Matrix Computation and its applications
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Lecture - 53
Some examples on the Jordan form of a given matrix and generalised eigenvectors

Hello viewers, welcome back to the course on Matrix Computation and its application. So, in the previous lecture we have started with the Jordan canonical form. So, today we are going to discuss a few examples based on how we can make the Jordan canonical form.

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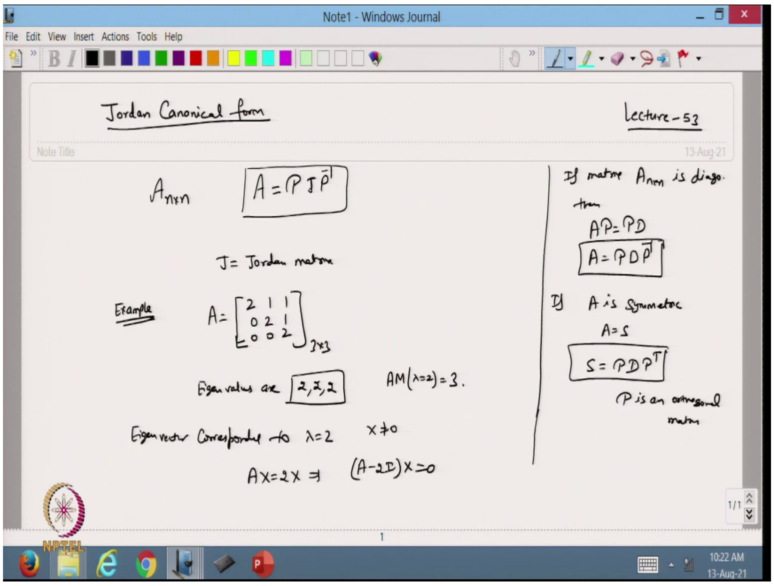

So, let us do this one. So, in the previous example previous lecture we have discussed that suppose I have a matrix A that is of n*n. So, it is a square matrix and in that case I can write the matrix A into this form. So, maybe this form I can write in the same way as we used to see in the diagonalization. Because we have seen that in the that if matrix A that is n*n is diagonalizable.

Then we can write A P = P D or maybe I can write A = PDP^{-1}, where D contains the diagonal the eigenvalues at the diagonal elements as diagonal elements. So, this is what we can do when the matrix is diagonalizable and we have also seen that if A is symmetric. So, maybe I can write A = S, S is symmetric then we have seen that we can write this as P and then so, maybe I can write $S=PDP^T$.

So, in this case I can say that this matrix S is orthogonally diagonalizable because the matrix P is an orthogonal matrix. Now after that we know that if I have a matrix A which is not diagonalizable then we can write its Jordan form and this is my Jordan form. So, where my J is Jordan matrix and then we have also discussed the generalized eigenvector in the previous lecture. So, now, today we are going to discuss one example based on this one.

So, let us take one example suppose I take a matrix A and this matrix I just take $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. So, this is a matrix 3 cross 3 matrix and this matrix you see this upper triangular matrix now from here I can find that the eigenvalues are 2 ,2, 2. So, it is the repeated eigenvalue. So, I can write here that arithmetic multiplicity of eigenvalue λ is equal to 2 is 3.

Now, based on this, if this is a repeated eigenvalue then this matrix may or may not be diagonalizable. So, let us see what is the dimension of the eigenspace of the eigenvalue 2. Now to find the eigenvector what we do is that let us find the eigenvector corresponding to eigenvalue λ =2. Now I know that this can be written as A X =2 X and from here I have $AX = 2X => (A-2I)X = 0$ that is an eigenvector.

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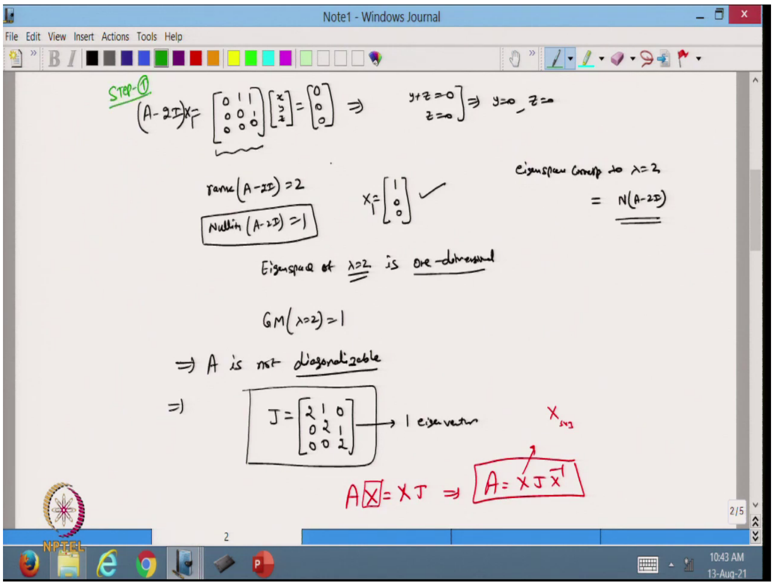

And from here I can write this as. So,

$$(A-2I)X = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y + z = 0, z = 0 => y = 0, z = 0$$

So, now, from here I get only one vector, this is freely free vectors and if you see from here then this matrix I can say that in this case the rank of (A -2 I) = 2 and from here also I can say that. Now, nullity of (A -2 I) is 1 because I will be able to get only one dimensional null space which is the solution of this one and from here now I get my y =0 and z =0. So, I can take my X as suppose I take

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

So, this is my eigenvector corresponding to the eigenvalue λ =2. Now, from here you see that the nullity of (A -2 I) =1. So, from here I can say that now because we know that the eigenspace or λ is equal to eigenvalue is one dimensional. So, it is one dimensional, basically it is the same as equal to the. So, I know that the eigenspace of eigenspace corresponding to eigenvalue λ =2 is equal to the null space of (A-2 I).

So, that is equal to the null space of this and null space of this will give you only this type of vector and scalar multiplication and its dimension is 1. So, I can say that eigenspace of λ =2 is one dimensional. So, from here I came to know that the geometric multiplicity the short form is geometric multiplicity of λ =2 in this case is just 1.

So, from here we found that we can say that this matrix A is not diagonalizable. Because this matrix is diagonalizable only when the geometric multiplicity would have been 3 then we can say that ok this algebraic multiplicity is equal to geometric multiplicity. So, the matrix will be diagonalizable.

So, in this case the matrix will not be diagonalizable and from here we will get the Jordan form. So, my Jordan form in this case will be $J=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. So, that will be the corresponding Jordan matrix in this case and we will get only one Jordan block and this I know that it is corresponding to 1 eigenvector and I am getting only 1 eigenvector from here now the thing is that.

So, from here we know that I can write my matrix A as I can write this matrix AX=XJ and from here I can write $A=X J X^{-1}$. So, I need to find the matrix X. So, X is a matrix that is 3*3 matrix. So, this matrix I need to find out. So, that matrix would be corresponding to the eigenvector and the generalizing eigenvector of the corresponding eigenvalues, so that we need to find that how we can find out this matrix X.

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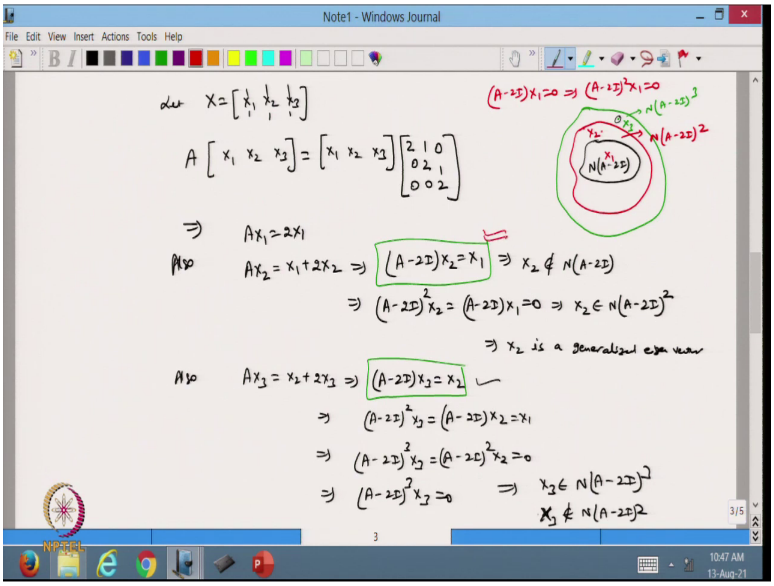

Now, from here now we get that our matrix. So, let my matrix $X = \begin{bmatrix} : & : & : \\ X_1 & X_2 & X_3 \\ : & : & : \end{bmatrix}$. So, these are the 3 vectors and X1 is the eigenvector I am taking this X1 same as X1 now from here I can write that the matrix $A[X_1, X_2, X_3] = [X_1, X_2, X_3]\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. And this is my Jordan form because I will get only this Jordan form in this case.

And from here I can find that my $AX_1 = 2X_1 => (A-2I)X = 0 => (A-2I)^2 X = 0$

and also from the second one I will get $AX_2 = X_1 + 2X_2 => (A-2I)X_2 = X_1$

So, from here I can say that this $X_2 \notin N(A-2I)$. Now from here as we have done in the previous lecture we can write this as square X_2 and that becomes $(A-2I)X_1$ and that is already 0.

$$=> (A-2I)^2 X_2 = (A-2I)X_1 = 0 => X_2 \in N(A-2I)^2$$

So, from here I can say that X_2 is a generalized eigenvector and this belongs to this. So, if you see from here let me take a picture. So, let we make this as the null space of $X_2 \in N(A-2I)^2$, but it does not belong to this one.

But So, from here if you see this the bigger one will be $N(A-2I)^2$ because all the vector belongs to this will definitely comes under the null space of $N(A-2I)^2$ from here I just multiplied by one more time if I am getting from here then because I know that $(A-2I)X_1 = 0$ and from here I can also write $(A-2I)^2X_1 = 0$.

So, X_1 is also belongs to the null space of this and from here that I can say that this null space is a subset of the null space $(A-2I)^2$. So, from here if you see I can say that suppose this is my X1 then somewhere here it is my X2. So, X1 and X2 are lying in the different null spaces, but X1 and X2 both are lying in the null space of $(A-2I)^2$. So, that is there.

Now, from here from the third one also if I multiply this one I can write my

$$AX_3 = X_2 + 2X_3$$
$$=> (A-2I)X_3 = X_2$$
$$=> (A-2I)^2 X_3 = (A-2I)X_2$$
$$=> (A-2I)^3 X_3 = (A-2I)^2 X_2 = 0$$
$$=> (A-2I)^3 X_3 = 0 => X_3 \in N(A-2I)^3, X_3 \in N(A-2I)^2$$

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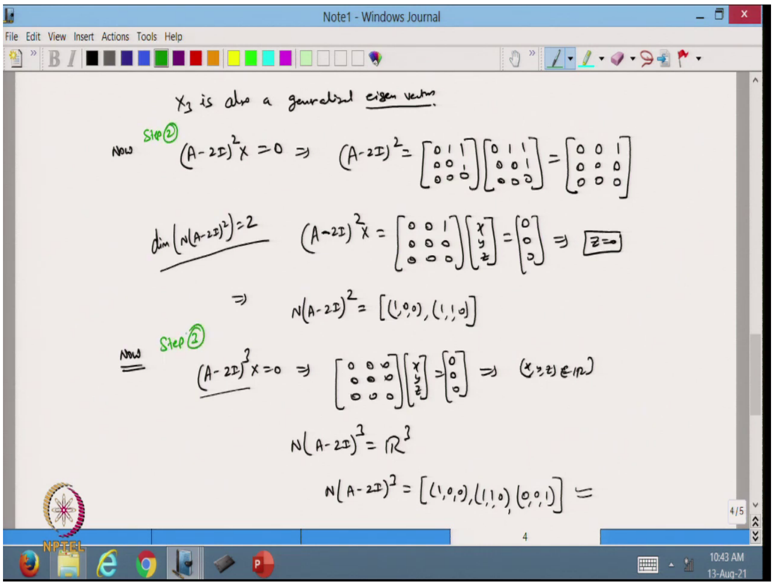

So, from here I can say that the X_3 is also a generalized eigenvector. So, let us find out how we can find this generalized eigenvector now I am able to find only X_1. So, from here I will take now I will try to find out what is my $(A-2I)^2$ because from here I know that $(A-2I)^2X_2.=0$

So, let us take X2 that is equal to 0 or maybe I just take some vector X. Let us take this one. Now, from here if you take the multiplication. So,

$$=> (A-2I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-2I)^2 X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} => z = 0$$

So, my free variables are x and y. So, based on this one I can find that the null space of $(A-2I)^2$ is spanned by the vectors.

So, I can take the vector because now I need to find out two vectors because the dimension of the null space is I can say that the dimension of null space of $(A-2I)^2$ is 2 now because the rank is one. So, two vector will span this the whole null space of $(A-2I)^2$.

So, what I can do is that I can choose X_1 what we have obtained from the eigen as a eigenvector corresponding to the eigen value and X_2 I can take the linearly independent of this one. So, I can take the first one I just take as span of (1 0 0). So, this vector I can take and another vector I will take whose z is 0 and will be linearly independent of this one. So, I may be taken (1 1 0). So, this will be spanned by this $N(A-2I)^2 = [(1,0,0),(1,1,0)]$

Now, I choose now we find out the so, from here I will get two vectors because now I need 3 linearly independent eigenvectors to construct the matrix X. So, now, from here on, this X I am just taking a vector and this capital X is a matrix. So, this X and that X is not the same one now suppose I take the third one. So, also I will take

$$(A-2I)^3 X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, from here I can say that my $(x,y,z) \in R^3$. So, that belongs to the whole R cube now because the $N(A-2I)^3 = R^3$. So, I can choose any vector. So, now, from here I can say that the $N(A-2I)^3$ is equal to the span by the vectors, now I have to choose another. So, suppose this is my $N(A-2I)^3$ and I can choose my X_3 somewhere here. So, I suppose I will choose X_3.

$$N(A-2I)^3 = [(1,0,0),(1,1,0),(0,0,1)]$$

So, this is the basis of null space of this one. Now, I got these vectors based on the null space, but I have not satisfied this condition because we have to satisfy this condition also for the corresponding eigenvectors and the generalized eigenvector. So, this is the relation between the generalized eigenvector and the eigenvector and this also I have to satisfy this condition. So, this will give you the Jordan chain.

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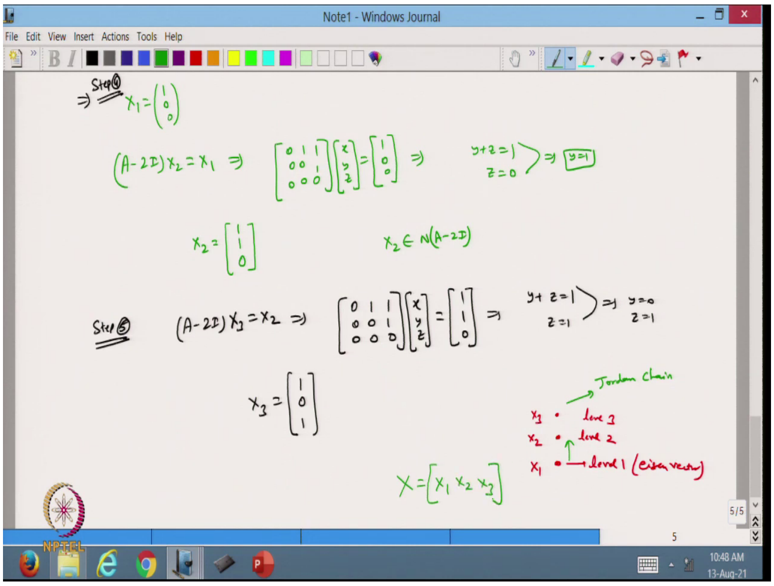

So, let us see how it is going to be, now from here I will get my X_1. So, $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Now, I need X_2. So, I have

$$(A - 2I)X_2 = X_1 => \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$=> y + z = 1, z = 0 => y = 1$$

So, in this case my y =1. So, I can choose my X_2 vector as now the condition is that y= 1. So, I have taken y =1, my z = 0 and the condition is that this X_2 should not belong to the null space of (A -2I) ok.

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, X_2 \in N(A - 2I)$$

,you can see from here because (A-2I) is here if I put (1 1 0) here. So, this will be (1 1 0). So, that will not be 0 it will be 1 here. So, you can see from here. So, we have checked this one and now. So, this is what I can say about finding these values. So, maybe I can write in the form of yeah. So, it can be written as step 1 finding the first eigenvector.

And then using this one then I have this value. So, it is step 2, step 3 and then from here I will get this is step 4 finding the generalized eigenvector based on this one and step 5. So, step 5 means I need to satisfy that $(A-2I)X_3 = X_2$ and from here if you see then I will get 0 0 1 and this is suppose my (x y z) and X_2 is (1 1 0).

So, I have taken my X_2 is (1 1 0) here because the condition was only z =0 and y=1. So, I have taken X_2 the same way and from here if you see I will get my z = 1 and that is it. So, that is the condition. And I want my X_3 to not belong to the null space of (A-2I) if you see from here this one yeah. So, from here $(A-2I)X_3$ should be X_2 ok.

So, I can choose my X_3 as so, my z =1 and y = 0. So, I should take the X_3 such that X_3 should not belong to.

So, X_3 is here. So, this is my X_3. So, it should not belong to the null space of X_2 or null space of X_1 and if you see the null space of X_2 has the third component 0 and third component 0. So, I should take this value. Maybe I can take here 1 no problem. So, now, from here I will make a chain that is called the Jordan chain. So, now, what we are getting at the first level; level 1 I will get my X_1. So, this is my eigenvalue eigenvector.

Now, from here I will get my X_2 that is a generalized eigenvector. So, this is my level 2. I have taken level 2. So, level 2 is coming from this condition by satisfying this condition. So, that is my level 2, now after doing this level 2 I got the next one is level 3 and this is my X_3. So, X_3 we have taken from this condition by comparing the last AX cube. So, the AX cube becomes X_2+ 2 X_3 and from here I will get this value. So, I go to all the three eigenvectors corresponding to the eigenvalue.

So, how I can write this is called Jordan chain. So, now I am ready to make the matrix X. So, X will be we have to write first we have to write the eigenvalue this one and then we have to go in this way. So, then I will write my $X = [X_1, X_2, X_3]$.

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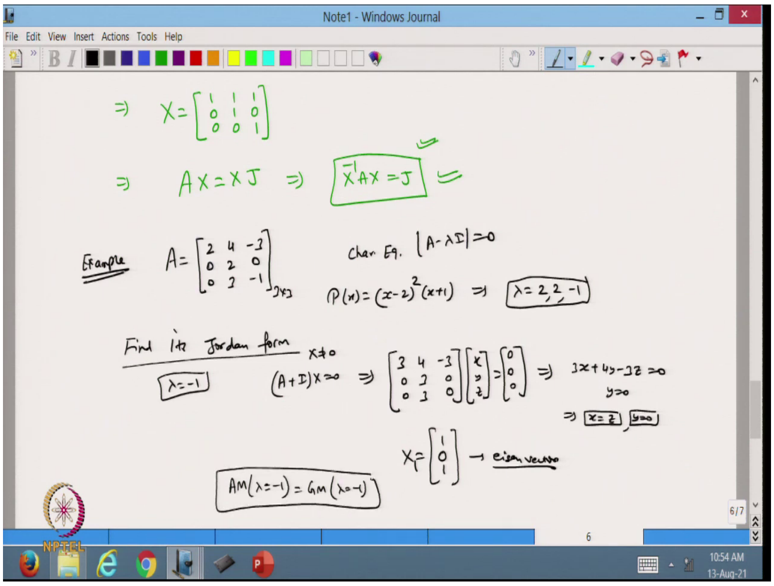

So, based on this one I will get the matrix X = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and from here you can see that if I take $AX = XJ$ and I should be able to and these are linearly independent. So, I can take my inverse.

So, I can from here I can write that $X^{-1}AX = J$ and these things we can verify that we should be able to get only J. So, that's why you can just verify from doing the calculation on the computer. So, this way we are able to find the Jordan canonical form for the corresponding matrix A and this is the way we are able to find the X.

Now let us do one more example. So, in this case we were able to have only one eigenvector corresponding to the eigenvalue. So, let us take another example. So, I have calculated these null spaces in their dimensions just for the clearance even though we can go directly from there. So, let us take one another example I will just take let me take the matrix A $A = \begin{bmatrix} 2 & 4 & -3 \\ 0 & 2 & 0 \\ 0 & 3 & -1 \end{bmatrix}$. So, this is my matrix A is 3*3 and if you find its characteristic equation.

So, I can write its characteristic equation P(x). So, it will be. So, characteristic equation means $|A-\lambda I|=0$. So, it will $P(x)=(x-2)^2(x+1) \Rightarrow \lambda = 2,2,-1$ because their sum should be equal to the trace. So, it is equal to the trace. So, from here now I want to find out its Jordan form. So, find its Jordan form.

Now, in this case we have a. So, let us start with the $\lambda=-1$. So, for $\lambda=-1$ we have only 1 eigenvalue and so, its algebraic multiplicity is 1 and will also get the geometric multiplicity that will be also 1 and this one we can find with. So, we have to do $(A+I)X=0$ where my X is not equal to 0 this is the eigenvector and from here if you see I can find. So, adding one here.

$$(A+I)X=\begin{bmatrix} 3 & 4 & -3 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x+4y-3z=0, y=0 \Rightarrow x=z, y=0$$

$$X_1=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So, now from here on, I can call it X_1 and this is my eigenvector. So, from here no problem the algebraic multiplicity of $\lambda=-1$ is equal to geometric multiplicity of $\lambda=-1$. So, no problem in this case we get this value.

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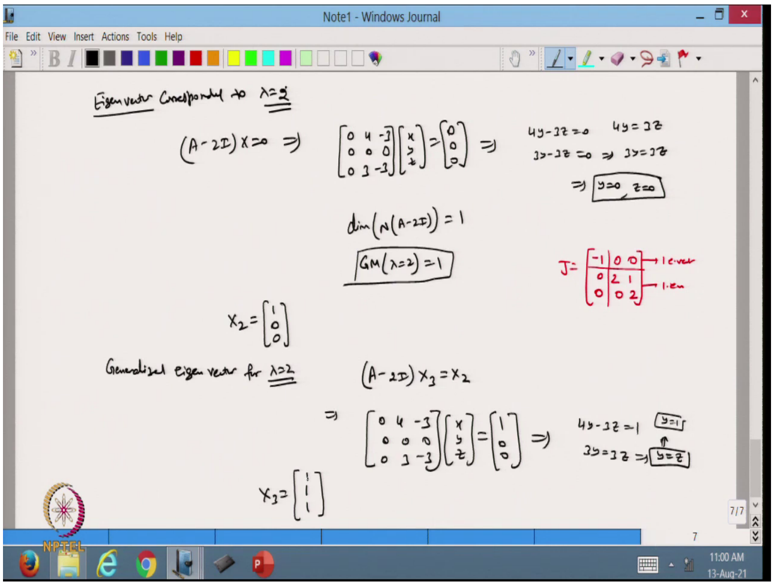

Now, let us find out the eigenvector corresponding to corresponding to $\lambda = 2$. So, in this case we have $(A-2I)X = 0$. So, let us take this one. So, from here if you see I will get

$$(A-2I)X = \begin{bmatrix} 0 & 4 & -3 \\ 0 & 0 & 0 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4y - 3z = 0, 3y - 3z = 0 => 4y = 3z, 3y = 3z$$

$$=> y = 0, z = 0$$

Now, if only one free variable is x. So, from here I can say that the dimension of the eigenspace is null space of (A-2I) = 1.

So, the geometric multiplicity of $\lambda = 2$ is 1. So, I will get only one eigenvector. So, I can choose that eigenvector as my X_2 and in this case my y =0 and z =0. So, I can take x as (1 0 0) and definitely this is linearly independent of (1 0 1). So, this is another eigenvector. Now from here we now need to find the generalized eigenvector. So, in that case I can find it directly. So, the generalized eigenvector for λ= 2. So, in this case we will take |A - λ I|. So, that is 2 basically.

So, I can take A- 2I some X. So, it is suppose X_2. So, I need to find X_3. So, I can write $(A-2I)X_3 = X_2$. So, this is the condition we need to find because in this case if you see my Jordan block will be of this form because that we can write very easily. So, if I have taken minus 1 first and then we have taken 2 and then 2. So, for minus 1 we are able to get 1 eigenvector and for this one we are able to get only 1 eigenvector.

So, this will be (0 0 0). So, that will be my Jordan block in this case. So, 1 eigenvector and 1 eigenvector and from here if you see I will get my X_3 this one and X_2 coming from here. Now from this one I will get |A- 2I|. So

$$(A-2I)X = \begin{bmatrix} 0 & 4 & -3 \\ 0 & 0 & 0 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$4y-3z=1, 3y-3z=0 \Rightarrow 4y=1+3z, 3y=3z$$

$$\Rightarrow y=1, z=1$$

And if it is y =1 then z =1. So, from here I can choose my X_3. So, this vector I can choose X_3. So, from this one my y =1, my z = 1 and I have only one vector, one free variable and that should be linearly independent of both X_1 and X_2. So, if you see X_2 is 0 0 here and X_1 is 1 0 1. So, I can take no problem. I may be able to take 1 third vector. So, that will be linearly independent.

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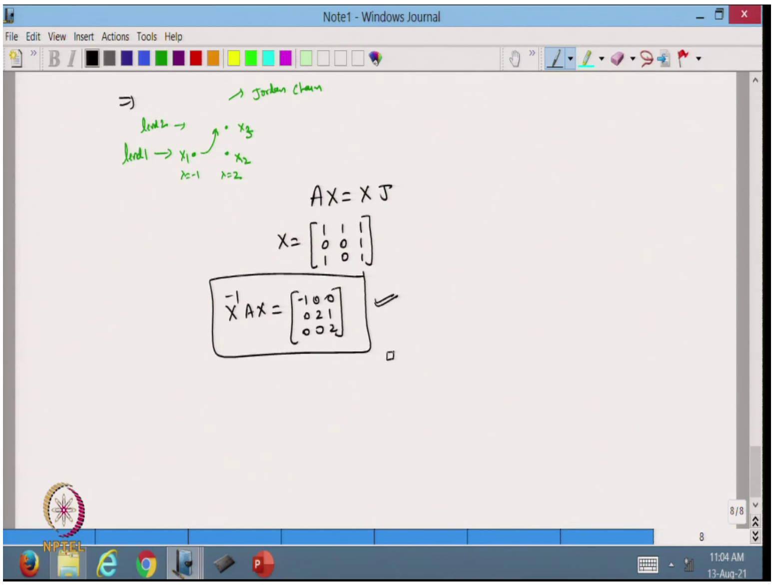

So, from here I will get my. So, now, I can make the chain. So, at level 1 I will get two eigenvectors that is corresponding to lambda minus 1 and this is corresponding to $\lambda = 2$. And after that I will get my level 2 and I will get my X_3. So, that gives my X_2 and this gives my X_1.

So, my so, this is my Jordan chain. So, I have to go in this direction. So, in this case my AX = X J. So, X is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

Because it should not belong to the null space of this one. So, I have taken 1 1 1 and let us see this one see if I take it should not belong to this. So, if I put 1. So, it is 1. So, never 0 this ok so, it is ok. So, this is my X and in this case if you find $X^{-1}AX = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ and that is the answer to this one.

So, this way we should be able to verify that this is my Jordan form for the given matrix A. So, this way we are able to find all the generalized eigenvectors and then we are able to write our Jordan matrix. So, this matrix is the Jordan matrix and this is upper triangular and you can see that it has the eigenvalue (-1,2,2). So, we can say that this is a similarity matrix and A and J are similar matrices. I mean the same eigenvalue.

So, this way we can write any matrix which is diagonalizable or not diagonalizable because if the matrix is diagonalizable then the J will be just the diagonal matrix, otherwise it will be in the Jordan canonical form. So, any matrix we can write in the Jordan canonical form based on the given eigenvalues. So, that is the way and we stop here.

So, in the lecture today we discussed how we can write a Jordan canonical form for a given matrix and then we also discussed how the Jordan chain is involved to write down the similarity matrix that we have represented here with the capital X. So, this way we can write the Jordan form for any square matrix A. So, I hope that you have enjoyed this lecture.

Thanks for watching.