Complex Analysis
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Module - 2
Complex Functions: Limits, Continuity and Differentiation
Lecture - 5
Analytic Functions; the exponential function

Hello viewers, last time we saw the definition of an analytic function, and I emphasize that these are important class of functions, and we will and rest of the complex analysis actually concentrates on its study. So, let me remind the viewer the definition of an analytic function.

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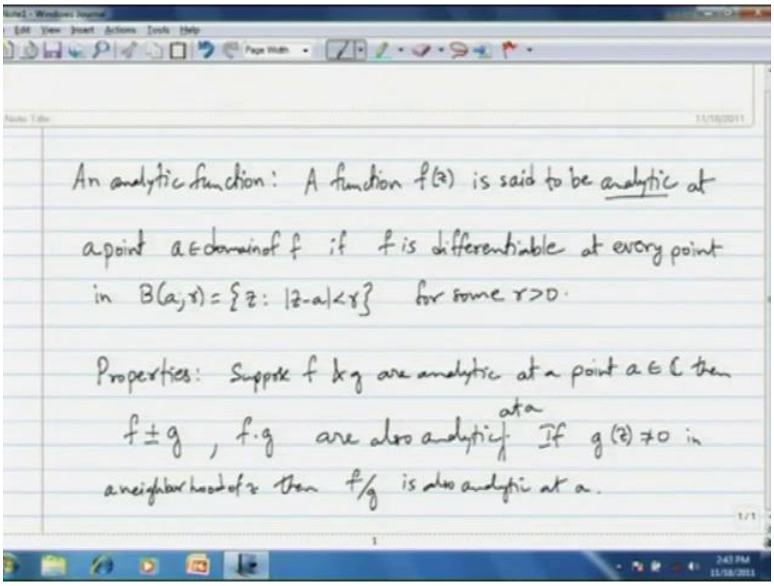

So an analytic function is the following; a function excuse me, f of z essentially it is a defined on a a subset of complex numbers is said to be analytic at a point a belongs to domain of f. If f is differentiable at every point in B a, r recall what that set is, that is the set of all z such that the modulus of z minus a strictly less than r for some r positive. So, a function, which is complex function which is differentiable at every point in neighbourhood of point a. So, B a, r is called r neighbourhood of the point a is said to be analytic at the point a. So, what this does is essentially this gives some room for function where it is differentiable so we will see that it has very deep consequences.

So, let me first start by giving some properties of analytic functions. So, here are some properties, so firstly suppose f and g are analytic at a point a belongs to c then f plus or minus g of g, so these are f of z plus g of z or f of z minus g of z. So, these functions and f times g the function f times g defined as f times g of z is f of z times g of z. So, this function are also these functions are also analytic.

If g of a is a not equal to 0, so let me say... Yes, if why do not I take if g of z is not equal to 0 in a neighbourhood neighbourhood of z then f by g is also analytic at a. So, I should have said analytic at a, so these, these functions are analytic at a and f by g is also analytic at a provided g of z is not equal to 0. So, this is useful to construct examples of analytic functions, once you know some analytic functions, so we can add two analytic functions and obtain a new analytic function etcetera.

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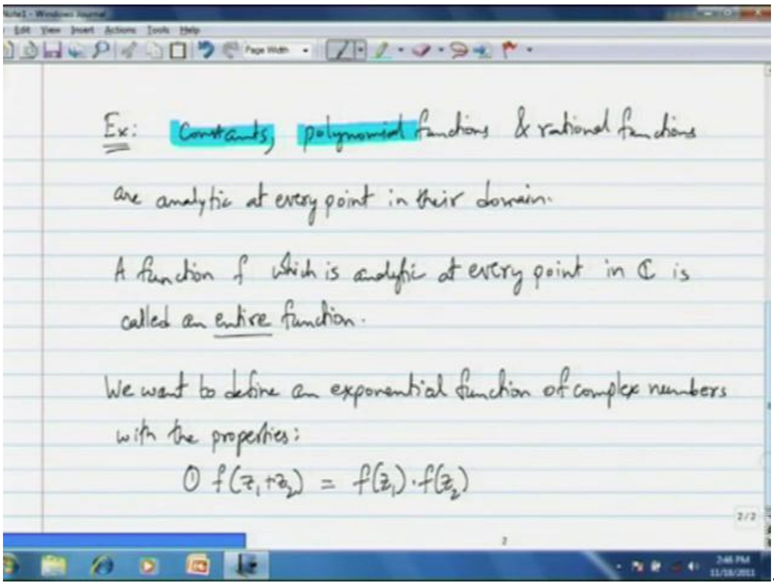

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So, what are some examples well we know we already know many examples of analytic functions. So, the 3 classes of functions we saw very earlier on, the constants, polynomials, polynomial functions and rational functions are all are analytic at every point in their domain. So, So already we have a large class of functions which we know are analytic. So, an analytic function or a function f which is analytic at every point in C, so in particular it is differentiable at every point in C is called an entire function. So, entire function is something which is analytic on all of C. So, in our example stated above both these constants and the polynomials are entire functions. So, today we will see another function namely the exponential function of a complex number.

So, this is motivated by the exponential function of real numbers and we will want to retain the properties of the exponential function of real numbers. Then we look at the definition of the exponential function of complex numbers when restricted to the real numbers in the complex plane. So, what we want is we want to define an exponential function of complex numbers with the properties the following desirable properties. Firstly if f is that function which we want to define we want f to have the property that f of z 1 plus z 2 is f of z 1 times f of z 2. Recall that the function the exponential function of real numbers has this property the exponentiation of or the exponential function of r 1 plus r 2 is the exponential of r 1 times exponential of r 2.

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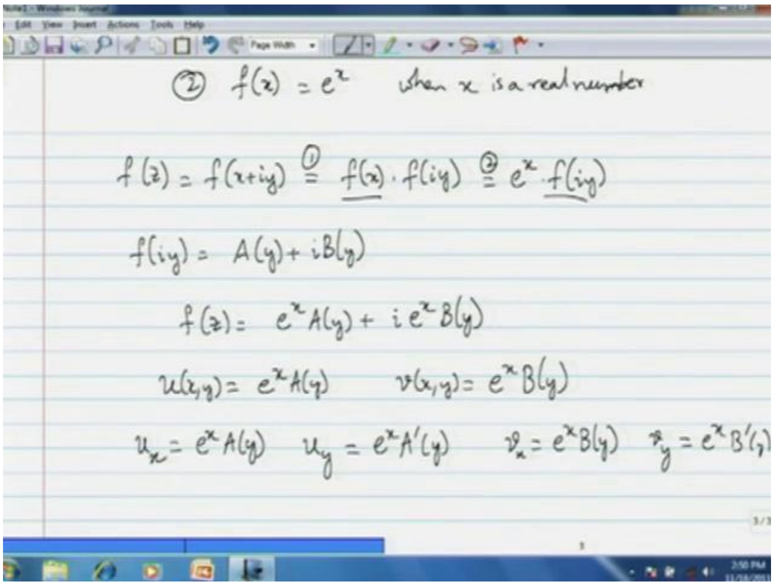

And we also want the property that f of x is e power x when x is a real number i e it is the real number contained in the complex plane. So, when restricted to a real number contained in the complex plane we want this new function to tally with the definition of the exponential function that we already know of real numbers. So, with these 2 desirable properties we want to make a definition, let us see what the definition should at least satisfy. So, in order to do that what I will do is I will consider f of z such a function f of z of course, it can be written as x plus i y, z can be written as x plus i y.

So, by property one by property one1 this should be f of x times f of i y, so and f of x now x is a real number so f of x should now by property two be e power x and then we have f of i y. So, such an f of z should like e power x f of i y for z equals x plus i y. So, further what we require is that well f of i y, if we write that as A of y plus i times B of y. So, I am writing the function f of i y this piece into its real and imaginary parts, so i am separating it into its real and imaginary parts, so then what i get is f of z should look like e power x A of y plus i times e power x B of y.

So, then since we want to define this exponential function in such a way that it is differentiable everywhere in the complex plane; so we also desire that this function will be should be differentiable at every point in the complex plane, so in particular the real and imaginary parts of this function should satisfy the Cauchy-Riemann equations. So, since these should satisfy the Cauchy-Riemann equations lets write them down u of x , y in this case is e power x A of y and v of x, y the imaginary part is e power x B of y. So, u x the partial derivative of u with respect to x gives us e power x A of y and u y is e power x A prime of y. v x is the partial derivative of v with respect to x is e power x B of y and the partial derivative of v with respect to y is e power x B prime of y.

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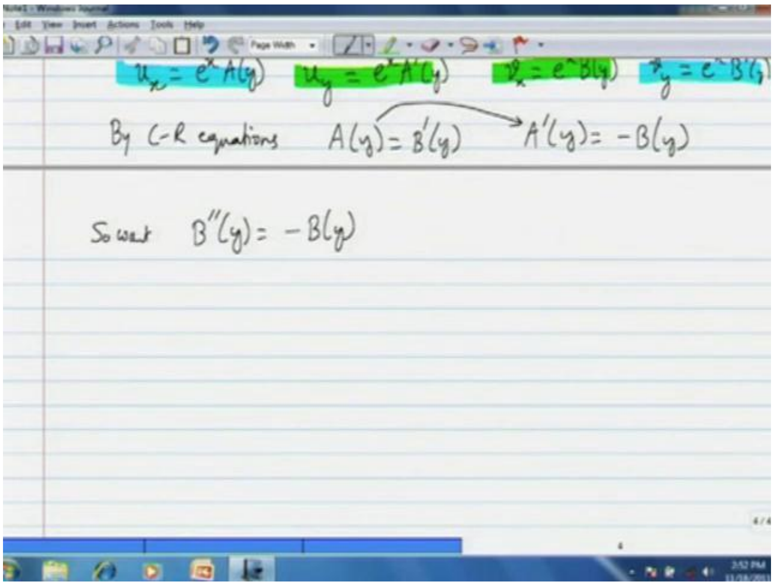

So, by by C-R equations we should have A of y should equal B prime of y, that is because your u x should equal v y and we want A prime of y should equal B of y. That is because or rather A prime of y should equal minus B of y, that is by the the second equation that u y is minus v x. So, then from these two we can conclude that we can substitute A of y in here and say that B prime B double prime of y should be equal to minus B of y.

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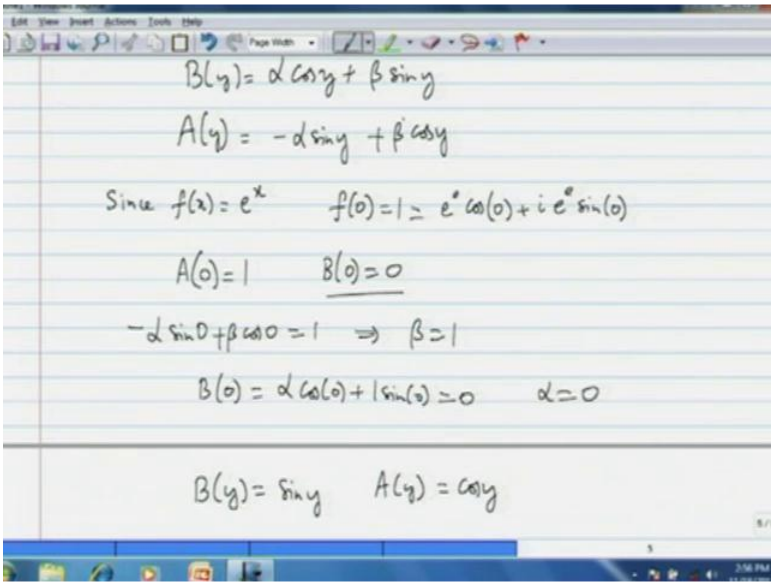

So, want B of B double prime of y to be minus B of y and we know such a function or what kind of function satisfy this relation, so you would want B to be B of y to be well alpha cosine y plus beta sin y. So, and then A of y now will be B prime of y by this relation by this relation here, so A of y should be minus alpha sin y plus beta cosine y. And further we also know that f of x tallies with e power x. So, in particular A of 0 so f of 0 is 1 is e power 0 cosine 0 plus i times e power 0 sin 0 so that should give us 1. So, well so what I want to say is, so we know that A of A of 0 should be 1 and B of 0 should be 0.

So, A of 0 equals 1 tells us that minus alpha sin 0 plus beta times cosine 0 is 1 which implies beta should be 1 and so then substituting beta equals 1 B of 0 equals 0 we get B of 0 is alpha times cosine 0 plus 1 time sin 0. So, we get this should be 0 so alpha should be 0. So, we get B of y is sin y and A of y is cosine y.

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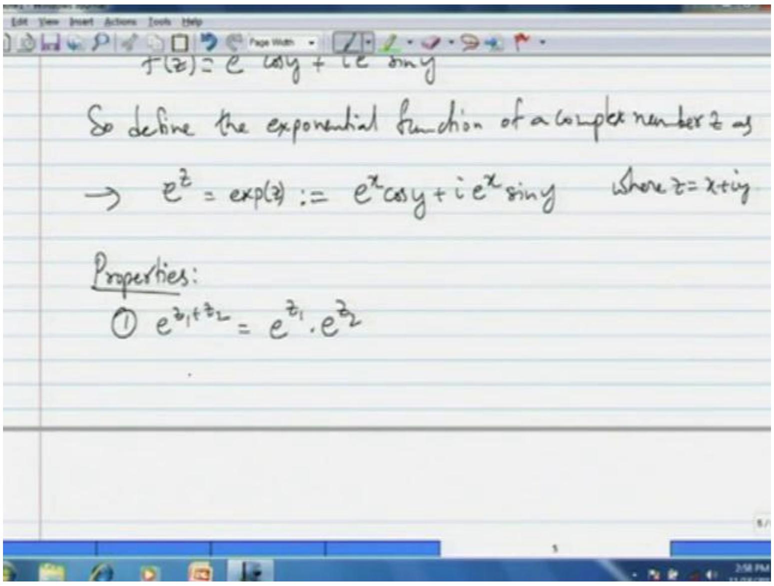

So, we want f of z we want f of z to be defined as e power x cosine y plus i times e power x sin y and hence we will define, so define the exponential function of a complex number z as e power z. So, e will stand for z, alternatively also written as e x p of z e x p of z. So, define that to be e power x cosine y plus i times e power x sin y where z equals x plus i y. So, we defined the exponential function owing to the motivation that we have seen so far. So, we will see that indeed it satisfies many of the properties which are counterparts of the real exponential function. So, here are some properties, so firstly f is so let me note down the motivation as follows e power e power z 1 plus z 2 is indeed e power z 1 times e power z 2.

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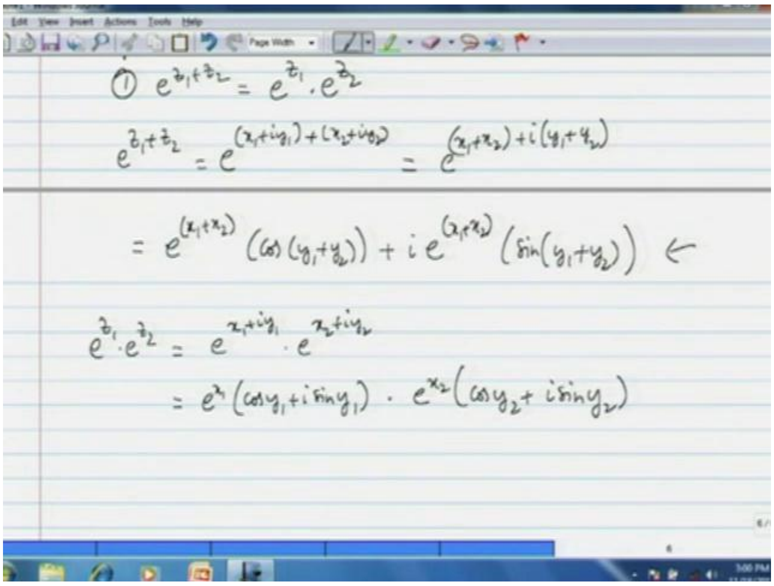

This was one of the properties we started off with and with this definition now this holds. So, let us check that so e power z 1 plus z 2 is if i write z 1 as x 1 plus i y 1and z 2 as x 2 plus i y 2 this is e power x 1 plus x 2 plus i times y 1 plus y 2. So, we get this is by the definition of exponential function e power x 1 plus x 2 times cosine of y 1 plus y 2 plus i times e power x 1 plus x 2 times sine of y 1 plus y 2. one can verify that e power z 1 times e power z 2 will tally with this. Let us see that, e power z 1 is e power x 1 plus i y 1 times e power e power z 2 is e power x 2 plus i y 2. So, now let us use the definition of the exponential function this is e power x 1 times cosine y 1 plus i sin y 1 times e power x 2 times cosine y 2 plus i sin y 2.

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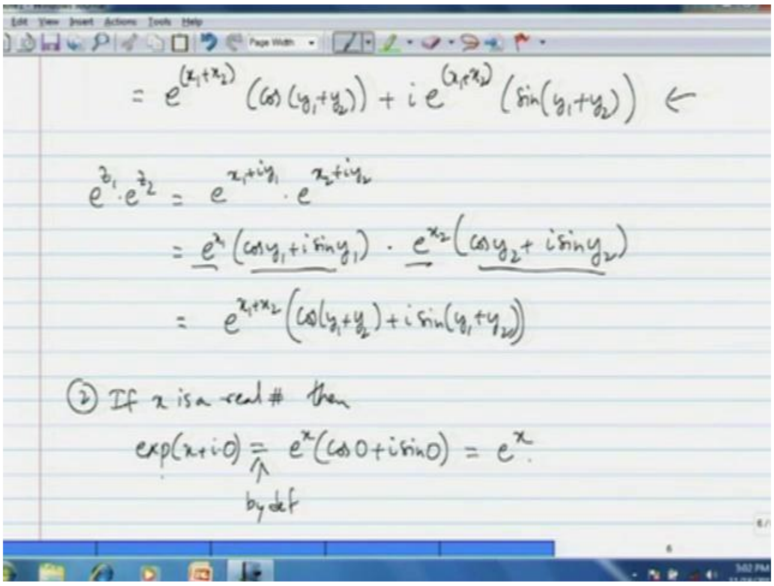

And we know that when we multiply cosine y 1 plus i sin y 1 with cosine y 2 plus i sin y 2 we do get cosine y 1 plus y 2 plus i sin y 1 plus y 2 and then this is the real exponential function now e power x 1 times e power x 2 so we do get a e power x 1 plus x 2. So, of course in this definition these are these are the real exponential functions here. So, we already understand that so indeed we, we get this property the property which we were motivated by. Likewise if, if x is a real number real number which is contained in the complex plane then e power x is the complex exponential function e power x.

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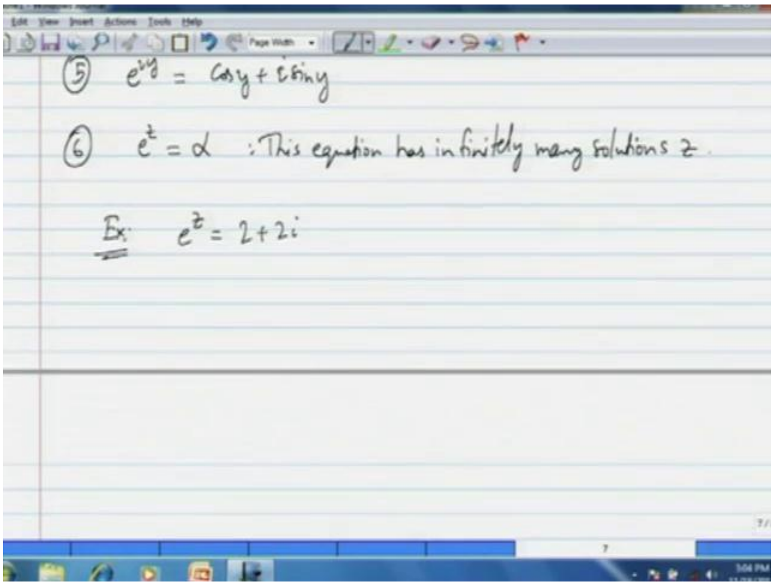

So, let me write that well or very artificially let me write that e x p of x plus i times 0 because that is what a real number in the complex plane looks like, this is equal to by the definition e power x where e power x is the real exponential function now times cosine 0 plus i sin 0. So, this is by definition so by definition. So, this indeed tallies with the real exponential function and there are other properties which follow. So, what are some other properties? The modulus of the exponential function is essentially the e raise to x where, where e is the real exponential function. Now I need not distinguish between real exponential function here because it tallies whenever we have a real number it tallies with the real exponential function.

So, the modulus of e power z is e power x because the modulus of the complex number cos y plus i sin y is 1 and the fourth of the properties is that e power z the way we have defined it is never equal to 0. That is because cosine y plus i sin y is never equal to 0, cosine because cosine y and sin y are not simultaneously 0 for any real number y. So, that is easy to see and then e power i y is cosine y plus i sin y, that is because you can just take x to be 0 and you get that. The sixth property, that if you have e power z equals alpha a complex number then if you look at this equation where alpha is a complex number, this has this equation infinitely many solutions z.

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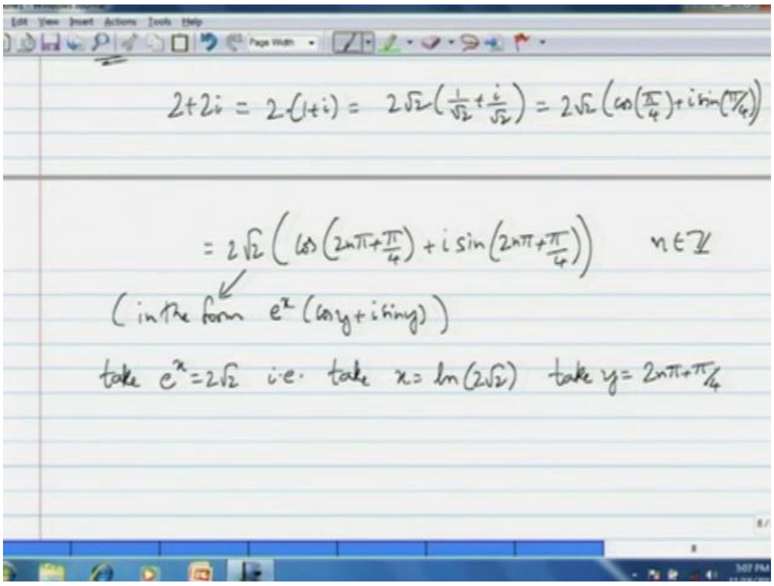

So, there are infinitely many complex numbers z which satisfy e power z equals alpha where alpha is a fixed complex number. So, how do you see this? Well, what we can do is let us consider an example here. So, let us look at e power z equals 2 plus 2 i for example, so what i can do is I will write 2 plus 2 i in its polar form. So, what I get is this is 2 root 2, well this is 2 times 1 plus i so this is 2 root 2 times 1 by root 2 plus i by root 2. So, which is 2 root 2 times cosine pi by 4 for example, plus i sin pi by 4 or this is likewise equal to 2 root 2 times cosine 2 n pi plus pi by 4 because cosine and sin are of period 2 pi. I can write this as sin of 2 n pi plus pi by 4.

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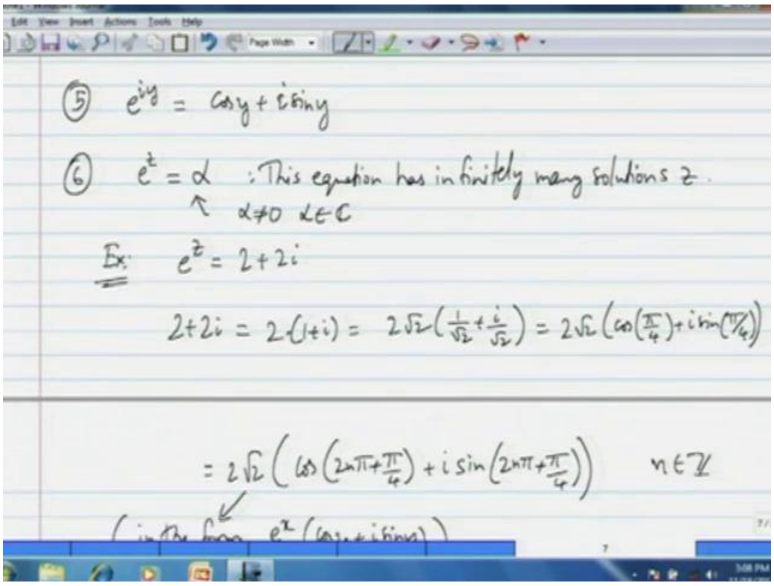

So, so that gives me I mean that least looks like in the form, this is in the form e power x times cosine y plus i sin y. So, of course here n is any integer now if I take e power x to be equal to take x e power x to be equal to 2 root 2 i e take x to be real logarithm of a 2 root 2 and take y to be equal to 2 n pi plus pi by 4. Then e power z equals e power x plus i y x l n 2 root 2 plus i times 2 n pi plus pi by 4 will be equal to your 2 plus 2 y for n belongs to 0.So, since there are infinitely many integers we can conclude that there are infinitely many solutions like that to the equation e power z is 2 plus 2 y and likewise this procedure can be repeated for any complex number alpha. any non zero complex number alpha.

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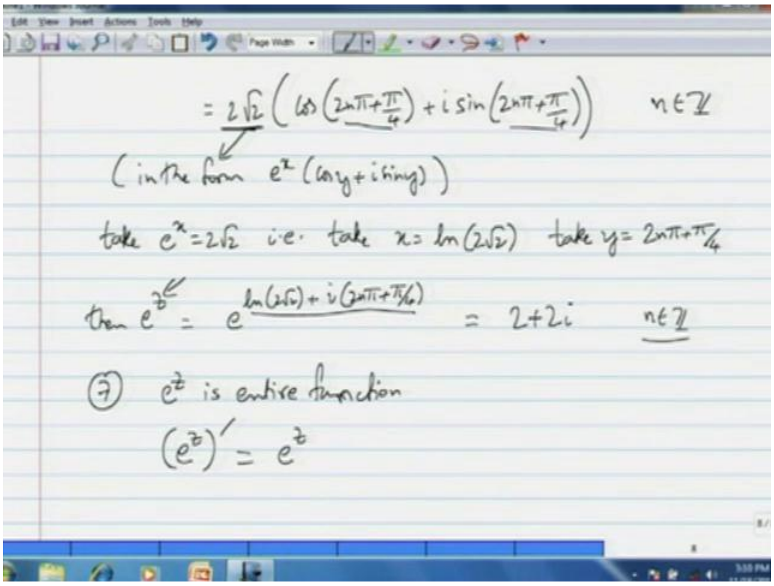

So, I should say alpha not equal to zero alpha belongs to c because e power z is not equal to 0. So, this procedure can be repeated, so when e power when alpha is not 0 this number here which appears to the front is not 0. So, you can take its logarithm it it is a positive real number really. So, you can take its logarithm and the argument of cosine, the argument of the complex number alpha essentially serves as the imaginary part of the solution z and since the argument as we saw earlier has infinitely many values you can you can conclude that equation has infinitely many solutions. So, let us see another property is number seven.

So, e power z is entire function that is because that is by design essentially we wanted to construct the exponential function such that e power z is differentiable everywhere. Remember we used the Cauchy-Riemann equations to to construct something which motivated our definition. So, the real and imaginary parts of this function e power z will now obviously satisfy the Cauchy-Riemann equations and at every point in the complex plane and hence and of course, it is also these partial derivatives are continuous. So, we can conclude that e power z is differentiable at every point in the complex plane. So, it is an entire function, furthermore the differentiation of e power z e power z prime is equal to e power z. So, let us use the fact that there are various ways to find the differentiation of an analytic function.

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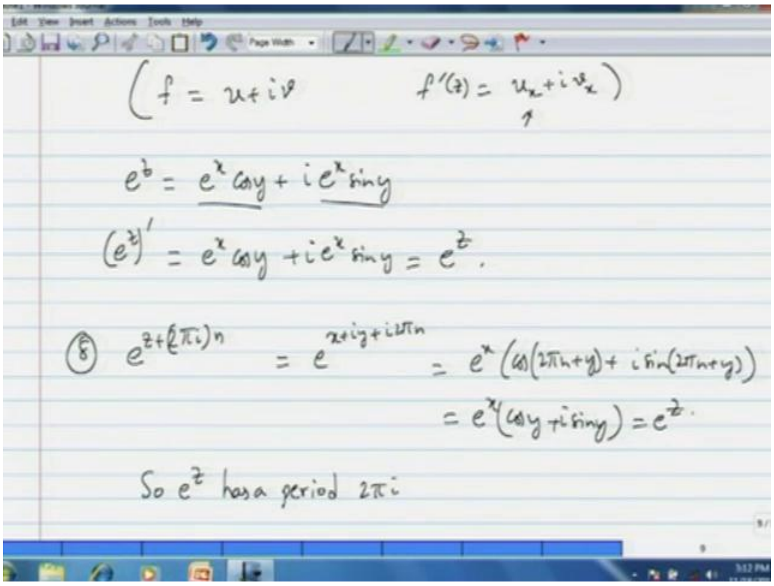

So, if f is an analytic function recall that f is an analytic function with real and imaginary parts u and i v then f prime of z, one of the ways to obtain it is take the partial derivative u with respect to x plus i times partial derivative of v with respect to x. So, e power z since this was e power x cosine y plus i times e power x sin y it is obvious that e power z prime we already proved that this is differentiable. So, we need to find the differentiation, this is e power x cosine y which is the partial derivative of e power x cosine y with respect to x plus i times e power x sin y e power x sin y is the partial derivative of e power x sin y with respect to x.

So, this is once again e power z, so e power z is differentiable at every point in the complex plane and its differentiation as itself. And of course, we can quickly observe that this is the periodic function with a period 2 pi i. So, if you have 2 pi i times n, so this is we observed e power x plus i y plus i times 2 pi n so this gives you e power x cosine 2 pi n plus y plus i sin 2 pi n plus y. So, of course this is equal to e power x cosine y plus i sin y which is e power z. So, e power z has a period 2 pi i n. So, these are some of the properties of the exponential function. Next let us look at the mapping properties of this exponential function. By mapping properties I mean let us try to see how this exponential function maps certain sets certain important sets in the complex plane to to its range.

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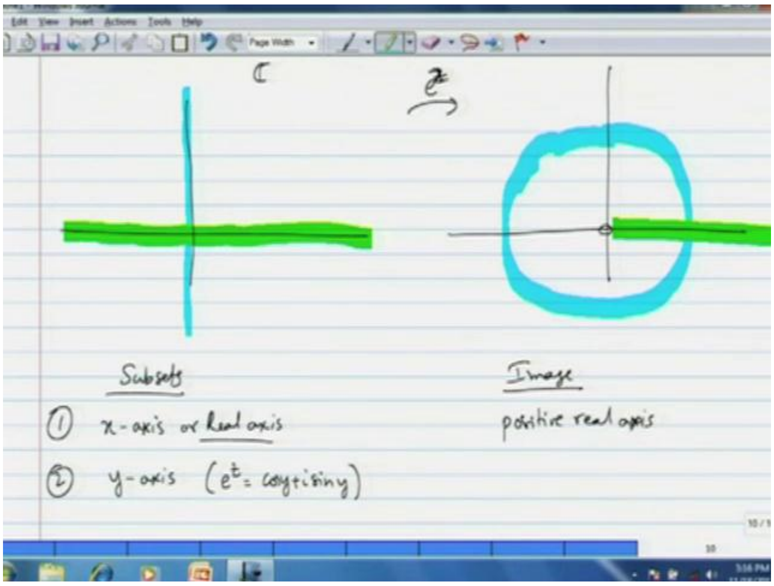

So, I will explain. So, here we will consider the domain of the exponential function namely all of the complex plane and let us try to see that see what it does to some subsets as follows. So, let us consider the following nine subsets. Firstly let us observe what it does to x axis or the real axis. So, this function this transformation is e power x e power z rather. So, when you consider the x axis or the real axis then the image of x axis, well we know the exponential function tallies with the real exponential function when we restrict ourselves to real numbers. So, what we get is positive real axis.

So, the exponential function the real exponential function is 1 to 1 and on to the positive real axis when you restrict to the, well the real exponential function is as that property, so the image is the positive real axis. So, let me denote this with the image of that being the positive real axis. So, of course, you skip 0. 0 is not in the image of the exponential complex exponential function. Now, the second subset we will consider is the y axis, let us examine where the y axis goes to? So, if you consider the y axis, its equation is x equals 0. So, e power z will now equal cosine y plus i sin y because x is 0, e power x is 1 so that you get e power z is cosine y plus i sin y.

Now these are points cosine y comma sin y for y real number for y any real number. So, its image is going to be the image of the the y axis is going to be points on the unit circle. So, that is because that is because the points cosine y comma sin y are on the unit circle are the set of all cosine y comma sin y such that y belongs to r is essentially equal to the set of all z belongs to complex plane with mod z, modulus of z equals 1. So, here equality means the equality of points in r 2 with the with the points in the complex plane. So, the image of y axis is the unit circle and let us now examine the image of any horizontal line. So, a horizontal line we have seen that the image of x axis is positive real axis. The image of a horizontal line horizontal line, the equation is y is constant.

So, e power z will look like e power x times cosine y naught plus i sin y naught. So, x varies x assumes all the positive real numbers and cosine y naught plus i sin y naught essentially determines the direction in which the positive real numbers proceed. So, to given example, example if you look at y naught equals pi by 4 y equals pi by 4 constant (( )) equals pi by 4. So, you are looking at e power z equals e power x times 1 plus i by root 2. So, this is a real multiple e power x is a positive real number. So, e power z is the real multiple of the complex number 1 plus i by root 2.

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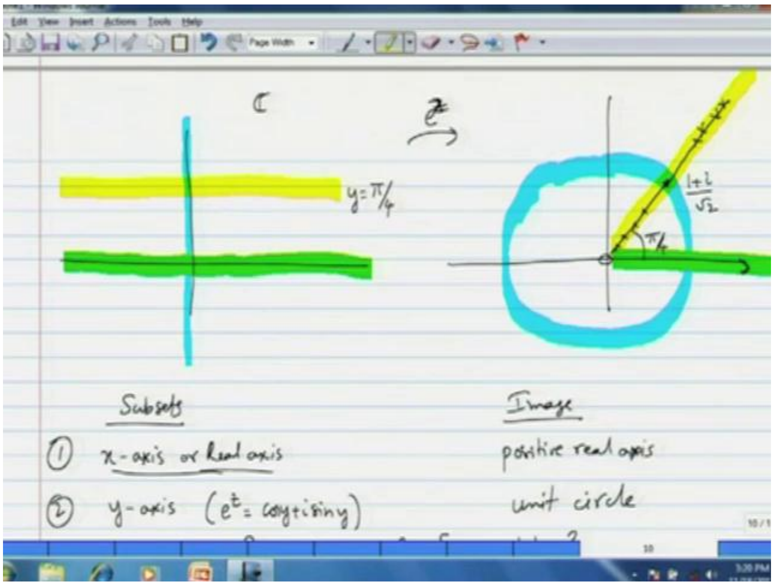

So, if you consider so once again let me take another colour, so for this horizontal line let me call that y equals pi by 4 for example, then the image of this is going to be well here is let us suppose 1 plus i by root 2 it is on the 45 degree line 45 degrees pi by 4. It is on the line which makes 45 degrees with the positive x axis. So, now we scale this so we multiply this 1 plus i by root 2 with any positive real number so if the positive real number is less than 1 you bring it closer to the origin and if the positive real number is greater than 1 you throw it away from the origin. So, the image of this yellow line is going to be this yellow line. So, it is the positive or it is the half ray starting at 0 and going all the way up till infinity except the that the point 0 is missing.

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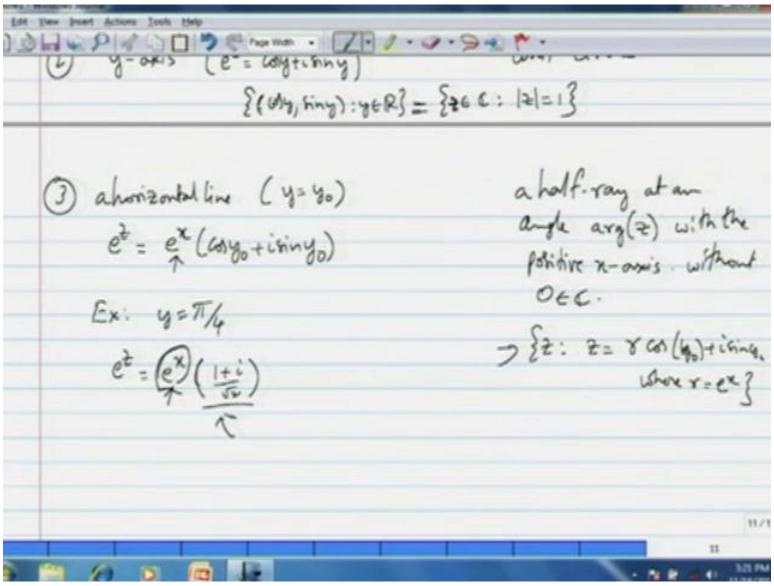

So, the image of such a horizontal line will be set of all z so set of all z such that such that e power what you say its better describe geometrically I will say this is a half ray at an angle argument of z with the positive x axis. So, without the 0 without the origin itself of course, I have given a set definition here as well you can say it is the set of all z such that such that z equals e power x or a real number times cosine of y naught plus i sin y naught where r equals e power x, so the image is set of all like that way.

So, that is the image of a horizontal line likewise let us say inspect where does a vertical line go to? The image of a vertical line has the equation x equals x naught. So, e power z then transforms to this line to e power x naught which is a fixed real number now fixed positive real number times cosine y plus i sin y and as we already discussed the point cosine y comma sin y in r 2 is on a circle, is on the unit circle. So, now the point e power x naught times a point on the unit circle is essentially the expansion of the unit circle or the contraction of the unit circle depending upon whether e power x naught is greater than 1 or less than 1.

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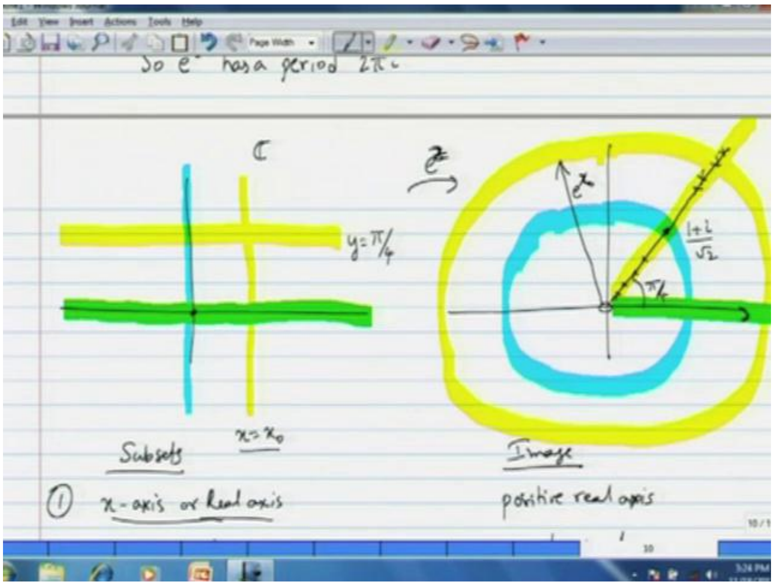

So, if you, if you go back to the picture, let me take a yellow line again or a or a yellow line again and if we look at vertical line like that, and whose equation is x equals x naught, then depending upon whether x naught is greater than 1 or less than or depending upon whether e power x naught is greater than 1 or less than 1.

So, since we see that x naught is positive in this case, it is beyond this point 0 so e power x naught will be greater than 1. So, the image of this yellow line will be circle it will be a circle of radius greater than 1 or exactly it will be a circle of radius e power x naught. So, the image of this vertical line is that circle. So, now let us look at one more important subset I did not complete the description here the image is a circle of radius e power x naught centred at the origin and if x naught is less than 0 then circle has radius less than 1 and if x naught is greater than 0 radius is greater than 1 .So, next we have the image of a strip now we look at image of regions so image of a strip, 0 less than or equal to y less than 2 pi. So, this is essentially the set z equals x plus i y such that y is in between 0 and 2 pi.

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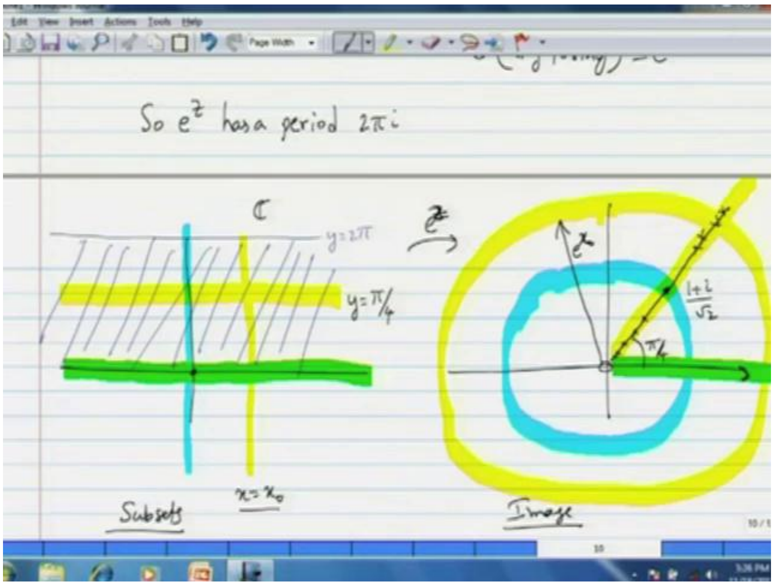

So, this is the collective image of all the horizontal lines lying between y equals 0 and y equals 2 pi including y equals 0 but, not including y equals 2 pi. So, let me go back to the picture so let me use a pen of some colour so fine. So, suppose this is, this is clearly not to scale, but let us suppose that is y equals 2 pi and the region now I am talking about is this hash region here. These it is the set of all horizontal lines, which go from 0 to 2 pi all the horizontal lines which go from 0 to 2 pi and so it is the collective image of all the horizontal lines and as we discussed the image of a horizontal line is half ray. Now as these horizontal lines move from y equals 0 to y equals 2 pi you get half rays which go from angle 0 with the positive x axis to the angle 2 pi the positive x axis.

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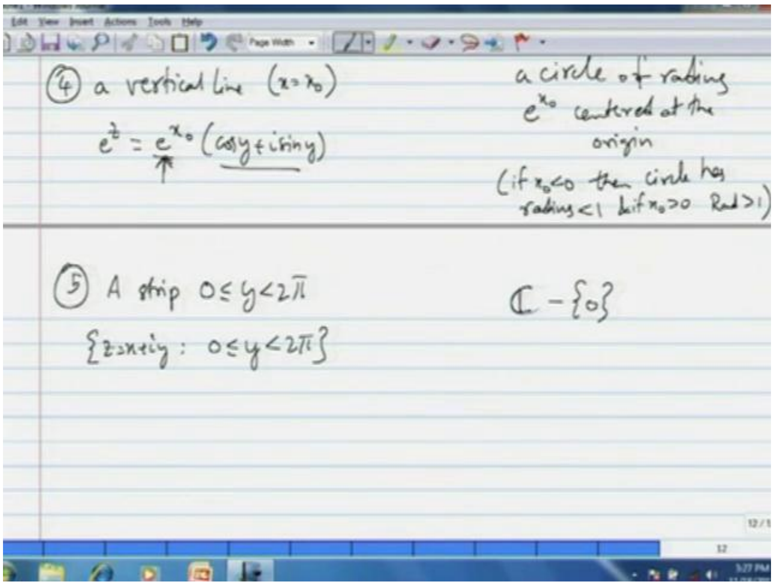

So, you essentially get all the rays, all the half rays which span the entire complex plane, but they of course, miss the point 0 itself. So, the image of such a strip is all of the complex plane I want to shift.

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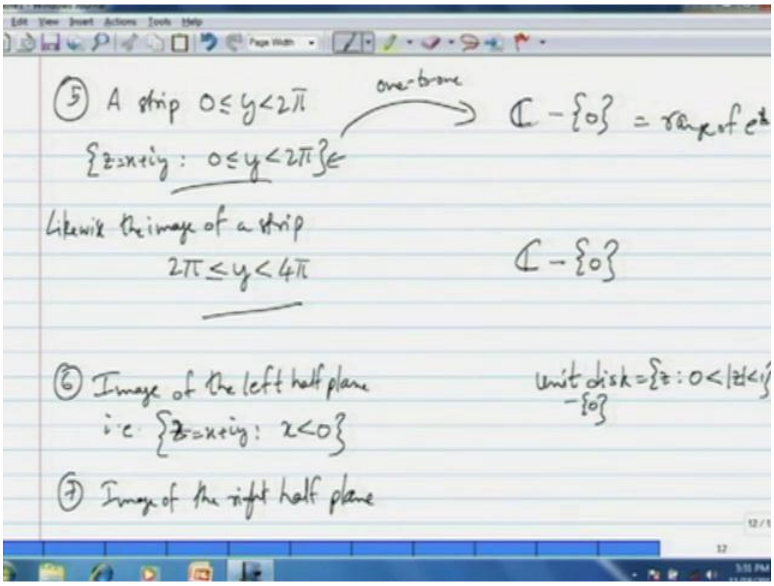

So, this is all of the complex plane minus the point 0 and this reflects. So, then before I say make that statement let me say that the image of likewise, likewise the image of a strip 2 pi less than equal to y strictly less than 4 pi is also all the complex number minus 0, because now you will begin at the angle 2 pi with the positive x axis, which will essentially be the positive x axis; and then you will take all the half rays half infinite rays starting at 0 and go around until angle 4 pi with the positive x axis.

So, you will essentially span all of the complex plane except the point 0. So, this reflects the periodic nature of the exponential function. So, these you can, you can spilt the entire complex plane into these strips of of y ranging from 0 to 2 pi and you can imagine the complex plane to be a stack of these strips and the image of each of the strip is all of the complex plane minus the point 0. So, what is also important is that the image of the strip onto C minus 0, which is all of the range of range of e power z, this is 1 to 1 the function e power z is 1 to 1 on this strip which is not a feature of the exponential function in general.

So, for example, we showed already that e power z equals alpha has infinitely many solution. So e power z is not a 1 to 1 function, but on this when you restrict the exponential function onto any of these kind of strips then the function e power z is 1 to 1. And then when you restrict the exponential function onto this strip there is a possibility of defining the inverse function of e power z which you which we would want to call the logarithm but that way we will do later. So for now, we will continue with these mapping properties, and let me say that the image one can observe of the left half plane i. e, the plane the set of all z equals x plus i y such that x is strictly less than 0.

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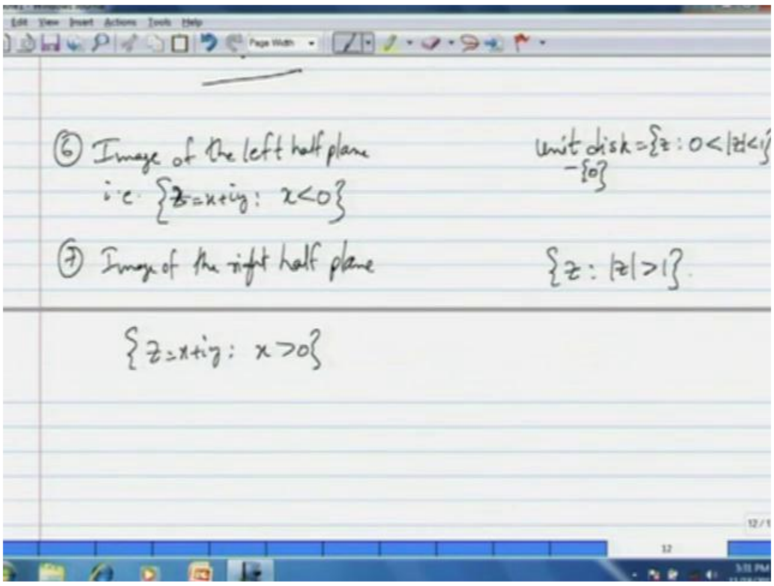

So, left to the y axis the left half plane the image of this is the unit disc, set of all z such that 0 strictly less than mod z strictly less than 1. So, this is unit disc minus the point 0 of course, the point 0 is not in the image of e power z and seven the image of the right half plane is going to be things outside the unit disc set of all z equals x plus i y, such that x is strictly greater than 0 the image of this is going to be, set of all z, such that the modulus of z is strictly greater than 1. So, these are the mapping properties of the exponential function. So I will stop here.