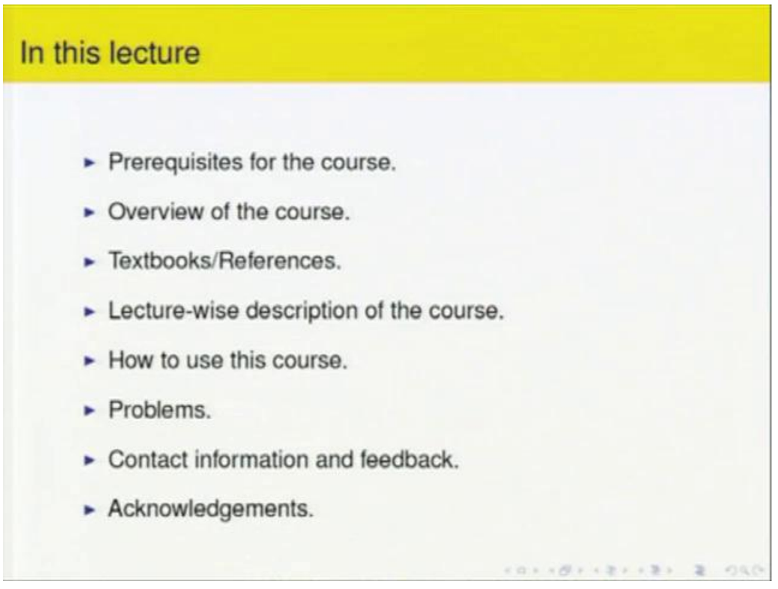
Complex Analysis

Prof. P. A. S. Sree Krishna
Department of Mathematics
Indian Institute of Technology, Guwahati

Introduction

Hello viewers and welcome to this video course on complex analysis. My name is Sree Krishna Palaparthi also P.A.S. Sree Krishna and I will be the instructor for this course. This course is a part of the national program on technology enhanced learning, the NPTEL project sponsored by the ministry of human resources development government of India. In this video or in this session, I will be talking about various aspects of this course, and I will try to put the course in context.

(Refer Slide Time: 00:58)

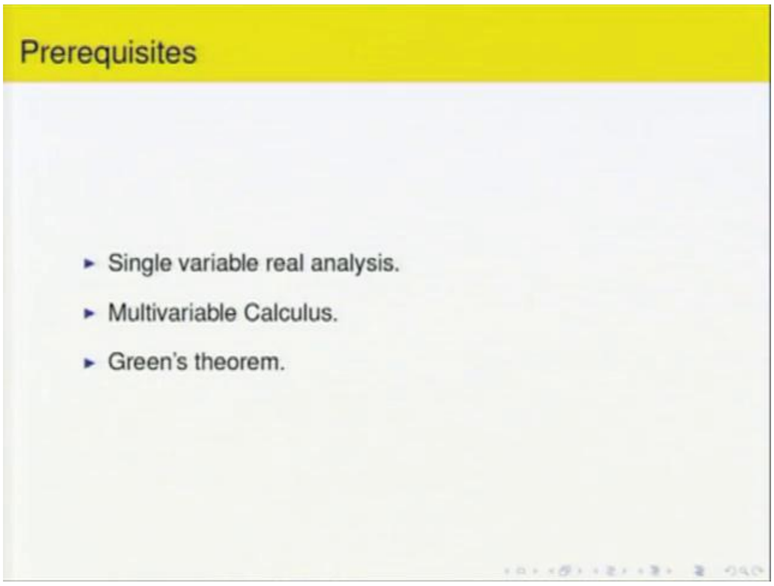


So, the various things that I will be talking about in this lecture are the prerequisites for this course, what you need to get started with the course, and then I will talk about the overview of the course. And then I will give some textbooks and references, which can be followed and then I will give a lecture wise description of the course, and I will inform how to use this course at two levels, I will say what that is elaborately. And I will also talk about problems I will say word about problems, and then I will give my contact information and solicit feedback, and finally I will end this lecture acknowledgments. So, feel free to skip a part of this video or the whole of this video if you feel appropriate.

(Refer Slide Time: 01:59)

So, I will begin with the prerequisites for this course.

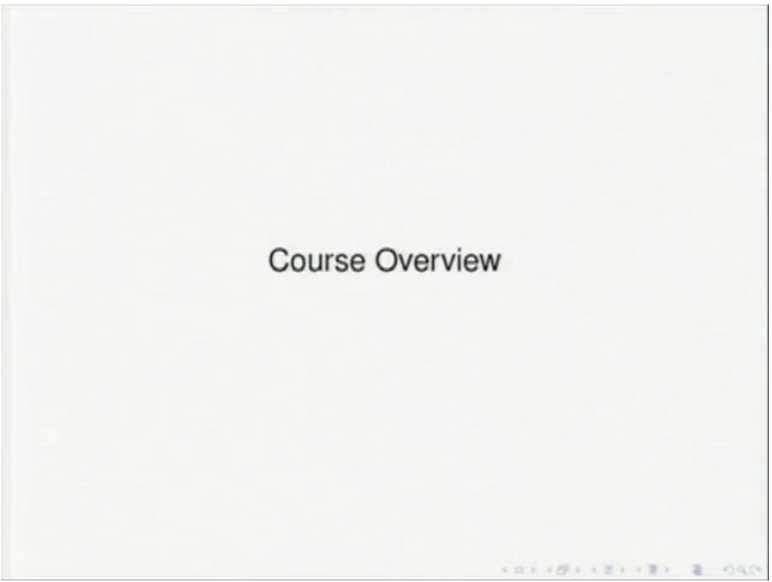
(Refer Slide Time: 02:05)



So, the prerequisites for this course are single variable real analysis. So, an introductory or course in real analysis functions of one real variable is essential, in order to follow the material in this course. So, the concepts of limits, continuity, differentiability, and integrability as applied to functions from subsets of real numbers to real numbers should be familiar to the viewer. One should also know or know and understand theorems like the intermediate value of theorem, the mean value of theorem Taylor's theorem, the fundamental theorem of calculus and such, which are taught in a first course on real analysis. And that is not all to follow some parts of this course, you will also need a first course in multivariable calculus.

So, the viewer should be familiar with the concepts of limits, continuity and differentiability as applied to functions from subsets of r to r 2 subsets of r 2 to r subsets of r 2 to r 2 at least. And it will definitely help the viewer to be familiar with the concepts of the line integral, surface integral, etcetera which are taught in a first course on multivariable calculus. In particular it will be useful for the viewer to understand Green's theorem and over and above these things, what one cannot quantify and what the viewer is expected to have is a certain mathematical maturity. If you have seen enough proofs in a first course on real analysis, then you can consider yourself to be well prepared for this course. And with these prerequisites, one can one can go over these lectures.

(Refer Slide Time: 04:33)



And now I will talk about the overview of this course. So, this is an introductory course in complex analysis. So, we will start with the algebraic properties of complex numbers. So, we will first introduce the complex numbers system itself and we will start with the algebraic properties of complex numbers. Then we will study the geometry of the complex plane. So, we will look at the geometry of addition complex, addition complex multiplication and then we study, the topology of the complex plane in some detail.

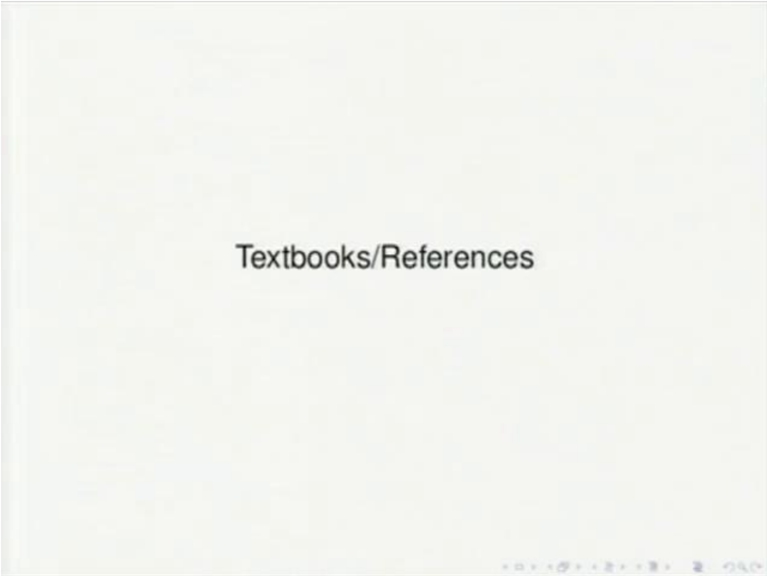
So, since topology is integral to the study of complex analytic functions, we will look into topology in some detail. And three lectures are allocated to the study of topology of the complex plane. And after a studying the topology of the complex plane, we introduce the viewer to the concept of a complex function, the complex valued function of a complex variable. And then we go on to introduce the concepts of limits of a complex function as the variable approaches, a complex number and the concept of continuity of a complex function, and the concept of a differentiability of complex function. And we will then introduce analytic functions, which are central to the study, central objects of study in complex analysis.

So, we will give enough examples of analytic functions, and then we go on to prove various versions of Cauchy's theorem, which one can consider to be the central theorem in complex analysis. So, we will prove various versions of this Cauchy's theorem, and we will derive the Cauchy's integral formula for function for a analytic function. And then we will also derive the Cauchy's integral formulae for higher derivatives of analytic functions.

So, in particular we will prove that an analytic function is differentiable, any number of times while deriving the Cauchy's integral formulae. And then we also prove some consequences of the Cauchy's integral formula namely, the Liouville's theorem and the fundamental theorem of algebra. And we also prove the converse to the Cauchy’s theorem namely, Morera’s theorem. Further in order to explore the properties of analytic functions, we introduce power series and then we prove the open mapping theorem and as a consequence, we prove the maximum modulus theorem. And then we take a detour into the study of Mobius transformations.

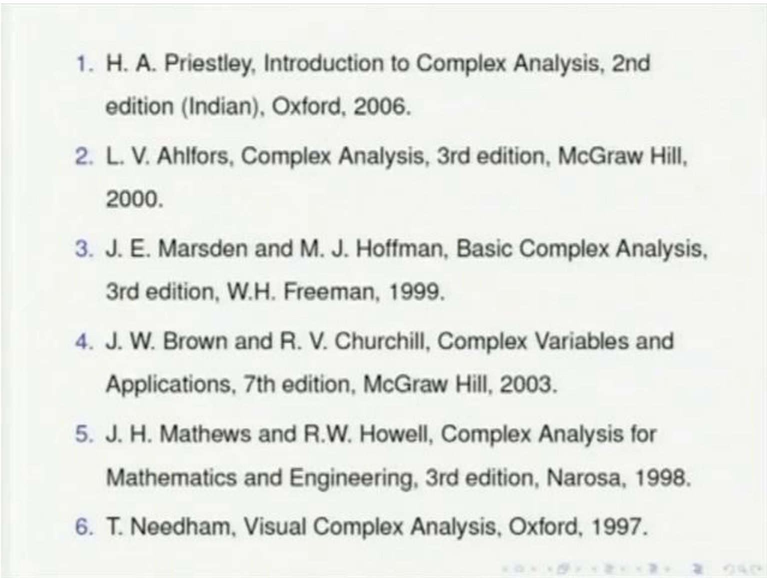
So, the Mobius transformations are an important class of functions, which are an important class of examples for conformal mappings. And we study the properties of Mobius transformations briefly, and then we go on to study isolated singularities of analytic functions. So, we classify the isolated singularities and prove some results about the behaviour of an analytic function, in the neighbourhood of different kinds of isolated singularities. So, we prove the Laurent's theorem and use it to prove the Cauchy’s residue theorem and finally, we end the course with some applications of the Cauchy’s residue theorem. So, that is an overview of this course.

(Refer Slide Time: 08:46)



Next, I will talk about the text books or references for this course.

(Refer Slide Time: 08:52)

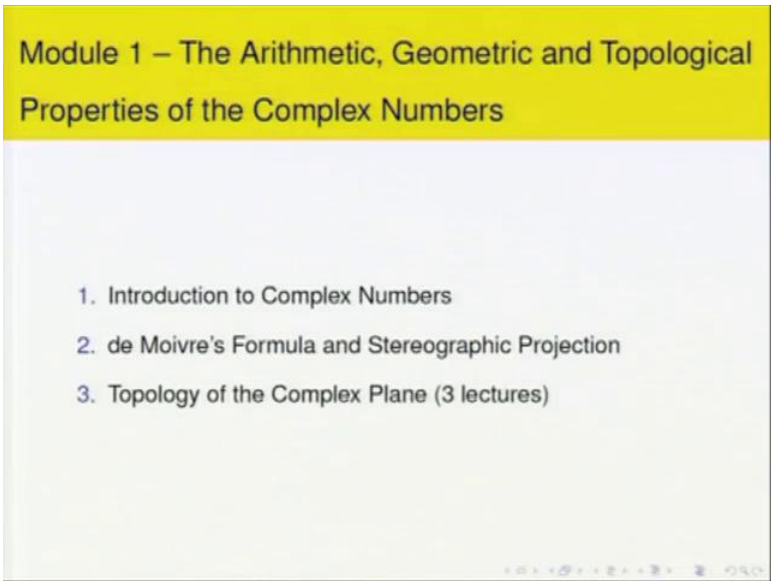


So, the viewer is strongly advised to take notes, while listening to the lectures giving the lectures and then supplement the understanding with reading, one of these textbooks listed here. So, the textbooks or references for this course are introduction to complex analysis by Priestley. And then complex analysis by Ahlfors, which is standard textbook and then one will notice that, the material which is presented in this course base a lot of similarity with Priestley, Priestley's text book and some similarity with Ahlfors text book.

And then there are other references for this course. So, the basic complex analysis by Marsden and Hoffman. Complex variables and applications by Brown and Churchill, which is a widely accepted textbook for a course at this level. And complex analysis for mathematics and engineering by Mathews and Howell, which is also a very popular book. And the references 3, 4, and 5 present the subject material in their own style and the viewer can pick anyone of 1, 2, 3, 4 or 5 according to his or her taste. And then there is this reference six visual complex analysis by Needham, which is a wonderful book which gives a visual feel to all the results, which are discussed in this course.

Let me inform the viewer that this text book six takes a completely different approach, then the one taken in this course, but it gives you a visual feel of all the results and is a pleasure to read. So, that is also listed as a reference here. Next, I will give a lecture wise description of the course module wise and lecture wise description of the course. So, the viewer can feel free to skip this particular part of the video if one likes.

(Refer Slide Time: 11:14)



So, let me begin with module 1. So, module 1 is about the arithmetic geometric and topological properties of complex numbers. So, in module 1, lecture 1 we introduce complex numbers perhaps, this is already familiar to many of the viewers and then we introduce complex number arithmetic, which is once again fairly easy. And then we introduce the conjugation of a complex number, and its properties. And then we introduce the complex plane and how to represent points from the complex plane.

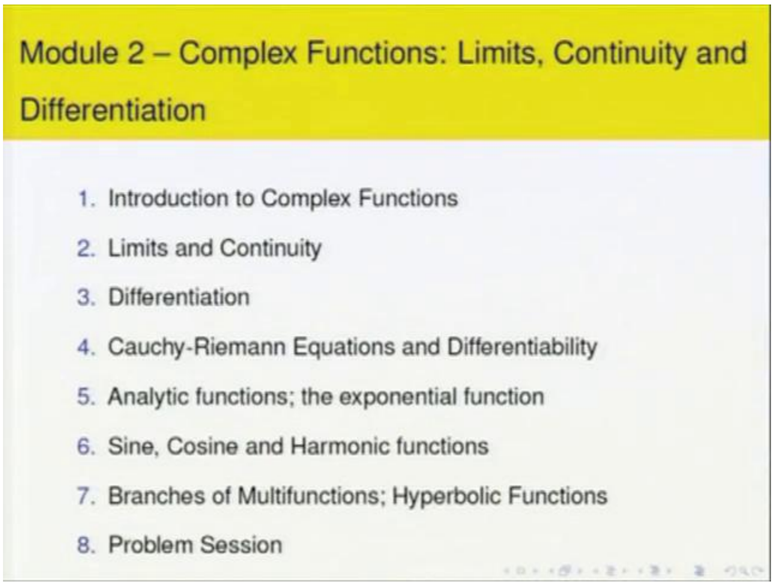
And we introduce the concepts of modulus and argument of a complex number and discuss their properties. So, this is a very introductory lecture and is in place here only for completeness. So, it is possible that many of the viewers are already familiar, with this material. If you are already familiar with this material feel free to skip, this particular lecture.

And then in the next lecture module 1, lecture 2 we continue with the triangle inequality which is one of the properties which is related to modulus. And then we talk about the geometry of addition and multiplication, in the complex plane. And then we introduce the De Moivre's formula for powers of integral powers of complex numbers, and then we through examples we demonstrate how to find the n eth roots of complex numbers, all the n eth roots of complex numbers, where n is a positive integer. And finally, we introduce the stereographic projection and introduce the point at infinity.

So, we give the concept of the complex plane union the point at infinity and such a construct is used at various points in the course, in particular we call this the Riemann’s sphere and we use it to study the Mobius transformations. So, that is module 2, lecture 2. module 1, lecture 2 sorry and then in module 1, lecture 3,4 and 5 we discuss the topology of the complex plane. So, as I already mentioned topology is integral to understanding the analysis of complex functions. So, topology is covered in three lectures.

So, what we discuss in these three lectures is the are concepts of open sets, closed sets limit points, and closure of a set, the interior exterior and boundary of a set, and we also discuss compact sets, connected sets and their properties and we also discuss complex sequences and there convergence. And that completes module 1 and prepares the ground for the study of complex functions.

(Refer Slide Time: 14:37)



We start module 2 with complex functions. So, module 2 is all about complex functions their limits continuity and differentiations. So, we start module 2 with introduction to complex functions. Once again module 2, lecture 1 is fairly easy material. So, it is quite possible that many viewers out there are familiar with, what is discussed in module 2 lecture 1. So, once again if you feel comfortable, you can skip module 2, lecture 1, but we will in module 2, lecture 2 we will start with the limits the concept of a limit of a complex function, as the variable approaches a complex number, we discuss the limits and then we also discuss continuity of complex functions. And then in this lecture we will also discuss limits involving in infinity and then we will discuss, what continuity of a function on a domain will mean.

And then in module 2, lecture 3 we introduce complex differentiation we define the derivative of a complex function when it exists. And we reconcile the definition of derivative of a complex function at a point with the definition of the derivative of a function of two real variables, which is a vector value in r 2. So, if that is if you take a function from subsets of r 2 to r 2, then it bears some likeness to functions from subsets of complex plane to complex plane, and then we reconcile the two using the Cauchy-Riemann equations.

So, the Cauchy- Riemann equations give a condition on a function from subset of r 2 to r 2 necessary condition for it to be complex differentiable, to be considered complex function and for it to be complex differentiable. And then we also give a geometric interpretation of the derivative in this lecture. And we also describe what conformality is and then say that these functions, which are differentiable and if the derivatives non zero then they are conformal. And in module 2 lecture 2 we talk about the Cauchy-Riemann equations, which are in necessary condition as I mentioned for functions from subsets of r 2 to r 2 to be complex differentiable as complex functions, when viewed as complex functions. Then we give sufficient conditions for such a function to be complex differentiable.

So, it turns out that as the Cauchy-Riemann equations are not sufficient for a function from a subset of r 2 to r 2 to be complex differentiable. So, in addition if it satisfies the condition that its partial derivatives are continuous, then when it is viewed it as complex function it is complex differentiable. So, we prove this proposition in module 2, lecture 4 and we give some examples, in this direction and in module 2, lecture 5 we introduce the important class of functions namely analytic functions.

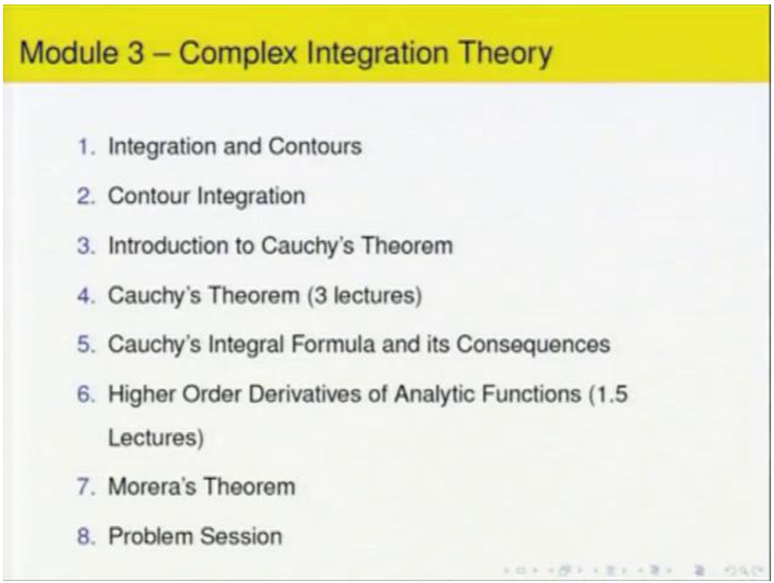
So, analytic functions are complex differentiable functions, in a neighbourhood of a point of course, differentiable at the point itself and also differentiable in a neighbourhood of that point. So, this class of this openness of the domain of differentiability is responsible for many important results, which we will prove in this course. We introduce the exponential function, the complex exponential function to the likeness to the real exponential function and we give some properties in particular the mapping properties of exponential function.

And then we talk about the sine, the complex sine function, the complex cosine function and then we describe some of their properties here and we then introduce the harmonic functions, in this lecture 6 in module 2 and in particular, we give couple of examples of how to find harmonic conjugates of a harmonic functions over a domain. In module 2, lecture 7 we introduce what we call as multi functions, we will see that argument is a multifunction on the complex plane minus the origin.

And we will introduce the logarithm multifunction, and we will see how to construct branches of analytic branches of these multi functions. So, we will see how to restrict the domains of these multi functions. Such that we can we can think of them as analytic functions on that restricted domain, single valued analytic functions by making appropriate choice choices.

Then we will describe analytic branches of the n-th root function of complex number and we will also describe the analytic branches of z power alpha, where z is a complex variable and alpha is a fixed complex number other than integers. So, we will then go on to discuss the complex hyperbolic functions, and their properties. So, we will end module 2 with a problem session, so after having introduced complex functions and the concepts of limits continuity and differentiation.

(Refer Slide Time: 21:12)



We introduce integration, complex integration along contours and we will use the complex integration theory to study properties of analytic functions. So, it turns out that integration, complex integration gives out many properties of analytic functions. So, we begin the complex integration theory, with how to integrate a complex valued function of a real variable and then we introduce contours in the complex plane. So, these are gadgets on which we will integrate complex functions, and then we define contour integration.

We give some properties of contour integration and then, we prove the complex version of the fundamental theorem of calculus. In lecture 3 in module 3 we give an introduction to a Cauchy's theorem, one can say that Cauchy's theorem is central to all of the complex analysis, and we will in this lecture module 3, lecture 3 we will prepare the ground for Cauchy’s theorem.

So, what we will do is we will prove couple of estimation theorems. Estimation of a contour integral, we will give a proof of a preliminary version of Cauchy’s theorem based on Green’s theorem. So, by adding additional hypothesis one can see that Cauchy’s theorem is Green’s theorem in disguise. So, we will do that in module 3, lecture 3 and in module 3 lecture 4. 4, 5 and 6 we will cover three different versions of Cauchy’s theorem.

So, in module 3 lecture 4 we will cover the Cauchy’s theorem for a rectangle, this is the Cauchy-Goursat theorem for rectangle, where we assume that the function is analytic on a inside a rectangle and we show that the integration of analytic function on the boundary of the rectangle is 0. And that we will do in module 3, lecture 4 in module 3, lecture 5 we will prove the anti derivative theorem and prove the disc version of Cauchy’s theorem.

We will also give the concept of the index of a closed curve, around a point in this lecture and in module 3, lecture 6 we will prove yet another version of Cauchy’s theorem known as deformation theorem, and this is the version of the Cauchy’s theorem that we are going to put to use a lot for the rest of the course. A word is an order here about module 3, lecture 6 and the proof is fairly technical and if one wants to believe the result and skip the proof one can do so without loss of continuity.

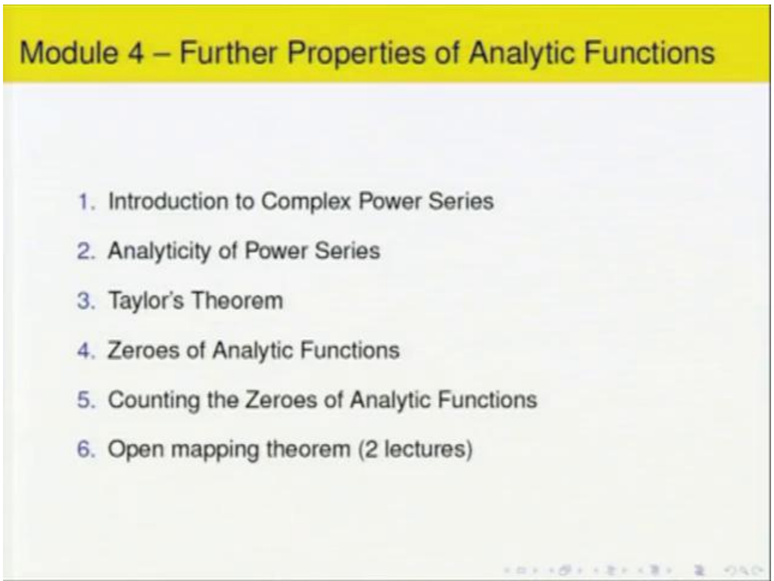
So, if you are a beginning student in a complex analysis, then you can feel free to skip module 3, lecture 6 and take the results from that lecture. Module 3, lecture 7 derives Cauchy’s formula and covers some of its consequences, in particular we prove the Liouville's theorem and the fundamental theorem of algebra. So, Liouville's theorem is surprising, it asserts that if you have a complex differentiable function on all of the complex plane, and if it is bounded then it has to be the constant function.

So, if you look at the real function sine x it is bounded and it is differentiable on all of the real line, but such a phenomenon cannot occur for complex analytic functions. So, if a complex analytic function is analytic on all of the complex plane, and if it is bounded in its modulus then it has to be a constant function, that is Liouville's theorem which is a surprising property and that we prove in module 3 lecture 7.

And we also prove the fundamental theorem of algebra, which tells that if you consider a complex polynomial that is a polynomial with complex coefficients, then it has to have a root in the complex numbers. What that means is it can be decomposed into product of linear factors both the complex field. So, it is a fundamental theorem of the algebra, and in module 3, lecture 8 and 9, we will derive the Cauchy integral formulae for higher order derivatives of analytic functions.

And we do that stepwise, we first give formulae for the first derivative of an analytic function and then for the second derivative and finally, generalise it to give and Cauchy’s integral formula for the n-th derivative of an analytic function. So, that also demonstrates that an analytic function can be differentiated any number of times. So, if a, if a complex function is analytic that is it is differentiable in a neighbourhood of a point at which it is differentiable, then it is differentiable in many number of times, which definitely is in star contrast with functions of real variables, which can be a differentiable n number of times, but not the n plus oneth time, there are examples of such a real functions. Then in module 3, lecture 9 we also prove the Morera's theorem, which is the converse to Cauchy’s theorem. We end module 3 with a problem session.

(Refer Slide Time: 27:42)



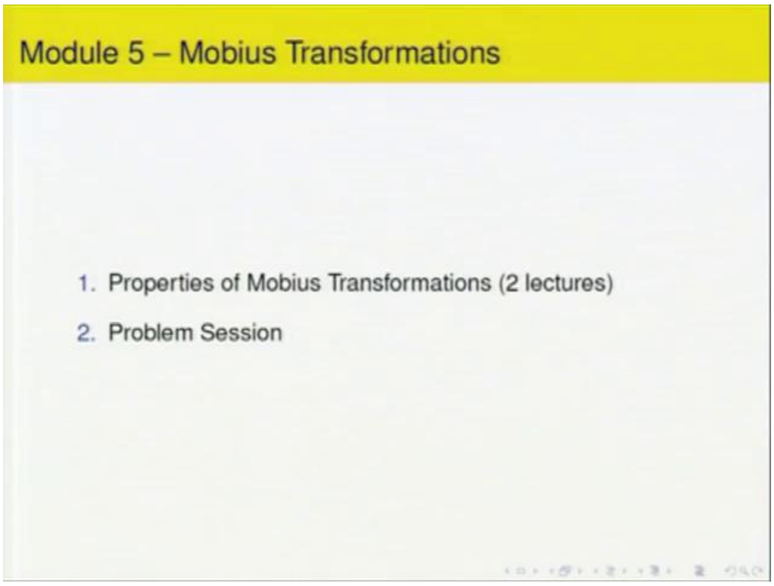
Module 4 is about further properties of analytic functions. So, we explore the properties of analytic functions further by using a complex power series. So, we begin module 4 with introduction to complex power series, and then we introduce these power series as examples of analytical functions. So, we prove the analyticity of power series in module 4, lecture 2, we in particular give a way to find the derivative of power series that is by differentiating them term by term. In module 4, lecture 3 we prove the Taylor's theorem for an analytic function, which asserts that at a point of analyticity, the analytic function has a power series expansion locally.

So, which is very important in order to prove some local theorems, and in module 4, lecture 4 we use Taylor's theorem to analyse the zeroes of an analytic function in particular, we show that the zeroes of a analytic function, which is not identically 0 itself are isolated. What that means is if you find a 0 of an analytic function in a neighbourhood, if you can find a neighbourhood of that point, where the analytic function is non 0, provided the function itself is not identically 0.

So, the Taylor's theorem allows us to make such conclusions and then we prove a theorem on how to count zeroes of an analytic function, inside a simple closed curve and then we prove Rouche's theorem in particular. Then we also use the Taylor's theorem to prove the open mapping theorem, which asserts that a complex analytic function is an open map, that is a it takes open sets to open sets. So, this is this shows the topological nature of these analytic functions.

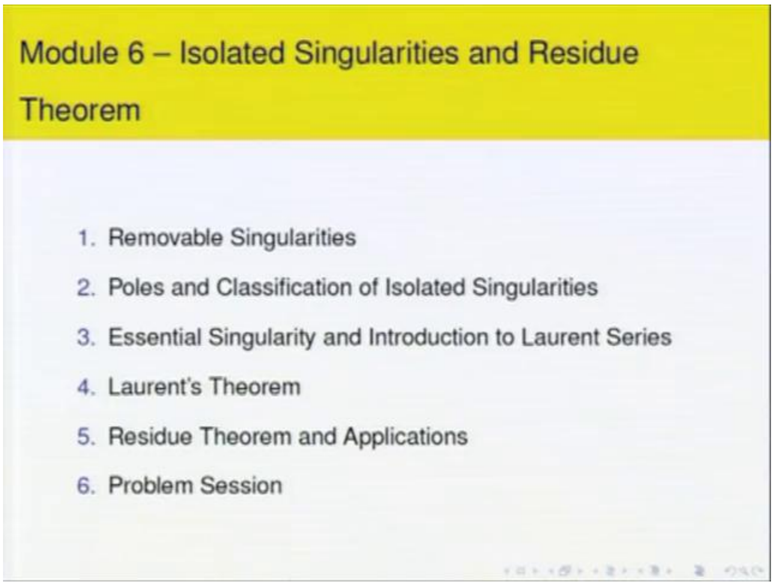
So, if the function is further one to one then it is a topological mapping. What that means it is one to one and on to its domain and then on to its range sorry, and it is also an open map. So, that is proved as a consequence of the open mapping theorem and we also give the maximum modulus theorem as a consequence of the open mapping theorem. So, this is a very important picture of the complex analytic functions that it, that they are open mappings of course, all this applies only to non constant analytic functions. So, if the function is itself a constant then definitely it is not an open map. So, we show the open mapping theorem over 2 lectures in at the end of module 4.

(Refer Slide Time: 31:09)



And in module 5, we take a detour into Mobius transformations. So, Mobius transformations are examples are important class of a examples, of conformal mappings. And we discuss the properties of a Mobius transformations over couple of lectures and end module 5 with a problem session.

(Refer Slide Time: 31:33)



And then we comeback to analytic functions, in general and then we discuss isolated singularities of analytic functions in module 6. In module 6, lecture 1 we define an isolated singularity and define a removable singularity, which is one of these isolated singularities and in module 6, lecture 2 we define what is called a pole for a complex analytic function. And then we classify the isolated singularities as the removable singularities poles or essential singularities. Then we define essential singularity based on the calcification actually.

Then we discuss the behaviour of an analytic function in a neighbourhood of essential singularity, in particular we prove the Casorati Weierstrass theorem, in module 6, lecture 3 and we give a brief introduction to Laurent's series for a function near a pole in module 6, lecture 3. In module 6, lecture 4 we introduce the viewer to double ended power series and we prove the Laurent's theorem and we use the Laurent's theorem to prove the Cauchy's residue theorem.

In module 6, lecture 5 and we end the course with a applications of residue theorem, in particular we will see the argument principle, and then we will apply the Cauchy's residue theorem to evaluate a certain real definite integral, in a real improper definite integral and we end module 6 with a problem session. So, that is an overview of what is covered in this course so, it is lecture-wise description of what is done in this course.

Next, I will talk about how to use this course I will basically classify, the target audience broadly into two classes, the a beginning undergraduate student in a complex analysis and advanced student in complex analysis. So, a beginning student can omit module 3, lecture 6 and take the results from module 3, lecture 6 and skip proofs of higher derivative formulae module 3, lecture 8 and 9 and also can skip the open mapping theorem. The proof of open mapping theorem which is covered in module 4, lecture 6 and 7.

So, a beginning student can follow the rest of the course by these, by following these omissions without loss of a much loss of continuity. An advanced student can definitely skip module 1, lecture 1 and module 2, lecture 1 which are introductions to complex numbers and complex functions respectively. So, an advanced student will definitely be prepared with these things. The advanced student can also skip module 4, lecture 1 if one is already familiar with complex power series. So, rest of the course can be followed well by an advanced student. So, that is rough classification of target audience and one can take first path or the second path in order to follow this course.

(Refer Slide Time: 35:46)

Next, I will discuss problems and problem solving sessions. Solving problems is integral to understanding the subject matter. So, one cannot overemphasize the importance of solving problems, in order to understand the subject matter. So, I have covered some problems at the end of some modules in problem solving sessions. So, try to pause after each problem is posed, and try to solve it yourself and also try to solve a various problems from the textbook. Solve as many problems as it takes to gain confidence in the subject matter. So, as I mentioned, I will mention once again that solving problems is integral to understanding the subject matter. So, it is a word about problems and problem solving sessions.

(Refer Slide Time: 36:51)

Next I will give my contact information, and I will solicit feedback.

(Refer Slide Time: 36:57)

So, my email is passkrishna@iitg.ernet.in. So, if you need to contact me feel free to email me I request the viewers to be considerate, when sending emails. I also request you to be patient in expecting a reply and I solicit a feedback. So, definitely constructive feedback is appreciated, and if you feel something needs to be commented on about the scores then please email me and I will be happy to respond, I will end this video with acknowledgements.

I first want to thank Doctor K V Krishna my colleague, who has initiated me into this NPTEL project and constantly motivated me to complete this project in bringing it to this state and I want to thank my recording staff Bikash and Dhruv, Jyothi, Shiv, Jyothi who have been cooperative and I want to thank my editor Ananth Sharma, who has been very patient with me while the video editing was going on, and I also want to thank the centre or education technology at IIT, Guwahati for providing all the infrastructure for this project and last, but not the least I want to thank my family, my wife and my son for being patient and very cooperative, while I was completing this course. With these acknowledgements I want to stop this video here.

Thank you.