Matrix Computation and its applications
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Lecture – 06
Spanning set

Hello viewers, welcome back to the course on Matrix Computation and its application. So, in the previous lecture, we have discussed many properties of the vector spaces. So, today we are starting with and continuing with the span of the set S.

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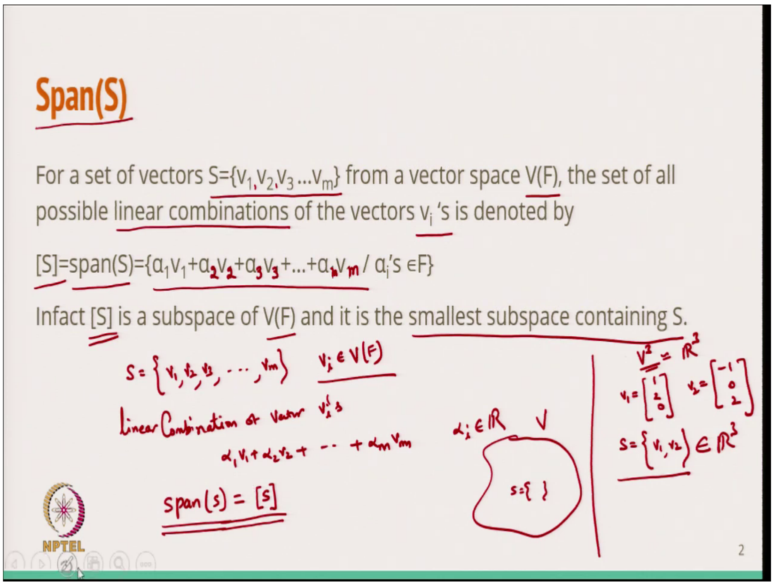

So, in the previous lecture, we have just discussed the concept of span of S. So, in this, we define the span of the set S. So, in the span of the set S, what do we have? We have a set of vectors.

Suppose, I take m number of vectors, and these vectors we are choosing from the vector space V(F). Then, the set of all possible linear combinations of the vectors $v_i's$. So, all the linear combinations we are taking here and that is denoted by the span(S) and we also represent with the [S], and what is that? it is a set of all the possible linear combinations, where $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + ... + \alpha_m v_m$.

So, this is what we are defining. Where all $\alpha_i's$ belongs to the field F. So, here we are taking all the linear combinations. And, after doing this one what we found is that, that the span of S is a subspace of the vector space V(F). And, it is the smaller subspace containing S. So, this one we want to discuss today. So, let us see what is we are going to discuss.

So, it is given to me, that S is a set that contains $v_1, v_2, v_3, ... v_m$, m number of elements. And, these $v_i's$ belongs to my vector space V(F). For example, if I take, let I take v 3. So, in v_3 maybe I can choose my v_3. So, v_3 I am taking. So, this one I can write like here v_3. So, this is what I am defining, which is equivalent to R^3.

So, I am defining a vector, v_1. So, this vector I can define as maybe [1 2 0], v_2, I am defining as suppose I define [-1 0 2] something like this one. So, I just take 2 vectors and I take S as {v_1, v_2}. So, these 2 vectors I am taking.

Now, this is a set that belongs to my V^3 or R^3. So, this is a set I am taking. Now, what do I want? So, this is what I have defined, that belongs to the vector field. Now, we know that, the linear combination of vectors v is equal to I am defining $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + ... + \alpha_m v_m$.

So, this is the linear combination I am taking, ok. And, where my α_i is are coming from the field. So, that is a real number in this case. Now, what I do? I choose all the linear combinations. So, I call it a span of S. So, this is a set that contains all this linear combination.

So, maybe I can show by the figure that let V is my any vector space and out of this I take a set S. So, this is my sum set S. So, now, I take all the linear combinations and wherever it is falling. So, this set will contain all the linear combinations. And, I represent that set by the close bracket with the set S. And, I call this as a set made up of all the linear combinations of these vectors.

Now, I want to check, whether it is a subspace or not, so, this one I want to check.

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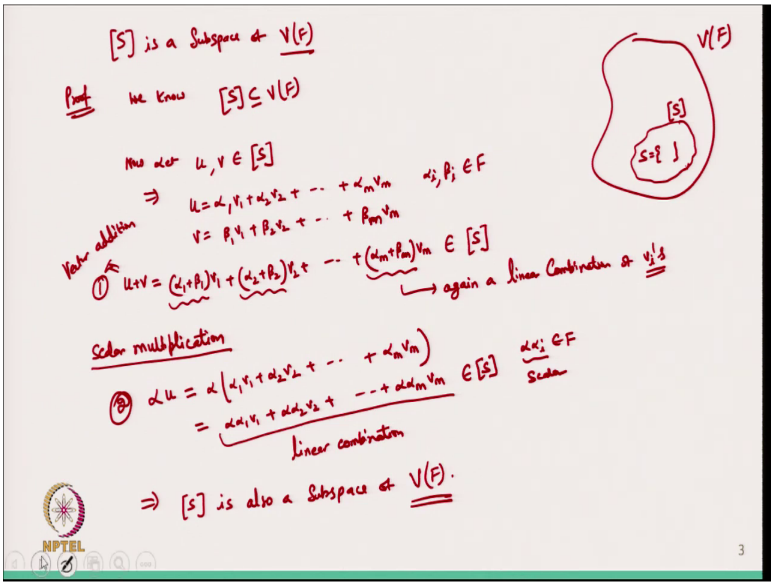

Now, we claim that this is a subspace of the vector space, whatever we have discussed here V(F). It means, that first I have some vector space V over the field F. Out of that, I have taken a set S made up of some vectors. And, then maybe I am getting another set I call it span(S).

So, span(S) is a subspace of V(F). So, this is what we are claiming here. Now, how we can do that? So, let what we are doing here is, we have to satisfy the two properties of vector addition and scalar multiplication, and if that is there then we can say that this is a subspace. So, we know that this is a subset of V(F). Because whatever the set S is there that belongs to the V and V is itself a vector space.

So, all the linear combinations will also belong to the vector space. So, it will be the set definitely. Now, let us take two elements. So, suppose I take one element u and another element v which belongs to span(S). This implies that, if this belongs to the span(S) it means; it should be a linear combination of the vectors that belong to the set S. So, let I am taking this one, v we are getting suppose it is equal to $\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + \ldots + \beta_m v_m$ because it belongs to this one.

So, definitely, it will be of this type, where my $\alpha_i's$ and $\beta_i's$ belong to the field. Now, what about u+v? So, u+v if I take the vector addition, then I can write this as an $(\alpha_1 + \beta_1)v_1$ because I can add component-wise. I can write this as $(\alpha_2 + \beta_2)v_2$ and this as $(\alpha_m + \beta_m)v_m$.

So, this one is there. Now, you can see from here, that $(\alpha + \beta)$ is a scalar, this is also a scalar, and then it becomes again the linear combination of the $v_1, v_2, v_3, ... v_m$.

So, now, from here I can say that this belongs to again this set because it is again the linear combination. So, again a linear combination of $v_i's$, means that it belongs to the set S. Now, so, this is vector addition. Now, we discuss the next property scalar multiplication.

So, I take any scalar α and I multiply to the element u ok. So, I can write it as $\alpha. u$. And, u can be written as $\alpha_1 v_1 + \alpha_2 v_2 +..... + \alpha_m v_m$, and this is a scalar. So, I can multiply this as $\alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 +..... + \alpha\alpha_m v_m$. So, I can write this one. Now, $\alpha. \alpha_i's$, that also belongs to the field F, because it is a scalar. So, from here this is again a linear combination.

So, it is really again the linear combination, then I can write that this belongs to the span(S). And, from here I can say that the span(S) is also a subspace of vector space V with the field F. Because, we know that if this first property, and the second property, if the two properties are satisfied, then that all other properties will definitely be satisfied because this set S definitely lies in the vector space V.

So, everything will be their inverse and will be their additive inverse, additive identity, everything will be there. So, if we satisfy these two properties, then from here, can say that this is also a subspace of the vector space V(F).

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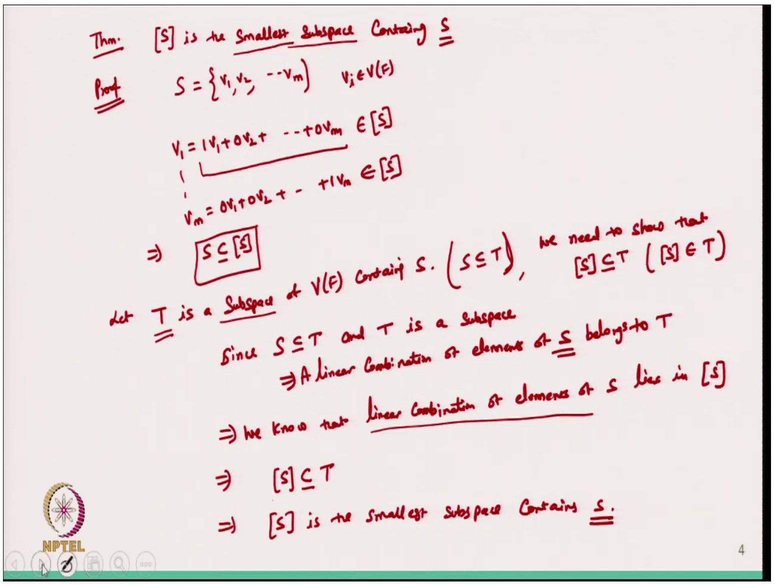

Now, the next thing comes is that, how I can check whatever we have claimed, that this is the smallest subspace containing S. So, it means I have to show that span(S) is the smallest subspace containing S. So, this is my claim or you can say it is a theorem and then I can do a proof also. Now, I know that this S.

So, we have chosen $v_1, v_2,..., v_m$ in S. So, this is what we have taken. Now, if I take S is equal to this, where v_i belongs to the vector space and so, this is what we have taken. Now, I can write my v_1 as $1.v_1 + 0.v_2 +...+ 0.v_m$ ok. It means v_1 is also a linear combination of these vectors. So, which implies it belongs to this one.

So, the same thing I can do for all. So, I can write v_m as $0.v_1 + 0.v_2 +...+ 1.v_m$. So, that also belongs to span(S). So, from here I can say that S belongs or is a subset of this one, so, this one I can say. So, from here I can say that this S is contained in the span(S). Now, the claim is that it is the smallest of subspace.

So, smallest subspace means let T is a subspace of vector space V containing S. So, let I take the subspace T such that T is contained in S, that is S is containing the subspace T. Now, we need to show, that, this is also contained in T, a subset of T ok. Or we can also say that this belongs to T this is also I can write. So, from here, this is how we can do that because T is a subspace.

Now, since the set S belongs to T and T is a subspace, which implies that a linear combination of elements of S also belongs to T. So, from here I can say there is a linear combination of the element of S. That also will be a set or maybe I can the element of S belong to T. Why? Because, T is a subspace and we know that if I take any element of T, then their linear combination definitely will be in the T.

Because, what is the linear combination? Linear combination is basically the two properties satisfying together; that is scalar multiplication and vector addition. And, both are satisfied, because it is a subspace. So, in this case, any linear combination if I take the element of S will belong to T.

So, if that is there which implies that linear combination of elements of S lie in this one span of S that if you take the linear combination, that will lie in the S. So, which implies that S will definitely also belong to T. So, it is also a subset of T or maybe equal also.

So, from here I can say that if the S belongs to there and then the span(S) also belongs to the T. And, definitely this is the smallest one because I cannot find any other T the subspace, which will be contained in this one. Because, I have chosen the subspace which contains the S and I found that the span(S) is also contained in T. So, from here I can say that this is the smallest subspace containing S.

So, S is always lying in the span(S). So, this is the way we can show that the span of S is the smallest subspace containing S.

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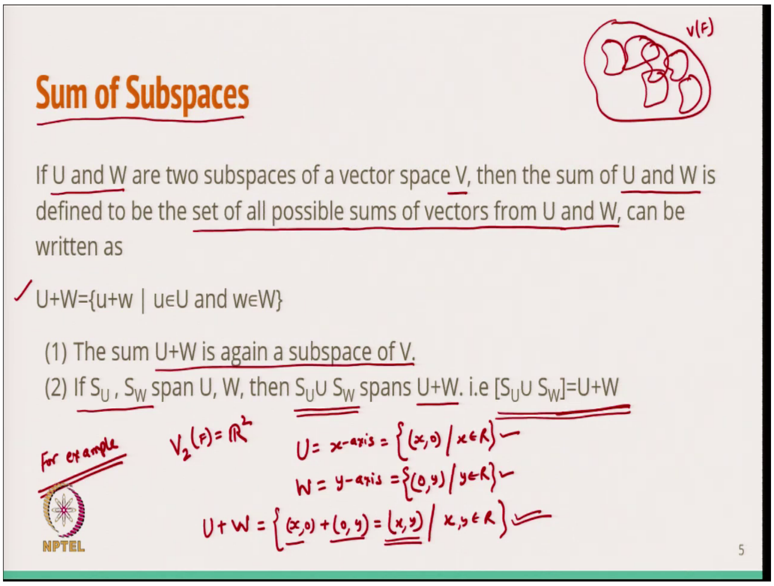

Now, after doing all these things, now we know that in a vector space I can have a large number of subspaces. So, I want to discuss what is going to happen? If I add two subspaces, suppose I have a vector space V(F) and let in this I have this one subspace is there, another subspace is there, another subspace is there, another subspace is there, this is another subspace.

So, I can have a large number of subspaces. So, I want to see what is going to happen if, I add two subspaces. So, if U and W if I take two subspaces U and W are the two subspaces of vector space V, then I want to discuss the sum of U and W. So, what is that?

So, that is defined to be the set of all possible sums of the vectors from U and from W. And, it can be written as like this one U+W is equal to the element u taken from the U and small w taken from this subspace W. So, I am taking one element from U plus another element from the W. Then, we want to show that, the sum is again a subspace of V.

It means, the summation is also subspace of V and if S_u and S_w span U and W. So, the union spans U plus W. That is if I take the union and then I will span this is equal to this U+W. So, this one we just want to discuss; now I want to show you what it will look like?

So, for example, now first I will discuss U+W. For example, so, let me show you the example. For example, I just take a vector space V_2 that is R^2. And, in this case, I just take a U as the x-axis. This means, I am taking all the elements (x, 0), where x is the real number, and I take W is supposed to as the y-axis. So, that will contain the element (0, y), where y is a real number. So, these two subspaces I am taking and it is the subspace.

Because, you can see that, (0, 0) is contained in U and (0, 0) contain in W also ok. And, all other satisfy sorry additive property vector addition and scalar multiplications are satisfied. So, this is a subspace of V_2 that we know, then I can define my U+W, as I am taking one element. So, that is the element I am taking (x, 0)+(0, y). And, if I add these together, then I will get (x,y), where x and y belong to the real line ok.

So, in this case, I am taking the linear combination of this plus this and I am getting this value. Now, our claim is that this is also a subspace of V ok because a vector space is also a subspace of itself. So, no problem, and the second one is that, if I take the spanning set of U and spanning set of W, then their union I can take as a spanning set of U+W. So, this one is just what we want to check. So, let us do this one.

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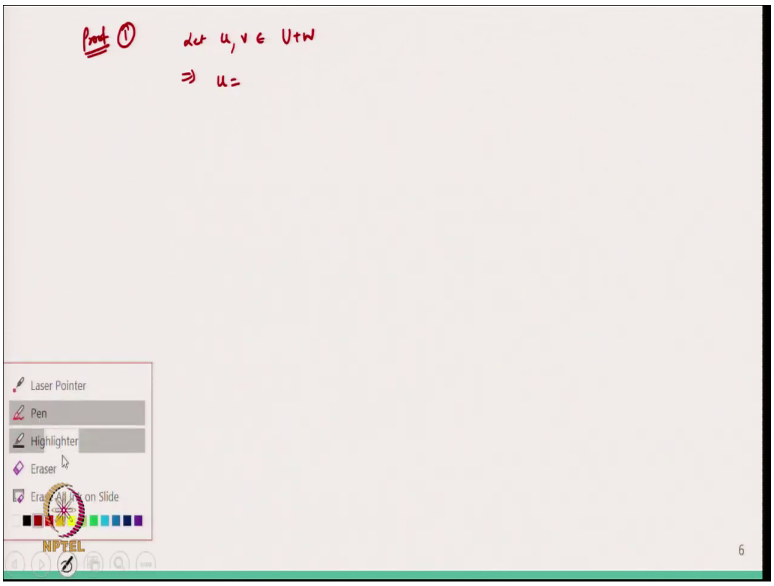

Now, the first property I want to prove, the first one ok. So, the first one is that the sum is again the subspace of V. So, this one I want to set. So, let me choose u and v belongs to U+W, ok, which implies my u is coming from this. So, I just choose two elements. And, then I want to see that, I choose one element U and another element V from here ok.

So, let us not take this one, I will give some other notation ok. So, I just take another notation.

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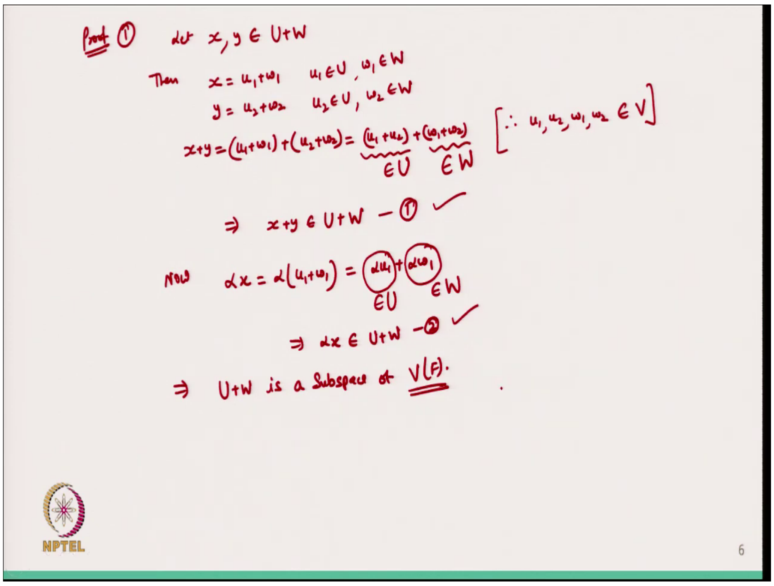

Let x and y two elements I am taking belong to U+W. Then, x can be written as some $u_1 + w_1$, where u_1 belongs to U and w_1 belongs to W. Similarly, y also $u_2 + w_2$ some element is coming from u and another element is coming from w.

And, this is also u_2 belongs to U and w_2 belongs to W, then I just want to see what will happen with x+y. So, x+y can be written as $(u_1 + w_1) + (u_2 + w_2)$. So, this one I also can write as $(u_1 + w_1) + (u_2 + w_2)$ because all these elements definitely come from V a vector space.

So, I can do this one because in the vector space is we know that associative property is satisfied. So, you can write down that, because $(u_1 + u_2) + (w_1 + w_2)$all belongs to V, and the associative property is satisfied. So, we know that we can add and shift the elements and can write like this one.

Now from here this $(u_1 + u_2)$, that is coming from the U. So, and that is the subspace. So, this also belongs to U. This is again the vector addition of two elements of W. So, this also belongs to W. So, from here I can say that this is from here I can say that x+y also belongs to U+W, because, I am taking one element coming from U and another element coming from W.

So, vector addition is satisfied ok. Now, another property so, I want to see what about $\alpha.x$. So, $\alpha.x$ can be written as $\alpha.(u_1 + w_1)$. And, this one I can write as $\alpha.u_1 + \alpha.w_1$. And this scalar multiplication, that it belongs to U, this scalar multiplication also belongs to W. So, from here I can say that the $\alpha.x$ also belongs to U+W ok.

So, from here I can say that another scalar multiplication is also satisfying. So, from here I can say that U+W is a subspace of vector space V because both the properties one and another property are satisfied. So, then from here, I can say that this is a subspace of the vector space V. So, this is what we wanted to show.

Now this one, the second part is there so, we will stop here and we will discuss this second part in the next lecture.

So, today we have started with some properties of the span(S) and then we have discussed that if we have subspaces then how their combination, like summation, is whether it is a subspace or not? So, that we have discussed and the other property we are going to discuss in the next lecture.

So, thanks for watching thanks very much.