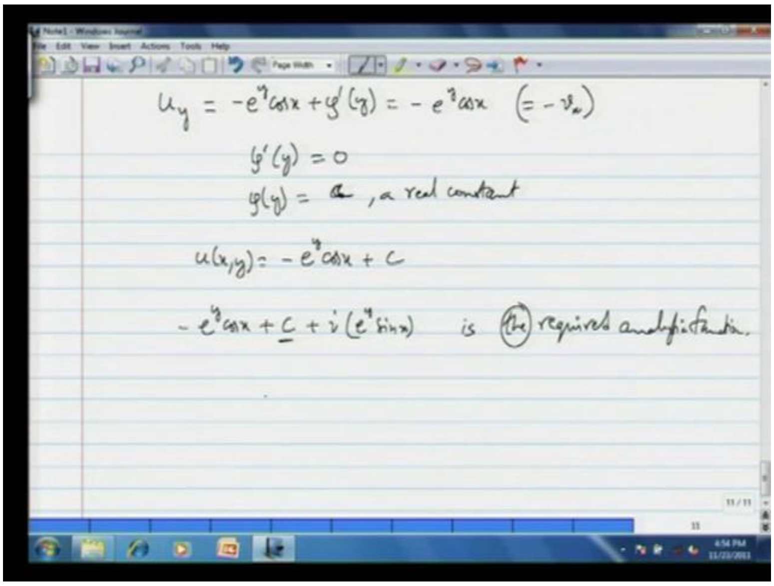

Using this, I will partially differentiate u with respect to y to obtain minus e power y cosine x plus phi prime of y, and by the Cauchy Riemann equations, this should equal the negative of v x which is minus e power y cosine x, so which is actually minus v x. So, comparing these two expressions I conclude phi prime of y is 0, or phi of y is constant, a complex constant, where ever a real constant. So, that gives us an expression of u of x y equals minus e power y cosine x plus a constant.

So, we can find a u like that and then u plus, so minus e power y cosine x plus C plus i times, i times v e power y sin x is the required analytic function. Notice that I used the definite article the required analytic function, but, actually there is an ambiguity due to this constant C. So, you should technically say is a required analytic function. So, this you can take any value of C, and that will satisfy the requirement. So, next let us see a context in which these we have connected the harmonic functions to analytic functions via the Cauchy Riemann equations. Let us see where these harmonic function are applied at least, or at least one application of it.

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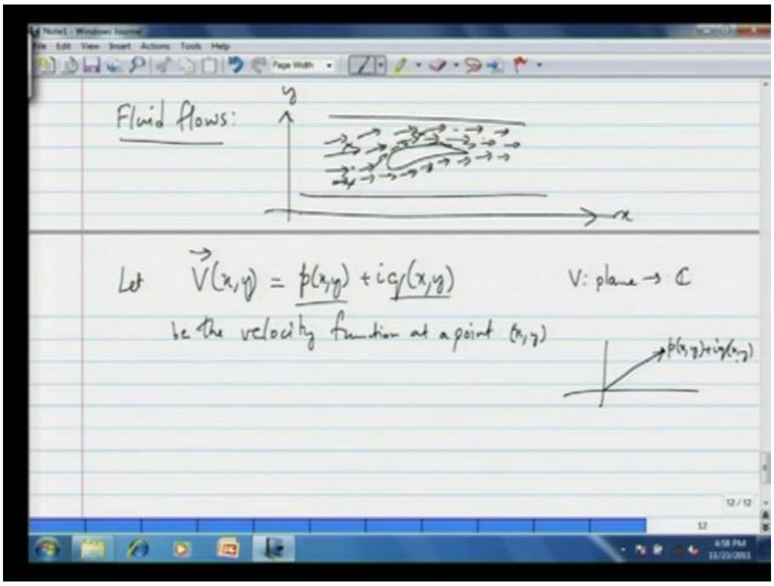

So, in the context of fluid flows, let us consider the application fluid flows. If you imagine a fluid flowing through some, some channel, so these these arrows represent a velocity at each point. Since I cannot draw arrow technically at every point which i see there, so these arrows are just indicative of velocities, so the velocity vector, so the velocity at a point x comma y, so I see a plane here, so let me imagine a plane here, the x plane and the y plane. So, then I co-ordinatize any of these points here, any of these points here, as x comma y, and let us imagine that a fluid is flowing through a certain, a channel, and we have an assortment of these x y planes stacked up.

So, let us also assume that the fluid flow on one of these planes looks exactly as same as the fluid flow on a plane parallel to it. So here, I have a stack of these x y planes as is shown and at each point x y, I am able to suppose write the velocity as a function p of x y plus i times q of x y, where I am pretending that I am in the complex plane and the velocity function is p of x y plus i times q of x y, you put a little bar to indicate a velocity. So, let this be the velocity function at a point x comma y, so the velocity is a vector, so v is actually a function from the plane to C, and I am interpreting points in the complex plane as vectors. So, right you, you remember how we can interpret a point in the complex plane as a vector you just join the origin to any given point x comma y and that becomes a vector.

So, in this case p of x y plus i times q of x y is that, so this is the image of the point x y, so recall that, so v is a function from the plane which we can once again pretend as a complex plane to the complex plane.

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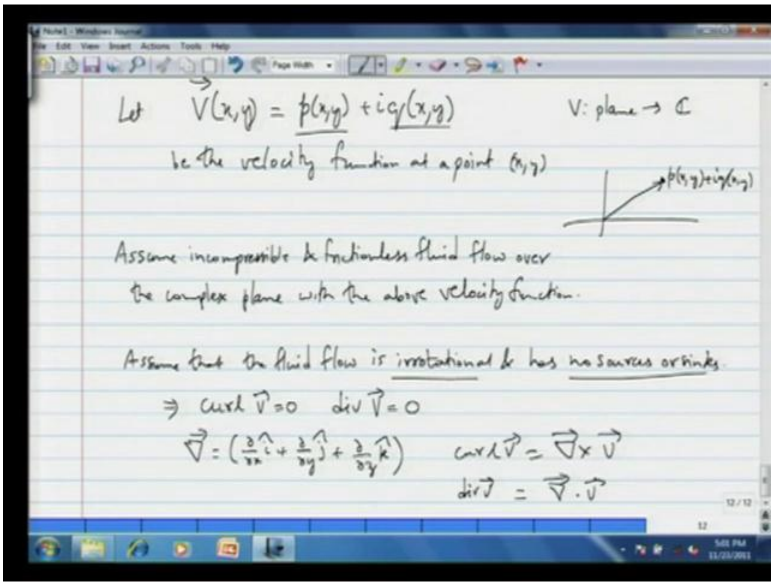

So, assume incompressible and frictionless fluid flow over the complex plane with the above velocity function. So, further assume that the fluid flow is irrotational and has no sources or sinks. So, this appears in fluid mechanics, so I cannot explain these terms in great detail, but, at least let me tell you that irrotational just means that the the flow is not, you know circulating around a point, or there are no sources or sinks means that the fluid is not generated at a point nor does it get absorbed at any point x comma y. So, what that translates to in terms of of this vector valued function is that the curl of the vector function v bar, and the, or sorry, the diversions. So, the irrotational component says that the curl is 0, and the diversions is in the, and the no sources or sinks says that the diversions of v bar is 0.

So, we can calculate the diversions and curl of these and equate them to 0, so the, the curl is the del cross, we will recall del, del bar is the vector dou by dou x i hat plus dou by dou y j hat plus plus dou by dou z k hat. Since we are just in the plane I can conveniently ignore this, well or I will just say that is the vector del bar. So, we can calculate del bar cross curl of v bar is del bar cross v bar, and likewise the diversions recall, diversions is this vector del this symbolic vector del dot v bar. So, these are expressions which help us, help us to calculate the curl and the diversions..

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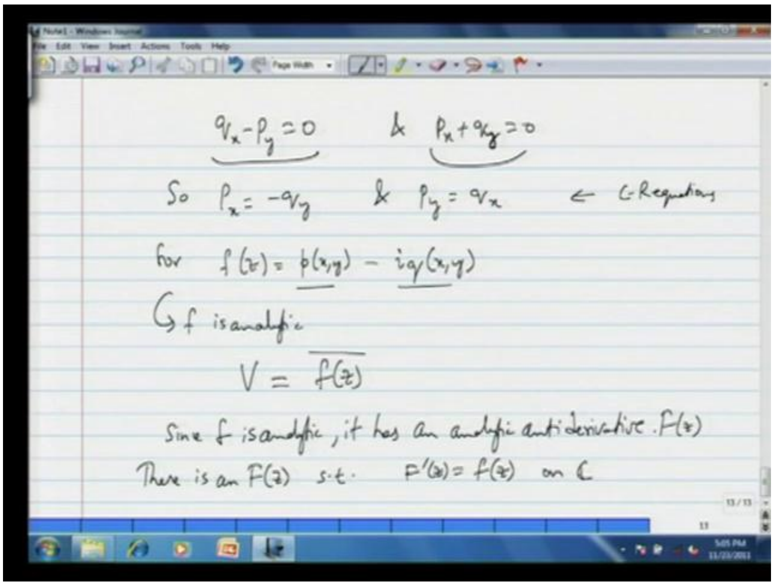

So, these lead us to the following equations, these imply that q x minus p y, like that is 0, and p x plus q y is 0. So, the curl is 0 gives us this equation, and the diversions is 0 gives us this equation. So, p y or rather p x is equal to minus q y and p y is equal to q x. So, if I take the function minus q, then we see that p and minus q, we see that these are the C R equations for, for f of a z equals p of x y minus i times q of x y. So, since p and minus q satisfy this CR equations, f is analytic. So, I am further assuming that p and q are differentiable or their further partials exist, and in the partial derivatives of p and q are continuous, so that I can make this conclusion, that conclusion needs the hypothesis that p x q y p y q x are all continuous. So, I will assume the continuity, so that I get f is an analytic function with p as the real part and q minus q as the imaginary part. So, then v is actually now the conjugate of f of z, right? Because, it is p plus i q, so v is the conjugate of f.

So, now f is analytic, so we will see further in the course that since f is analytic, it has an analytic anti derivative when restricted to appropriate domain, so we have a an analytic anti derivative, let us assume that is capital f of z. So, f prime there exists there is an f of z such that f prime of z is equal to f of z on, on that C.

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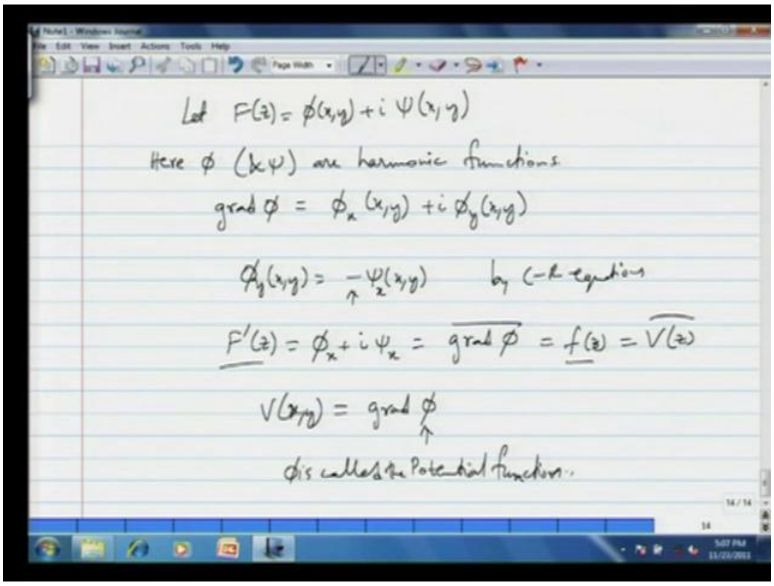

So, now you let f of z equals phi of x y plus i times psi of x y, and capital f is analytic, so here phi and of course,, psi are harmonic functions. After some simple calculations, we can actually see that the gradient of phi, well let us calculate it, gradient of phi is phi x of x y plus i times phi y of x y, and by the Cauchy Riemann equations, phi y of x y is actually minus psi x of x y, by C R equations. So, we get all in all f prime of z is phi x plus i times psi x.

One way to calculate the derivative of F, capital F. So, this gives us the gradient of phi conjugate, because I have a negative here, I get gradient of phi conjugate. But this is f of z conjugate which is v of rather this is a f time of z is f of z, so this is v of z conjugate, so we conclude that v of x plus x comma y is actually the gradient of a harmonic function phi, phi. So, the velocity function is can be interpreted as the gradient of a harmonic function, and this function is called the potential function, phi is called the potential function, and there are interesting properties of this potential function, and one can study further properties in a fluid mechanics course. So, let us see a quick example here.

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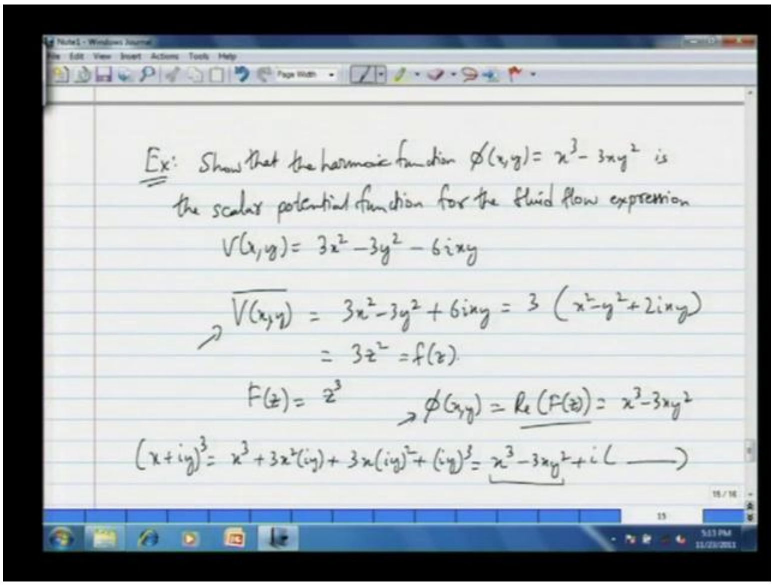

So, show that the harmonic function phi of x y equals x cube minus 3 x y squared is the scalar, potential function for the fluid flow expression v of x y is 3 x squared minus 3 y squared minus 6 i x y. So, v is, v of x y when we take its conjugate, we realise that looks like 3 z squared; so the conjugate of v of x y is 3 x squared minus 3 y squared plus 6 i x y, which is three times x squared minus y squared plus 2 i x y, which is 3 z squared; and which is an analytic entire function. It is an entire function, so let us call this f of z, so capital F of z is the anti derivative of this analytic function which is z cube, z cube the anti derivative of 3 z squared and, and it works on all of C.

So, the real part phi of x y of this analytic function capital f of z is indeed x cube minus 3 x y squared, that can be done by expanding z cube x plus i y cube, this is x cube plus 3 x squared i y plus 3 times x i y squared plus i y cube. So, which gives us x cube among other things x cube minus 3 x y squared for the real part and in the imaginary part, I will not work it out, so you get this. So, indeed the real part of this analytic function is x cube minus 3 x y squared, and it is harmonic function, because it is a real part of an analytic function and and that gives you the required phi for this velocity function v of x y.