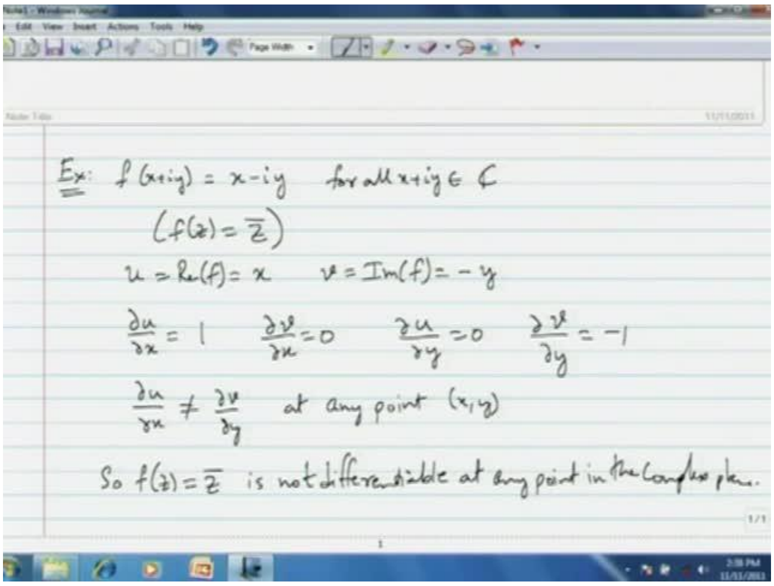
Complex Analysis

Prof. Dr. P. A. S. Sree Krishna
Department of Mathematics
Indian Institute of Technology, Guwahati

Module - 2
Complex Functions: Limits, Continuity and Differentiation
Lecture - 4
Cauchy-Riemann Equations and Differentiability

Hello viewers, last time we saw that Cauchy-Riemann equations where necessary for a function to be differentiable. So, let us start this session with an example where Cauchy-Riemann equation fail. So, that we can conclude that the function is not differentiable.

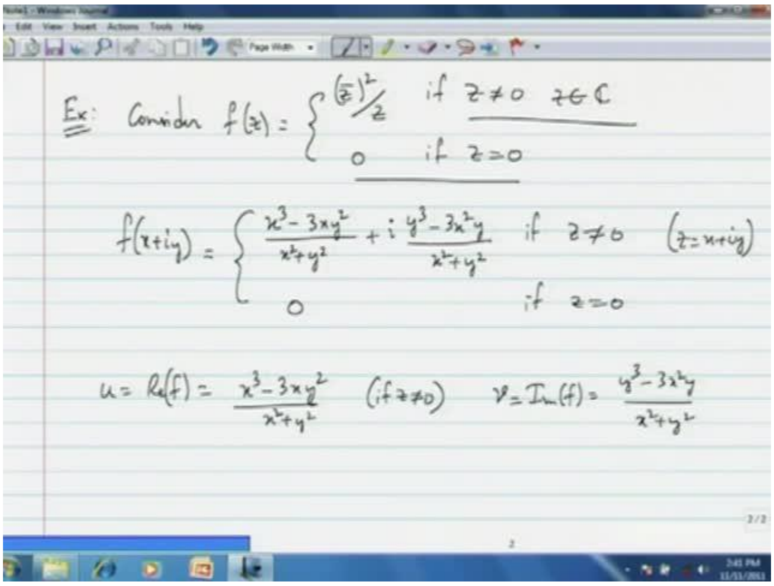
(Refer Slide Time: 00:50)



So, let me start with this example f of x plus i y. So, z is x plus i y equal x minus i y. This is the function which is essentially the conjugation of a complex number. So, this is this definition is for all x plus i y belongs to the complex numbers. So, f of z is z bar, so in this case u which is the real part of f is the function x and v which is the imaginary part of the function f is minus y. So, that the partial of u with respect to x is 1 and partial of v with respect to x is 0 partial of u with respect to y as 0, and the partial of v with respect to y is minus 1. And clearly dou u by dou x is not equal to dou v by dou y at in point x comma y.

So, since the Cauchy Riemann-equations are not being satisfied at any point z in the complex plane, we conclude so f of z is equal z bar is not differential at any point in the complex plane. So, Cauchy-Riemann equations are sometimes useful to note, when a functions is actually not differentiable.

(Refer Slide Time: 03:01)

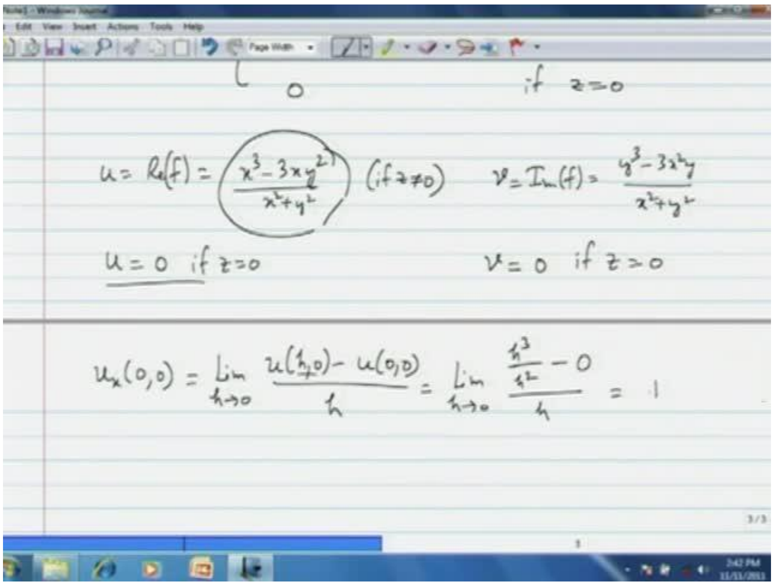


So, next we will see an example this is an example of an different kind. In this example we will see that the Cauchy-Riemann equations are satisfied, but something strange happens. So, consider this function f defined as follows f of z is the conjugate of z square divided by z, if z is not equal to 0 and it is equal to 0, if z is equal to 0. So, we can write this in the form f of x plus i y. So, this definition is clearly for all complex numbers because this is for z not equal to 0 and this is for z equals to 0, so that counts all complex number.

So, when we are write this in the form f of x plus i y, in order to separate the real and imaginary parts of the function, what we get is get x cube minus three x y square the viewer can work this out by writing said as x plus i y, and then this is divided by x square plus y square plus i times y cube minus three x square by divided by x square plus y square.

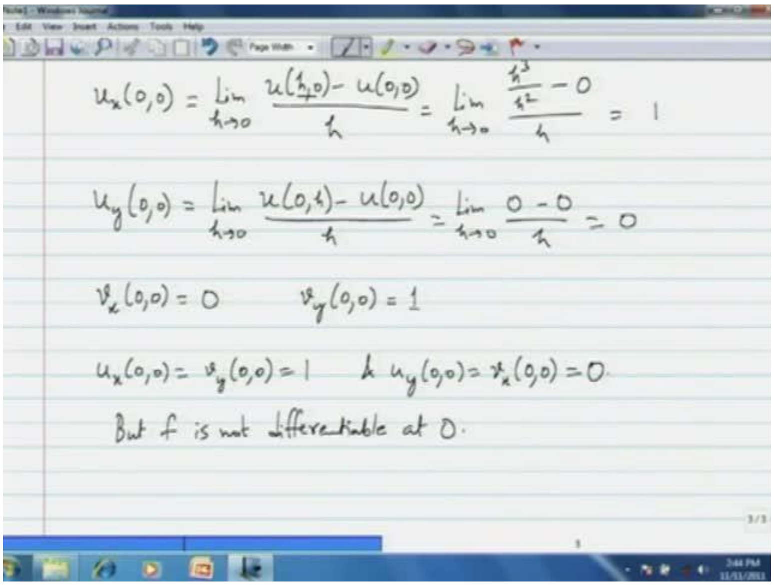
When or let me write if z is not equal to 0 here said is a x plus i y and this is 0 if z is equal to 0. So, since i z is equal to x plus i y. I am using z equal to x plus i y. So, with this with this definition we see that at the point z equal to 0. Let us try to calculate the partial derivatives of, let me write down exclusively explicitly the real part of f and the imaginary part of real part of f is x cube minus three x y square by x square plus y square, and if z is not equal to 0 and v is the imaginary part of f is y cube minus three x square y by x square plus y square.

(Refer Slide Time: 05:51)



So, using this now calculate the partial derivatives of u v etcetera, at the point 0,0. So, the partial derivative of u so let me also write the other case u is equal to 0. If z is equal to 0 this is useful for us v is 0, if z is equal to 0. So, the partial derivative of u with respect to x at the point 0, 0 is the limit as some h goes to 0 u of h comma 0 minus u of 0 comma 0 divided by h. So, which a is so we can calculate easily limit as h equals to 0, let substitute x equals h to y equals to 0 for the first expression u of h 0, we get h cube divided by h square. So, I am substituting h comma 0 in this expression. So, we get h cube by x square by h square rather minus u of 0, 0 value of 0,0 is 0 divided by h. So, this is as we can see the limit is 1.

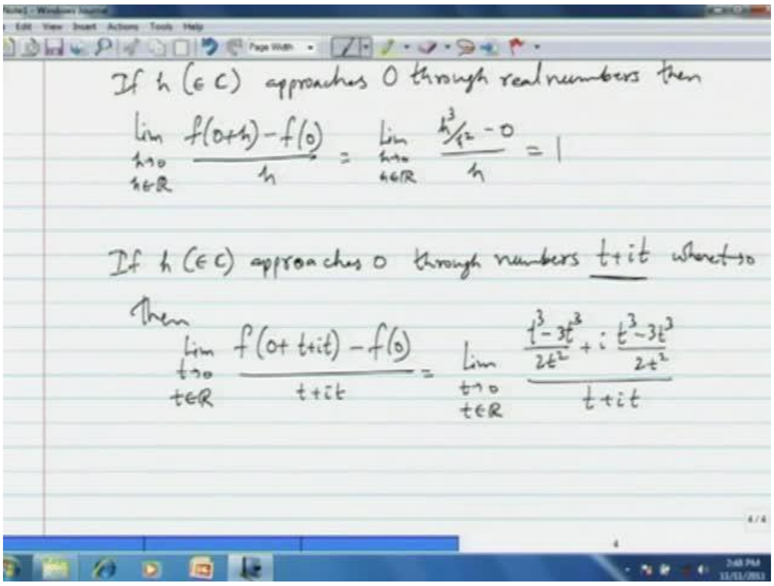
(Refer Slide Time: 07:15)



Likewise we calculate u y of 0, 0 which is the limit as h equal to 0, u of 0 comma h minus u of 0 comma 0 divided by h. So, this gives us limit as h equals to 0 and when we substitute 0 comma h in the expression. For the u here we get 0 and then minus 0 by h square actually. So, that is 0 and then this is 0 by h which is 0. Likewise we calculate v x of 0, 0 which will turn out to be 0 and v y of 0, 0 which will turn on to be 1.

So, indeed u x equals 0 v y or let me u x at 0, 0 is equal to v y at 0, 0 is equal to 1 and u y at 0, 0 is equal to v x at 0, 0 is equal to 1, 0 rather, but what is interesting is that, but f is not differentiable at 0. This has to do with the fact that partial derivative only respective directions and we can approach 0 in several directions. So, let us see y f is not differentiable at the point 0.

(Refer Slide Time: 09:08)



So, here if h approaches. Now, h is complex number so h belongs to complex number approaches 0 through real numbers, then limit as h goes to 0 h belongs to real numbers. So, h is complex number, which is a real number now f of 0 plus h minus f of 0 by h is limit as h goes to 0, h belongs to r. Since, h is real number it only has well the from the definition of f we can calculate what f of 0 plus h is we get h cube by h square minus 0 by h which gives us 1.

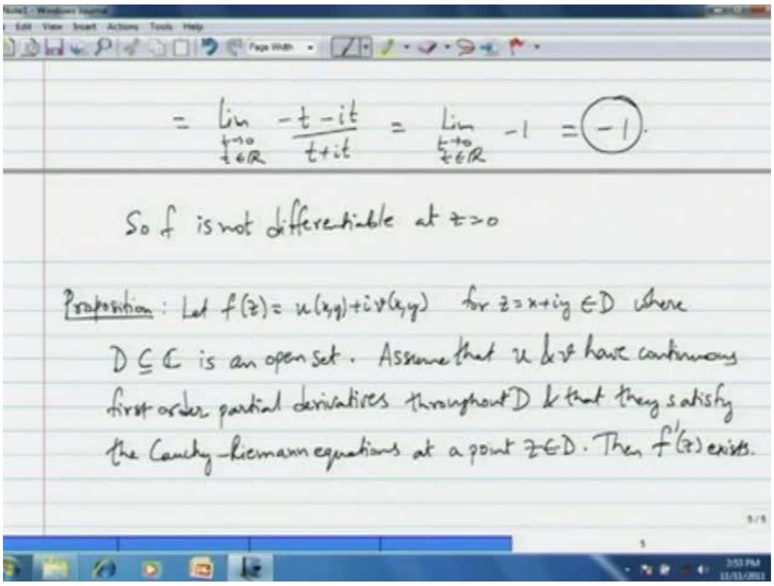
So, now it would not be very helpful to see what happens when approaches 0 through purely imaginary numbers, that would gives us a same kind of result, but instead what we will do is let us take h in the complex number approaching 0. So, if h approaches 0 through numbers of the form t plus i t a 1 parameter complex number t plus i t. So, these are complex numbers which lie on a 45 degrees line with the positive x axis.

So, this are on the line y equals x in the in the plane x y plane. So, they are they can be parameterized as a t plus i t. So, let us assume that h approaches 0 through a those kind of numbers than where t approaches 0, of-course when h approaches 0, this one parameter number t plus i t should be such that, t approaches 0 in order for it to go to 0 or in order for it to approach 0.

So, where t goes to 0 then let us see what happens to this limit, limit t goes to 0 t belongs to r t is are real. So, that t plus i t is a complex number like I described and then you get f of 0 plus t plus i t. Now, notice h is t plus i t minus f of 0 divided by h so in this case I have t plus i t. So, this gives me the limit as t goes to 0 t belongs to real numbers.

We have used the formula for f in terms of the real and imaginary parts to get t cube minus t 3 square by 2 t square plus i times t cube minus 3 t cube divided by sorry this is t cube minus 3 t cube here, this is t cube, t cube minus 3 t cube by 2 t square plus i times t cube minus 3 t cubes divided by 2 t square divided by t plus i t. So, it is symmetric.

(Refer Slide Time: 13:03)



And lets calculate the portion we get this is the limit as t goes to 0 t belongs to r (( )) t cube minus 3 t cube is minus 2 t cube divided by 2 t square. So, that gives me minus t and then plus i times likewise minus t. So, this is minus i t divided by t plus i t, so this is limit as t goes to 0 t belongs to r. Since, t is not equal to 0 I can divided this numbers to get a minus 1 and then this limit is minus 1.

So, we have two different directions of approaching 0 and this quotient the limit of the quotient gives us different results as we can see. So, once it is 1 and another time it is minus 1. So, f is not differentiable at z equal to 0 right, because this quotient limit should give us the same number independent of which direction h approaches 0. So, here we pick two directions one is the 45 degree line and another is the real numbers through which h approaches to 0, but we get different answers. So, f is not differentiable at this point z equal to 0.

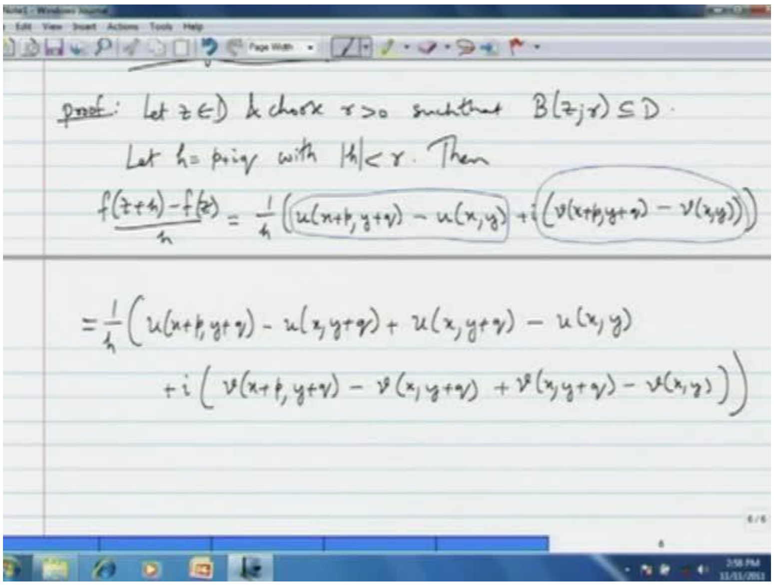
So, this tells us that Cauchy-Riemann equations, can be held, but still the function may not be differentiable or in the other words, the Cauchy-Riemann equations are not sufficient conditions to ensure that the function is differentiable at a point. So, there is a slight modification that we can make in order to ensure that when, the Cauchy-Riemann equations hold the functions is indeed differentiable.

So, the next proposition tell us what that condition is so here is proposition so let me let me give the punch line right away. When we assume that this partial derivatives u x u y u x v y which participate in the Cauchy-Riemann, equations are continuous around the point of interest in addition to the Cauchy-Riemann equations, holding at that point then we can conclude that the function is differentiable at the point of interest.

So, the next proposition makes this concrete here is the position. Let z equals, let f of z equals u of x y plus i times f x y for z equals x plus i y belongs to say d, where d contained in c is an open set and is the domain of x at u and v have continuous first order partial derivatives throughout d and that and also assume, that they satisfy the Cauchy Riemann equations at a point 0 then the conclusion is that f prime of z exists that is f prime z exist in our other words, f is differentiable at the point.

So, here the I mean in addition to u and v satisfying in the Cauchy-Riemann equations, we also assume that u and v have continuous first order partial derivatives in d throughout d. And d is an open set so actually the conditions can be weaken quite a lot all we requires is that u and v, have partial derivatives in a neighborhood of z at the point z and in a neighborhood also. So, let us see the prove of this theorem.

(Refer Slide Time: 18:38)

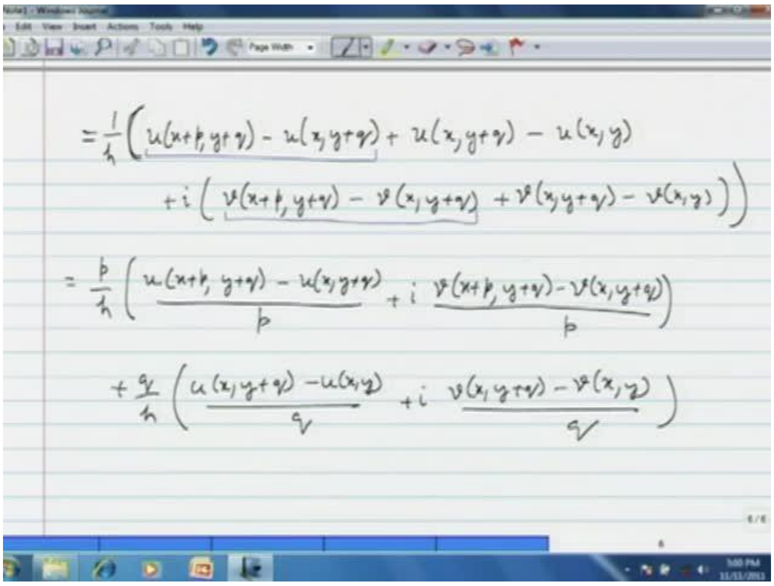


Let see y this is true so let us first locate lets first locate a single point in d and choose r positive such that, the ball of radius r around z at the point of interest is contained in d. We can always do that because b is an open set. So, it always contains wall of some radius around a given point, which is contained in the set. So, such an assumption is fair and then let, let h be p plus i q. Let us a write the real and imaginary parts of h as p and q with a modules of h strictly less than r. So, that when I take z plus h i can be sure that z plus h is in this bowl b z r, then let us look at the quotient which appears in differentiation f of z plus h minus f of z divided by h.

So, this is 1 by h times well u of x plus p comma y plus q minus u of x comma y plus i times v of x plus p comma y plus q minus v of x comma y. Now, I am going to add a couple of terms here so in order to apply being value theorem of one variables to this functions. What I will do is, I will add and subtract a term to each group here. So, take this group here and then what I will do is add and subtract u of x plus p comma y plus q minus u of x comma y plus q.

So, this is the new term that I subtract and then add the same u of x comma y plus q minus u of x y and all of this which I have written now, is the one which appears in the box in the, in the blue colored box above, plus i times likewise plus i times and then instead of this v of x plus p y plus q minus v of x y. What I will do is I will write v of x plus p comma y plus q minus v of x comma y plus q the very same thing, I have done to you plus v of x comma y plus q minus v of x y and end this parenthesis. All this is multiplied by one by h assume for the timing be that both p and q are non 0, p and q are 0 the appropriate factors will not appear.

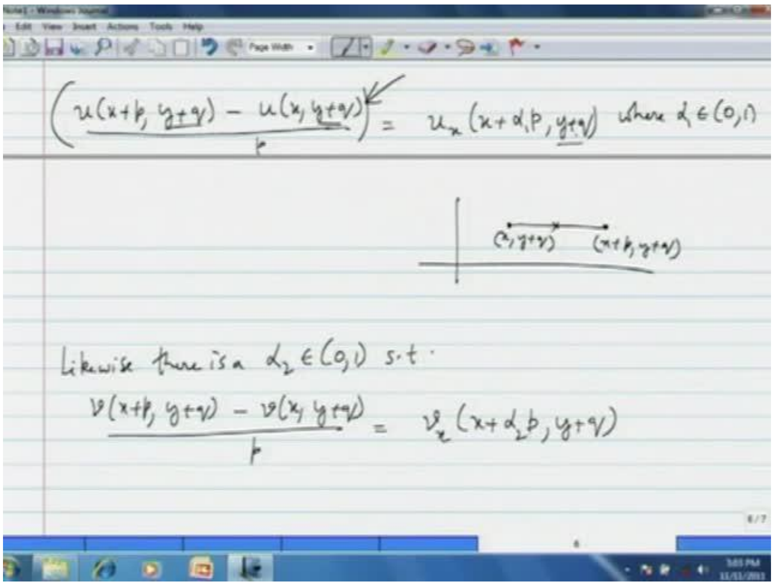
(Refer Slide Time: 23:10)



So, then what I will do is I will multiply and divide by the number p the real number p to the groups here. So, let us again highlight so to this and to this i times this I will multiply and divide p you will see in a moment. Why I am doing that I want to apply the mean value theorem of one variable. So, this is u of x plus p comma y plus q minus u of x comma y plus q divided by p. So, that p settles score with that p plus i times.

Likewise, v of s plus p comma y plus q minus v of x comma y plus q divided by p and then I have likewise plus q by h for whatever is remaining, I will multiply and divide by q. So, that I get u of x comma y plus q minus u of x y divided by q plus i times v of x comma y plus q minus v of x comma y divided by q.

(Refer Slide Time: 25:01)

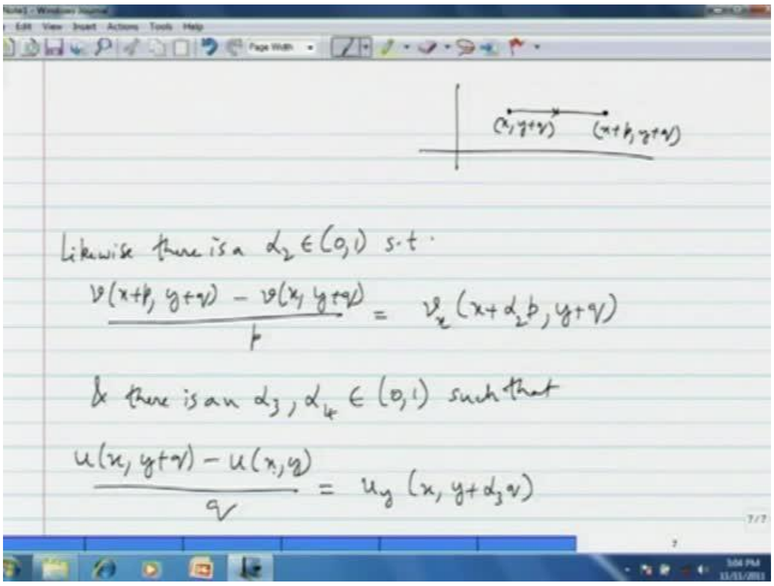


So, now you see that this is the candidate for mean value theorem. So, u of x plus p comma y plus q minus u of x comma y plus q divided by p will equal u, the partial of u with respect to x you see that only the x variable is varying, in the numerator you see that x plus p changes to x. So, since u is use the partial of u with respect to x exists. What you can do is you can write this as u x of x plus sum alpha one times p comma q, where alpha 1 belongs to 0 comma y.

So, there exists an alpha 1 such that this, this quotient is this quotient here is equal to the partial of u with respect to x. At the point x plus alpha 1 p comma q so to once again emphasis the point here is the point x comma y plus q, and then here is the point x plus p comma y plus q it is possible that point is two the left of this point, if p is negative does not matter, but there on the same straight horizontal line.

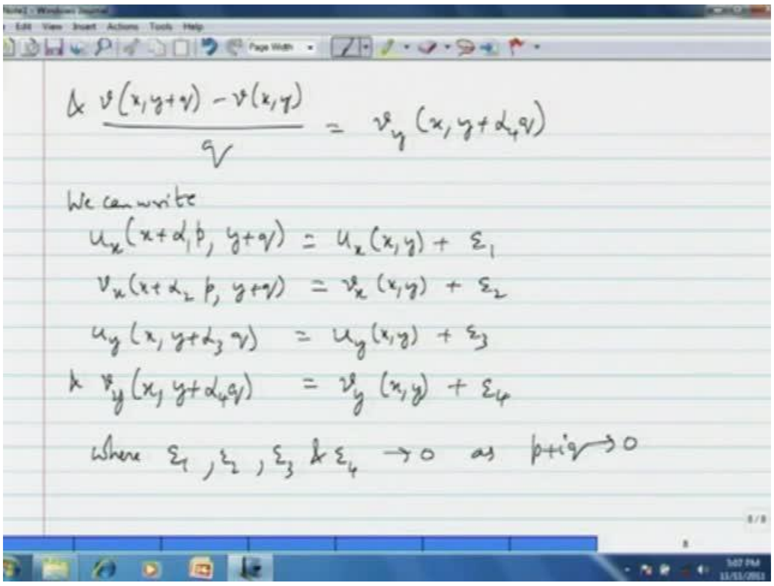
So, there will be a point in some where there between such that, that the partial is u with respect to x at the will equal this quotient that is the mean value theorem, one variable mean valuable theorem except that. Since, we have a two variable function we replace that the partial derivative instead of instead of single variable derivative. So, then likewise there is alpha 2 belongs to 0, 1 such that u of or let me say v of x plus p comma y plus q minus v of x comma y plus q divided by p is equal to v x. The partial of v with respect to x at the point x plus alpha 2 p comma y plus q, I really apologize this should by y plus q not q here I ignore that point, because it is the same y plus q. So, y plus q that should have been y plus q.

(Refer Slide Time: 28:19)



So, then and there is an alpha 3 comma alpha 4, which are in 0, 1 for the very same reason. Such that u of x comma y plus q minus u of x comma y divided by q is equal to u y this times of x comma y plus alpha 3 q. So, this time you notice that it is y which is changing x, the x coordinate remains constant at x. Now, you are looking at a vertical line passing through x y and x comma y plus q and somewhere in between there is a point x comma y plus alpha 3 cube. Such that the partial derivative at the point with respect to y of q will equal this quotient, which is to the left of this equation. So, that is the mean value theorem once again.

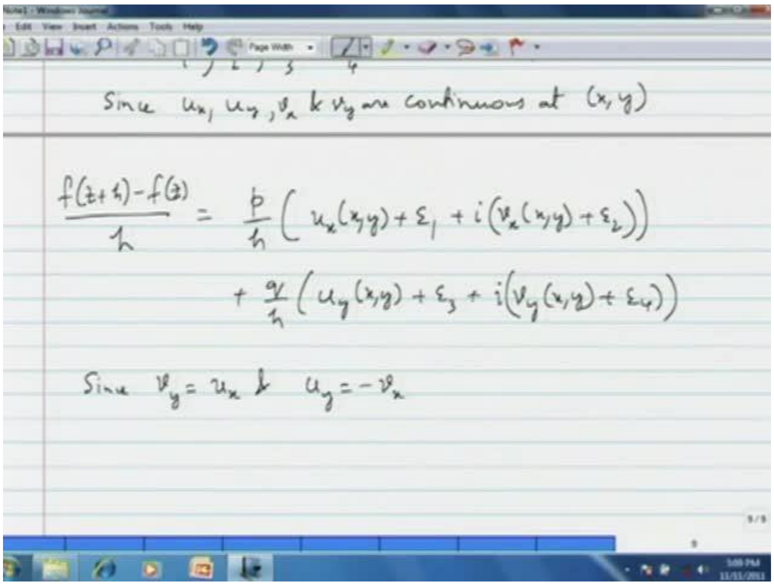
(Refer Slide Time: 29:36)



And v of x comma y plus q minus v of x comma y divided by q is equal to v y of x comma y plus alpha 4 q. So, with all this, we can we can make substations in the in quotient f of z plus h minus f of z by h, but we want more than this. Let us look at a for example, u of u x of x plus alpha 1 p comma y plus q. Since, the partial derivatives u x u y v x v y are continuous. Now, what we can do is we can write the partial derivative u x plus at the point x plus alpha 1 p comma y plus q as u x, the partial of u with respect to x at the point x comma y plus a quantity epsilon 1, which is really small as long as p n q are small as long as h approaches 0, h is close to 0.

So, you can write I will say we can write, this and then v x of x plus alpha 2 p comma y plus q as v x of x comma y plus epsilon 2, and likewise u y of x comma y plus alpha 3 q equal u y of x comma y plus epsilon 3, and u v y at the point x comma y plus alpha 4 q equal v y at the point x comma y plus epsilon 4. Where epsilon 1, epsilon 2, epsilon 3 and epsilon 4 approach 0 as p plus i q approaches here. So, that is the that is something we know because of continuity. This we can do since u x u y v x and v y are continuous at x comma y. So, here is where we use the continuity of these functions of this partial derivatives.

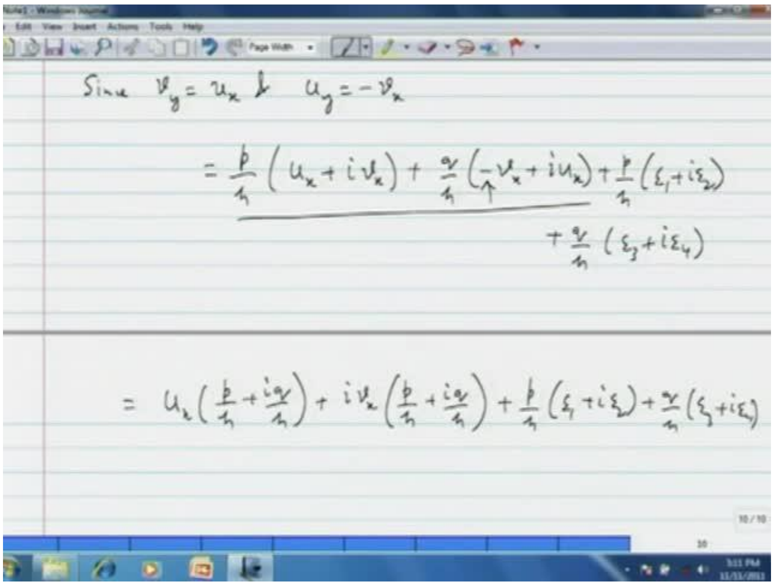
(Refer Slide Time: 32:09)



And with that now, we are ready to substitute into the quotient f of z plus h minus f of z divided by h that quotient. Let me please go back, so that quotient k into this here this much here, and we are now able to write each of this quotients as follows. So, this is equal to u x so firstly t by h there is a co-efficient p by h for the first part. So, you get u x at the point x comma y plus epsilon 1 plus i times v x at the point x comma y plus epsilon 2 plus q by h times u y at the point x comma y plus epsilon 3 plus i times v y at the point x comma y plus epsilon 4.

So, now the other part of the hypothesis or the first part of the hypothesis is that, u x u y v x v y this partial derivatives satisfy the Cauchy-Riemann equations. Since, u x equals v y so let me write it in the other way around since v y is equal to u x and v x is equal to or let me say u y is minus v x.

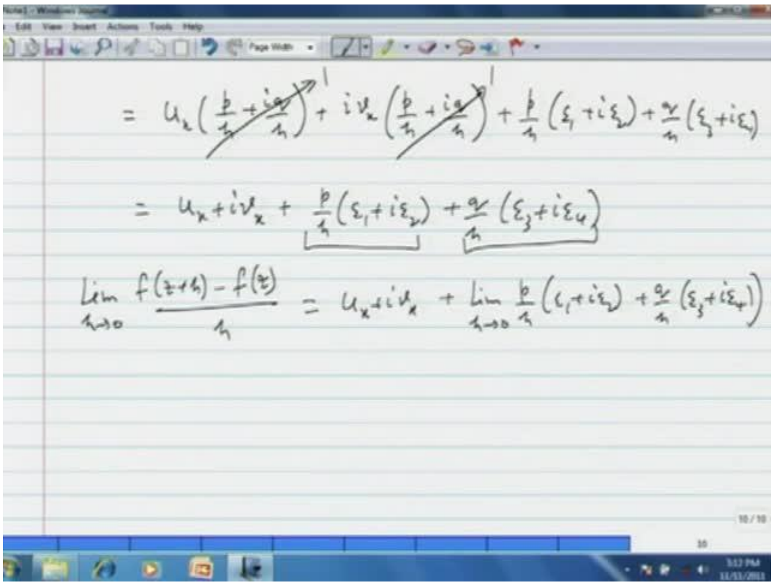
(Refer Slide Time: 34:35)



I will substitute this into this quotient. So, this quotient is equal to p by h times u x, let me drop the x comma y because the same thing appears all over. So, let me just write u x plus i v x plus q by h times u y is now minus v x plus i v v i is now u x. So, then this is plus p by h times epsilon 1 plus i epsilon 2 plus q by h times epsilon 3 plus i epsilon 4, this are the error terms.

Now, I will put this terms together so this is p by or u x times p by h plus i times q by h and then I will factor out i times v x and then I get p by h and then minus 1 can be written as i times i. So, this is plus i times q by h. So, I am writing that minus 1 as i times i then there are error terms p by h epsilon 1 plus i epsilon 2 plus q by h times epsilon 3 plus i epsilon 4.

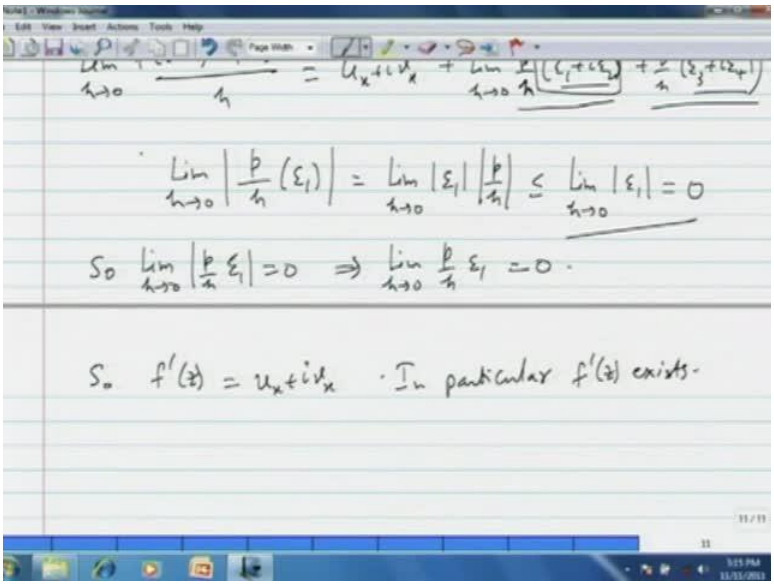
(Refer Slide Time: 36:26)



Now, when I take the limit as h approaches 0 p plus i q is h. So, I can cancel h is not equal to 0. So, this and this are both 1 so I get this is equal to u x plus i v x. So, assuming h is not 0, I am cancelling right away and then I get plus p by h, the other error terms epsilon 1 plus i epsilon 2 plus q by h times epsilon 3 plus i epsilon 4. Now, notice that when I take the limit as h goes to 0 of f of z plus h minus f of z by h. I am looking at the limits of these two guys as h approaches 0.

So, what I get is this is u x plus i v x at the point x y plus the limit as h approaches 0 p by h times epsilon 1 plus i epsilon 2 plus q by h times epsilon 3 plus i epsilon 4. So, if it were for just one of this limits without the support of epsilon 1 plus i epsilon 2 then in an earlier example in last section, we actually saw that we can be in trouble, we just if we just take a real part of h divided by h then we are in trouble, but it is really it is epsilons which help us to resolve the troubles.

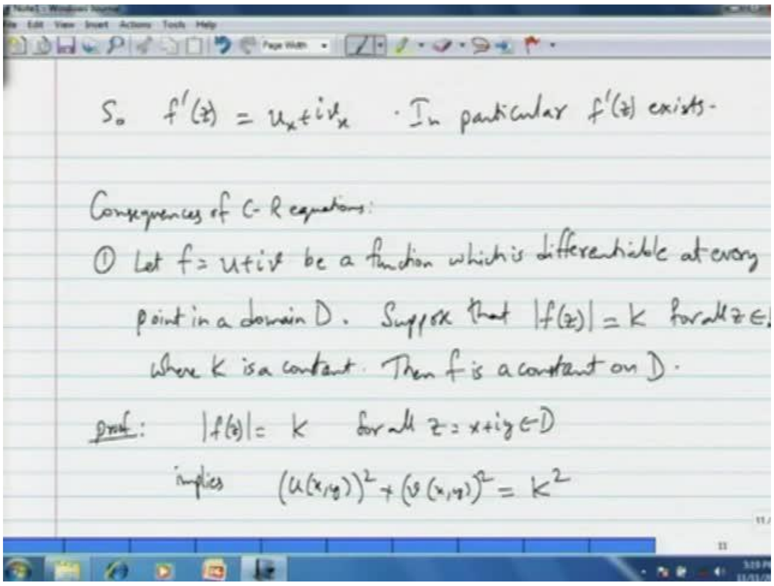
(Refer Slide Time: 38:19)



So, what I meant to say is that let us concentrate and on one thing. Let us look at the limit as h goes to 0 of the modules of the quantity p by h times epsilon 1. So, the limit h approaches 0 of p by h itself does not exist, but when we have epsilon 1 we can do more this is a limit as h goes to 0 of the modules of a epsilon, 1 times the modules of p by h, which I can definitely say is the limit as is less than or equal to limit as h approaches to 0 of the modules of epsilon 1 because the modules of the p by h is at most one in the bowl of radius r around z.

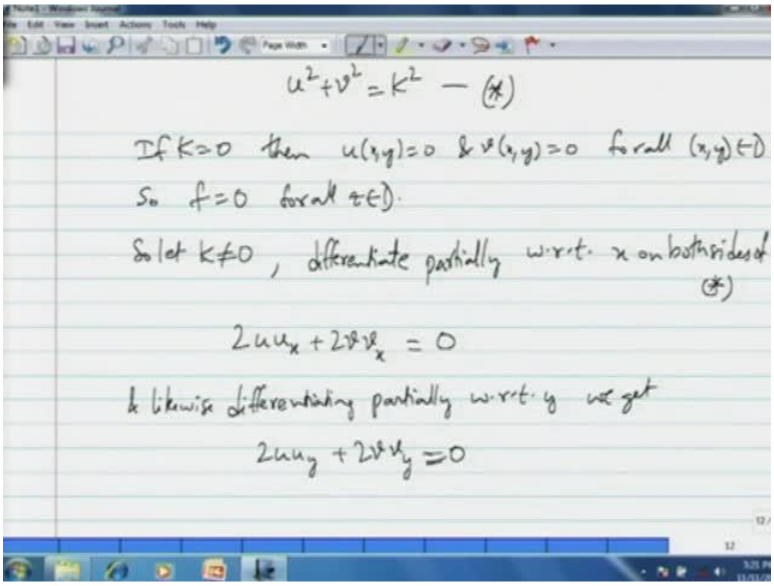
So, this is this we know is 0 by continuity of the partial derivatives. So, this is where the continuity of the partial derivatives is playing a role and then likewise, the other limits p by h times i epsilon 2 and other two limits, they also are equal to 0 by the way. So, I should conclude that limit as h goes to 0. So, the limit as h goes to 0 the modules of p by h times epsilon 1 is equal to 0, which implies the limit as h approaches 0 of epsilon where p by h times epsilon 1 is equal to 0. So, we conclude that f prime z is equal to u x plus i v x. In particular f time of z exist. So, continuity of these partial derivatives ensures that f is differentiable at the point in addition to Cauchy Riemann equations of course.

(Refer Slide Time: 40:28)



So, here is first one let f equals u plus i v b at function, which is differentiable on a domain d or let me say differentiable at every point in a domain D. Now, suppose for all suppose that for all z in d the modules of f of z is a constant. Where k is a constant, then we can conclude that, then f is a constant on D. So, we will see why so the modules of f is constant. So, let me just say proof of this fact the modules is constant for all x plus i y in d means that, implies that u square plus v square, the square of the modules u square by u squared. I mean is the function u of x y square plus v of x y square is k square. The modules square it is also constant.

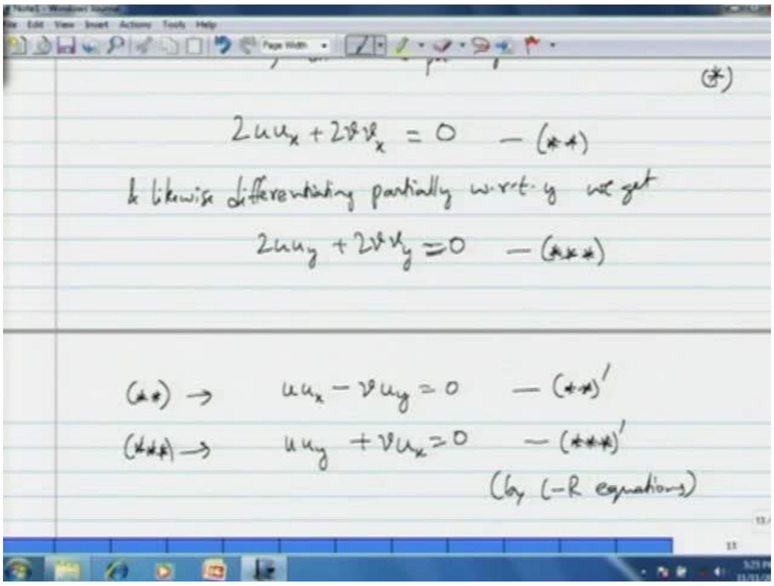
(Refer Slide Time: 42:50)



So, let me simply write it as a u square plus v square suppressing, the x comma y is k square. So, the left hand side is the function, where as the right hand side is the constant. So, let us rule out the simple case first. Suppose if k is equal to 0 then the sum of squares of a 2 numbers is 0. So, then u of x y is equal to 0 and v of x y is equal to 0 for all x comma y in d, what that implies is so f is already. So, if is actually equal to 0 that we can infer directly from here the modules is constantly 0 means, that f is itself 0 for all z belong to d.

So, assume that so let k naught be 0, this will be useful in doing the following. So, then we can look at this equation 1. So, or we call this star so differentiate partially with respect to x on both sides of star. So, you get 2 u u x plus 2 v v x is equal to 0 right hand side is the constant so its differentiation is 0. Likewise differentiating partially with respect to y or differentiating partially with respect to y, we get 2 u u y plus 2 v v y is equal to 0.

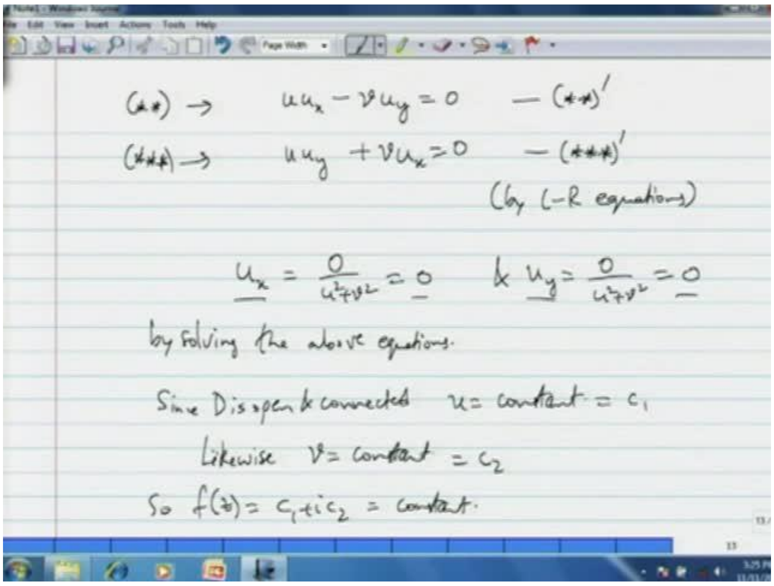
(Refer Slide Time: 45:22)



Note that these are this equation, let me give their names this is star, star and this is star star, star. Note that this equations hold at every point at x comma y in domain D. Since f is differentiable at every point in the domain D, we can we can use the Cauchy-Riemann equations, which now this u and v satisfy. So, let us write everything in terms of u x and u y.

So, from star, star I get the following I get u u x plus v v x is minus u y. So, this is minus v u y is equal to 0. So, let me call this star, star prime and then star, star, star implies u u y plus v y is u x. So, I will write that as v u x is equal to 0, star, star, star prime by the Cauchy-Riemann equations. So, I am using the Cauchy-Riemann equations. So, now we can treat this as equations in linear equations, in u x and u y and try to solve for u x and u y.

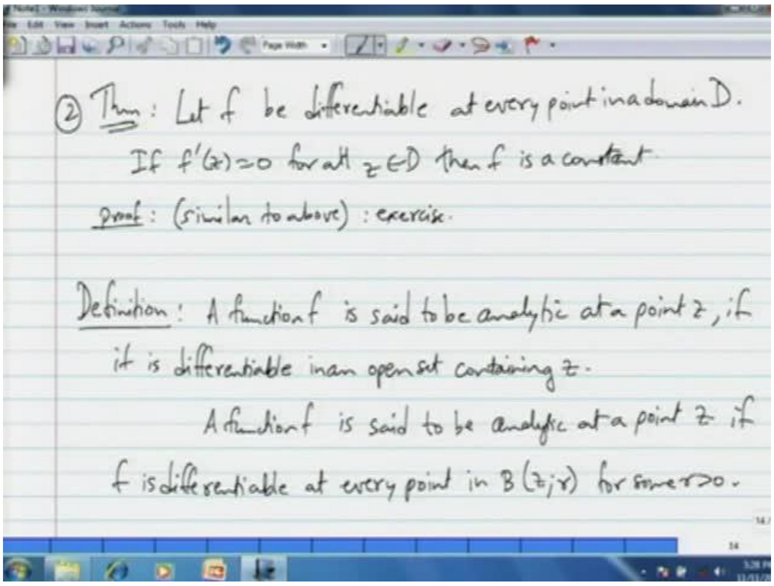
(Refer Slide Time: 46:52)



So, we get so here the co-efficient are u an v so we get that u x is solving that way we get u x is 0 by u square plus v square. We can assume v square plus v square is not 0. So, we can divide because k is not 0 so u x is 0 and likewise u y is 0 by u square plus v square, which is also 0 by solving the above equations. So, since d is open and connected, these domain are open and connected.

We conclude that u is constant if the, if both the partial derivatives are 0 at every point in an open connected set, then by theorem and multi variable calculus, we can conclude that the function u is constant. Likewise using the Cauchy-Riemann equation, in another fashion we conclude that v is constant. So, this is let say c 1, this is c 2. So, f of z is c 1 plus i c 2 which is constant. So, this is one conclusion in Cauchy-Riemann equations.

(Refer Slide Time: 48:39)



And then yet another in a similar fashion, we can also conclude the following. Let f be differentiable at every point in a domain D. So, it is an open connected set d, if f prime of z is equal to 0, for all z in d then f is likewise a constant. And the proof of this is similar to the first consequence. So, since I this is the second consequence of the Cauchy-Riemann equations. So, proof is similar to above try to do this as an exercise, the viewer is asked to this an exercise.

And in both this consequences we know that the hypothesis is that, the domain d is an open connected set and that is important and we already see that, the openness of the domain is important in more than one fashion and so differentiability along with openness will make the following definition or differentiability. In an open set is will make for the following definition, a function f is said to be analytic at a point z, if it is differentiable in an open set containing z.

So, this can be said in an another fashion, a function is said to be analytic at a point z if it is differential be in a neighborhood of z. So, let me also note that so a function f is said to be there is an alternate definition said to be analytic at a point z, if f is differentiable at every point in B z r for some r positive. And its seems like I had given two different definitions, but they are one at the same because if you have an open set containing z, where f is differentiable there is a bowl in it of positive radius, which contains z and vice-versa

So, the equality of these two definitions can be easily seen from the definition of a open set. So, we will see that the analytic functions are functions of interest differentiability along with differentiability, in a neighborhood is an important for studying various properties. And these functions the class of analytic functions will be off interest, when it comes to complex analysis. So, the next the next few lectures or in the rest of the lectures, we will concentrate on analytic functions and their properties.