Matrix Computation and its applications
Dr. Vivek Aggarwal
Prof. Mani Mehra
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture - 07
Properties of subspaces

Hello viewers. So, welcome back to the course on Matrix Computation and its Application. So, in the previous lecture we have started with that the summation of the subspaces and we are continue with this one in this lecture. So, this is a lecture number 7.

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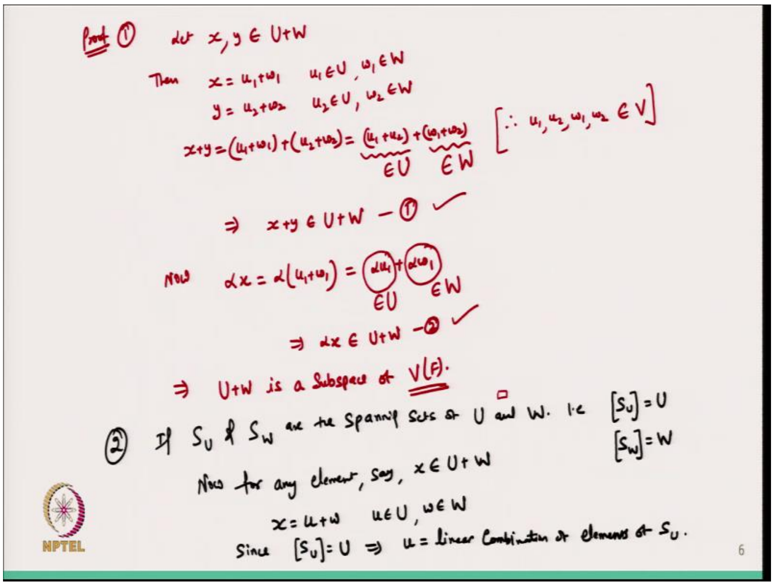

So, in the previous lecture we have shown that if I have a two subspaces U and W then their sum is also subspaces of the vector space V.

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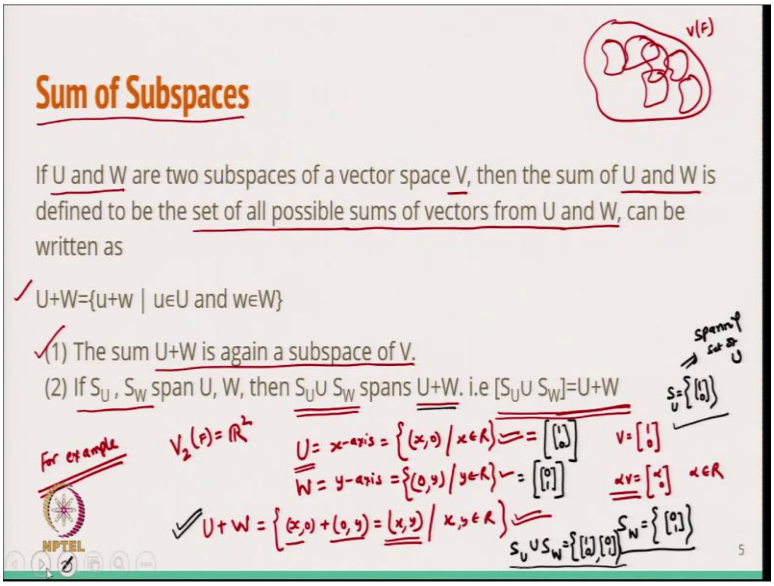

And also we know that if we are taking the spanning set of U and spanning set of W then their union will be the spanning set for U plus W. So, this is we are going to discuss. Now just I want to show you that what is the meaning of spanning set of U. You see that this is the I am taking one subspace that is U and from here I can say that just I take one vector V as suppose I take one vector 1 0 and if I take all the linear combination of the vector V.

So, you will see from here that this vector. So, I can write alpha V. So, this will be equal to or maybe alpha or maybe. So, it will be alpha 0 and I can choose my alpha as a real line. And if you see from here all the elements is come will be same as this one because it is a just I am writing the vector in the column form, but this is the same.

So, from here you can say that I can write this as I can write this vector 1 0 and then I can make it the span. So, this is equal to this one and from here I can say that this vector as if I take S as containing only this vector 1 0 then this is the spanning set of U.

Similarly, I can write the spanning set of V as a taking one vector and then I am taking the span of this. So, that is I am getting the subspace W and in this case also I can say that the spanning set of this contain only one element this is 0 1. So, this is the way we can define the spanning set of a subspace.

So, it is written that if this and this spans U; it means my this is my S U and similarly I can define as W and that as W will contain this vector. So, it is a one vector it is another vector ok. Now, I want to show that what is going to happen if I taking the union. So, you can see from here I can have this vector and this vector and if I divide define S U union as W then it is going to have these two vectors together 1 0 and 0 1.

So, this is I can define as the union. Union means I am taking this vector and this vector together. So, our claim is that it will span U plus W and this one clearly you can check from here that this two vector is going to span U plus W from here; because if you take any element that can be written as a linear combination of this and so, from here you can check that this U plus W is spanned by this one ok.

So, the here we can verify this, but let us prove this one. So, this is ok, now we satisfy the next property. So, first is ok. Second one I want to show that if S U and S W are the. So, this is I am taking that the spanning set. So, these are the spanning sets of U and W ok.

Now, from here so, it; that means, S U span that is equal to U and S W that is equal to W ok. So, let us see. Now for any element say x belongs to U plus W I can write as x is equal to some u plus w, where u belongs to U and w belongs to W, which is again a subspace ok. So, this is I am taking element phenomena.

Now, from here I can write that since spanning set of U is this one that is U which implies that this u is equal to the linear combination of elements of S ok S U because u is belonging to U and that is a spanned by the set S U. So, definitely u is a linear combination of the element of S U.

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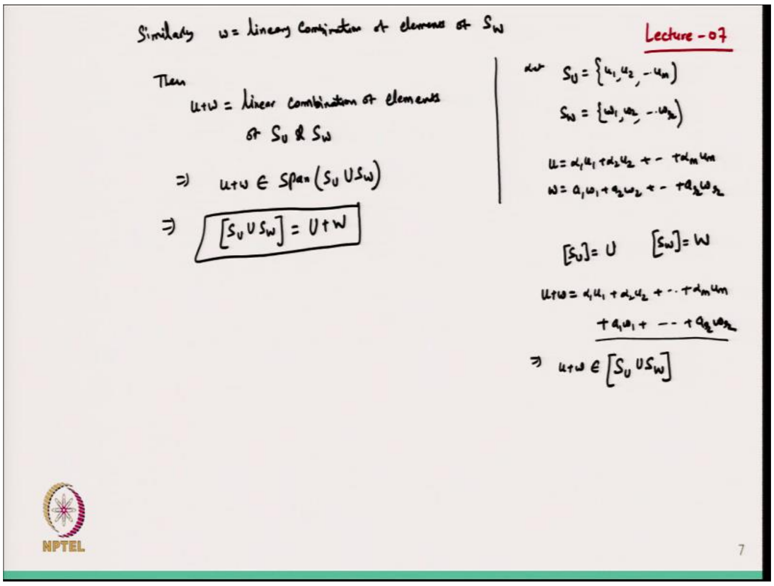

Similarly, w is the linear combination of elements of S W ok. So, maybe I can show from here that let S U contains may be u 1, u 2 up to u m and S W contains suppose w 1, w 2, w r. So, these are the sets that is contained in the spanning set of U and W.

Then I can from here I can write that the u is written as suppose it is written as alpha 1 u 1 plus alpha 2 u 2 alpha m u m and w can be written as some may be a 1 w 1 plus a 2 w 2 and a r w r ok and S U span is U and S W span is W. Now, if that is there then u plus w is also a linear combination is also linear combination of elements of S U and S W.

Because now if I write d u plus w this one can be written as alpha 1 u 1 alpha 2 u 2 alpha m u m plus a 1 w 1 up to a r w r, this one I can write ok. So, it means there is a linear combination of this one. So, it is a linear combination of the element of S U and S W.

So, from here you can check from here that u plus w can be written as that belongs to the span of U union span of W because u 1 u 2 u m that belongs to S U and w 1 w 2 w r that belongs to S W. So, if I take the union then their linear combination will be definitely from this one.

So, from here I can write that u plus w belongs to the span of S the U union S W ok and from here and this is true for all elements. So, what we have this starting with that the span S W and this is basically span the whole element ok. Now, you choose any element u plus w that will be spanned by the element from this one.

So, from here I can say that S W the span this will be equal to U plus W. So, it will be spanned by the spanning set of u union spanning set of W ok. So, this one we can find out.

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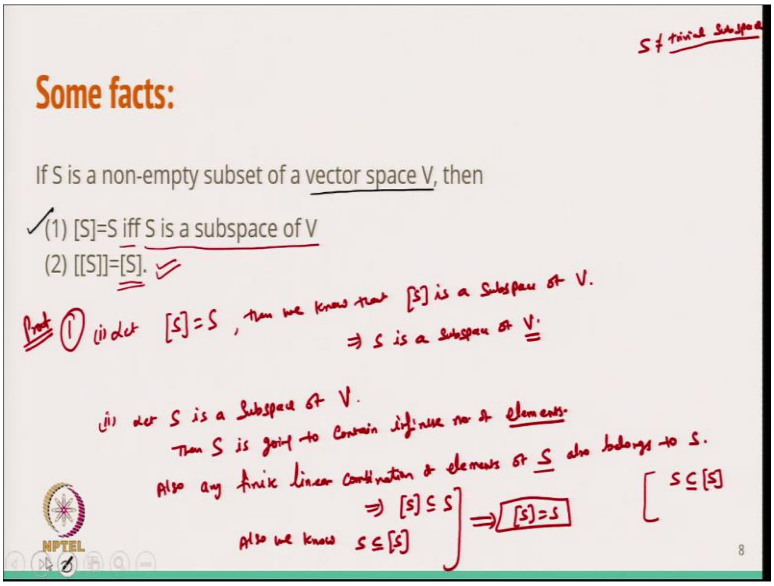

Now, we want to discuss some facts that if S is a non empty subset of the vector space V then the first property is that span of S is equal to S if and only if S is itself a subspace of V. Because till now we have taken only S as a subset of the vectors space, but now what will happen if S is a subspace of V?

And then if I take the subspace and then I take again the span of that one that is also equal to this one ok. So, this one we want to show from here I can prove it just with the argument we can prove. So, first one we want to prove. Now, suppose let we take this property is equal to S then we know that this is a subspace of V because just now we have discussed.

And if it is subspace equal to something then it means that S is also a subspace ok. So, that is proved that S is subspace of V ok. Now, I want to define the other term. So, this is the one way ok. So, I can write first way. Second way is let S is a subspace of V. So, let us S is a subspace of V.

So, it is a subspace of V and we already know that either S is a trivial subspace trivial subspace means it contains only zero element or it is a non trivial and if it is a non trivial then it is going to contain infinite number of terms elements ok. So, definitely we are here.

We are not taking that S this is not equal to trivial subspace because that is understood. So, I am not taking that S is a trivial subspace. So, it is non trivial and this is a subspace. Then S is going to contain infinite number of elements ok. Now, what is the meaning of then linear combination of the set S? Now which imply also; so, whenever we talk about this one then also any finite linear combination.

Finite linear combination means, I am choosing the finite number of elements and taking the linear combination then any finite linear combination of elements of S. So, because I have chosen that S is a subspace of V also belongs to S. Why? Because S is a subspace.

So, if I choose any linear combination of the element of S that will also contain in the S that is there because that is a subspace which implies that S is equal to this because we know this one. See I can show that we also already know that S belongs to a subset of this one that we already know from here. So, from here I can write this property in them. So, if I taking any element of S that I know that that belongs to this one ok.

But in this case if I take the any linear combination, so, from here I can write that this is contained in S because I am taking all the linear combination of the element of S finite linear combination and that is also belongs to this. So, from here I can say this. So, also we know that S is contained in this one.

So, from these two property, I can say that this is equal to S ok. So, if the S is a subspace then it does not matter. I write S or the span of S, both are same and the second property is just the extension of the first one. Once the S in this case S can be any set. So, if S is a subspace then it is equal to S and taking the gain span does not matter.

So, this property just the extension of this one. So, this is a I think now clear that if S is a subspace then it is equivalent to its span.

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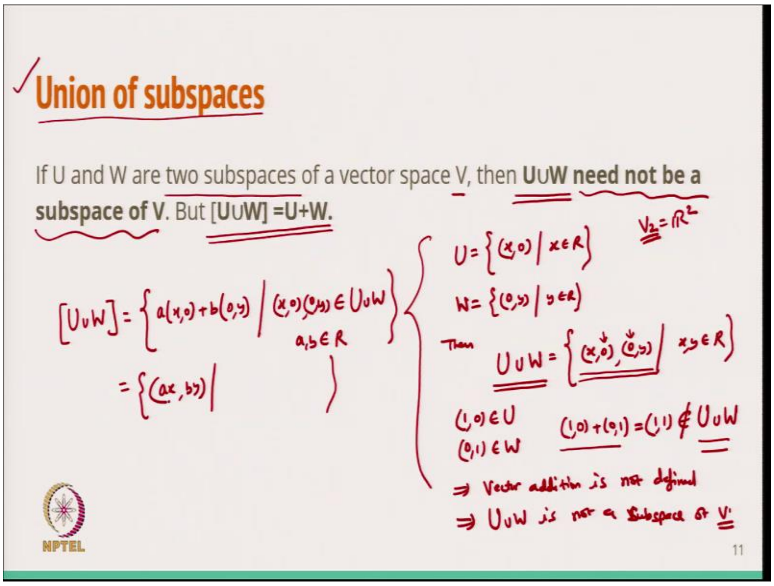

Now, so, after doing this one now we discuss the next term that is what is going to happen when we deal with the union of the subspaces. Till now we have only discussed about the summation of the subspaces. What we are going to have in the union of subspaces?

So, in this case if I choose U and W are the two subspaces of a vector space V then I define U union W then it is written that it need not be a subspace of V, but if I take the span of this one then we already know then this is going to be U plus W. So, this is called that the how the union of subspaces can be dealt here.

Now, let us define this one again U plus W. So, I just give you the example because it is written that it need not be true. I will choose the same example I have started with. So, in the previous examples we have started with U, we have taken already taken this as x, 0 where x belongs to a real line and W we have taken as 0, y where y belongs to real line and that and we have taken the V 2 that is equal to R square I have taken ok.

So, I have taken U and W then I just take U union W. So, U union W will contain all the elements x, 0 and 0, y all the elements where x and y belongs to the real line or the field what I have taken here. So, this is my union I have taken. I have taken all the elements of U and all the elements of V such that x and y belongs to R.

Now, I want to check whether it is a subspace or not. So, this is I want to check. So, I just take element 1, 0. So, 1 0 belongs to U right I take 0, 1 that belongs to W then let us see their combination. So, 1, 0 if I take plus 0, 1 then it will be 1, 1 and 1, 1 is not there. So, that does not belongs to U union W.

Because in U union W only this type of elements are there in which either the second element is 0 or the first element 0 in the vector. So, that is a two dimensional vector. So, it contains only those type of elements in which either of the element in the vector is 0.

But, if I choose two element U from U and W and I take their linear combination or the vector addition that is 1, 1. So, it does not belong to this one. So, from here I can say that vector addition is not defined. So, from here I can say that U union W is not a subspace of V, it is not a subspace ok. So, this is what it is written that need not be a subspace of V.

So, this is the example to show that how it is not a subspace of this. Now, so, if it is not happening then it is written that if I take the union and then the span. So, let us see what is going to happen here. So, I will take U union W span. So, I am taking this one as. So, these are the elements, I am taking their linear combination.

So, I just write a x, 0 plus b 0, y such that x, 0; 0, y belongs to U union W and a, b belongs to the real number R that is a field and this will become a x, b y and all are the same things are same.

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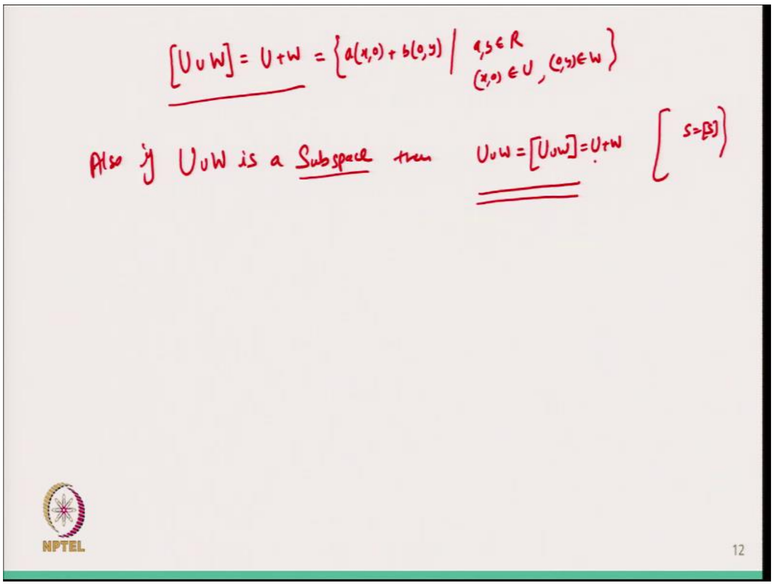

So, now from here you can see that this is just the same element any element from here that belongs to the that U union W span will be again equal to U plus W. Because that combination becomes a element a x, 0 and element b from W such that a, b belongs to U plus W ok, sorry U to the field real number and x, 0 belongs to U and 0, y belongs to W ok.

So, based on this one we are able to show that U union W is need not be a subspace and if it need not the subspace then if I take the span of this one then it is going to be U plus W. And also if U union W is a subspace, suppose it is a subspace because it need not be, but suppose it is a subspace then we know that U union W is same as U union W span.

Just now before that we have shown that S is equal to S span if S is a subspace and then again itself and this is equal to this one, so, no problem in that case. So, from here I can write then it will be U plus W. So, U union W will be again that will be U plus W. So, this properties is very crucial in the case of the subspaces the union of the subspaces.

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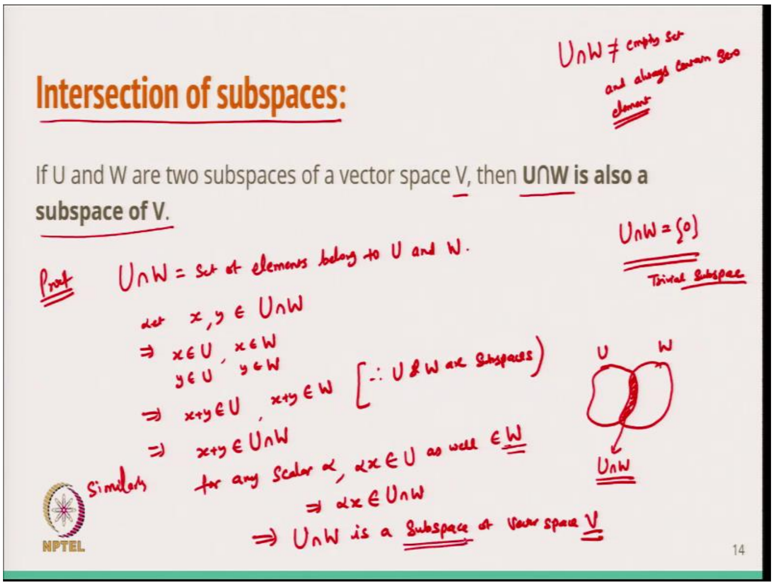

Now, after discussing this one, now we want to discuss what is going to happen if I discussing instead of union we will discuss the intersection. What is going to happen in the case of intersection? So, let us see if in this case. So, now, we are discussing the intersection.

If U and W are the two subspaces of the vector space V and then I define their intersection, then it says that is also a subspace of vector space V. So, in this case it is always true that it will be a subspace of V because we know that as we have already discussed that 0 element will be contained in each of the subspace.

So, from here one thing is true that if I take any subspace then definitely it cannot be empty set. It cannot be empty set, contains at least it contains it cannot be empty set and always contain 0 element. And you know that if U intersection W is just the 0 element then it is the trivial subspace, but it will be the subspace that we have already seen ok.

So, in this case also it will be the trivial subspace. Because you know that from if I take two subspaces one subspace here and this suppose I take another subspace U and W then their intersection is going to be this part in this case. So, this place that is U intersection W so, it say that it is always subspace of V. So, how we are going to define this one? So, it is very simple. So, I can just give the proof. Now, these are the two subspaces. So, I will define what is U intersection W. So, it is set of elements belong to U and W both ok.

Now, let I take U and, but I take not U, I just I write x. Let x and y belongs to U intersection W which implies that x belongs to U and x belongs to W; y belongs to U and y belongs to W because it is the intersection. So, from here it implies that x plus y also belongs to U because U is a subspace and x plus y also belongs to W because U and W are subspaces.

So, from here I can show that x and y belongs to U intersection W because it is containing both. So, definitely it will be here. So, the vector addition is satisfied. Similarly, if I choose any alpha and then I take x then from here; so, if I am taking any x, it can be written as now it can be just I just write. Now, for any scalar alpha, alpha x belongs to U because x I am taking intersection belongs to U intersection.

So, alpha x is belongs to U as well belongs to W because it just the scalar multiplication and U is itself a subspace and W is also subspace and from here alpha x will belongs to U intersection W. So, I have taken the x from this and I have taken the scalar multiplication that is also belongs to this. So, from here I can say that U intersection W is a subspace of vector space V. So, it is always subspace of the vector space V ok.

So, I will stop here. So, today we have discussed some important properties about the subspaces. We started with the union of the subspaces and we found that the union it need not be a subspace of the vector space V and then we have discussed the intersection of the subspaces. So, we will continue these things in the next lecture. I hope you enjoyed this lecture.

Thanks for watching. Thanks very much.