Matrix Computation and its applications
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Lecture - 52
Jordan - Canonical form

Hello viewers, welcome back to the course on Matrix Computation and its application. So, today we are going to discuss a very important topic i.e Jordan - Canonical form. So, let us start with that.

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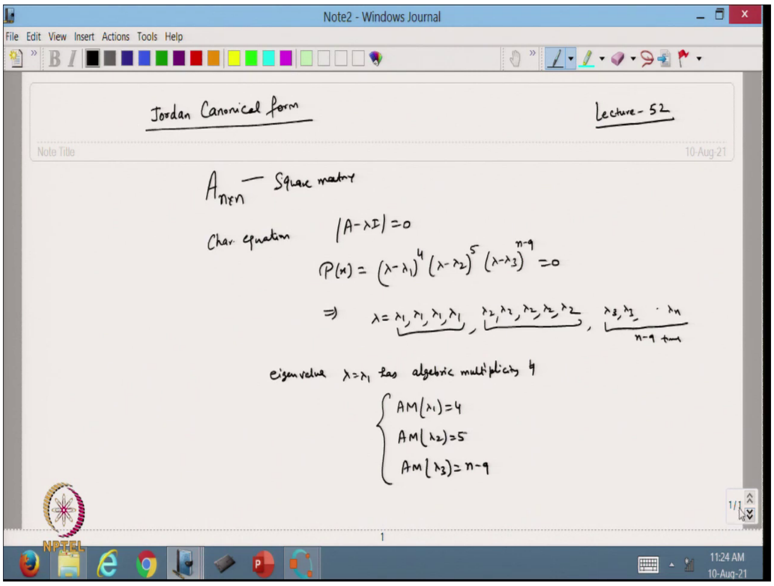

Now suppose I have a matrix A; that is of order $n * n$. So, it is a square matrix and then we can find out its eigenvalue and eigenvectors. So, I can define its characteristic equation. So, the characteristic equation is represented by taking the $|A - \lambda I| = 0$ determinant equal to 0. So, that is the characteristic equation.

A_{n*n} square matrix then :
Characteristic equation is defined as:

$$|A - \lambda I| = 0 \quad P(x) = (\lambda - \lambda_1)^4(\lambda - \lambda_2)^5(\lambda - \lambda_3)^{n-9} = 0 \quad \Rightarrow \lambda = \lambda_1, \lambda_1, \lambda_1, \lambda_1 \ \Rightarrow, \lambda_2, \lambda_2, \lambda_2, \lambda_2, \lambda_2, , \lambda_3, \lambda_3, \ldots \lambda_n$$

eigenvalue $\lambda = \lambda_1$ res algebric muliplicits 4

$$\{AM(\lambda_1) = 4 \; AM(\lambda_2) = 5 \; AM(\lambda_3) = n - 9$$

So, in this case I will say that the. So, these are the eigenvalues. So, I can say that the eigenvalue λ is equal to lambda 1 has algebraic multiplicity has algebraic multiplicity 4 because it is 4 times it is repeating or in the short form I can write that algebraic multiplicity just for the short form for lambda 1 is 4. Similarly, I can write down the algebraic multiplicity of lambda 2 is 5 and algebraic multiplicity of lambda 3 is n minus 9. So, we can write the algebraic multiplicity. Now, I want to find its eigenvectors.

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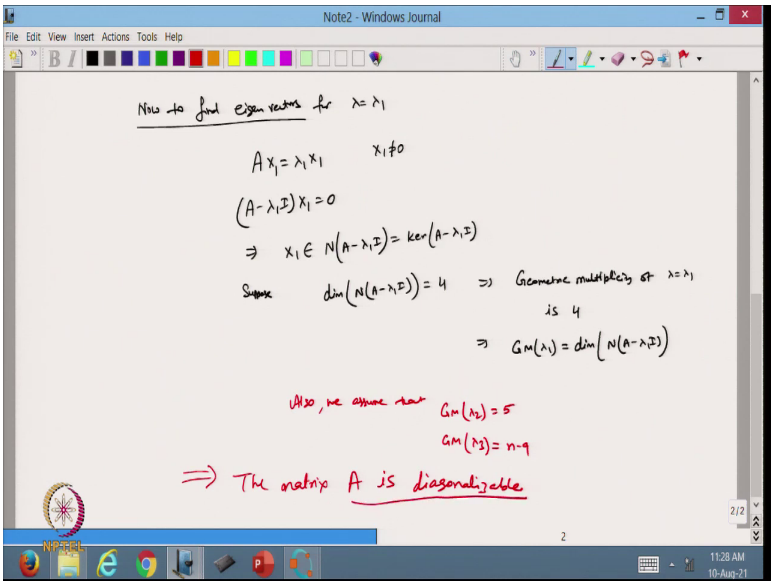

So, now to find eigenvectors, what I need to do is that. So, let us find the eigenvectors. So, first I take for lambda is equal to lambda 1. So, what I need to do is that I will take matrix A and then I need to find eigenvector x corresponding to lambda 1. I should write x 1 ok. So, and x 1 is not 0 that we already know.

Now to find Eigenvectors for $\lambda = \lambda_1$

$$Ax_1 = \lambda_1 x_1 \qquad\qquad x_1 \neq 0$$
$$(A - \lambda_1 I)x_1 = 0$$
$$\Rightarrow x_1 \in N(A - \lambda_1 I) = ker(A - \lambda_1 I)$$

Suppose $dim(N(A - \lambda_1 I)) = 4 \Rightarrow$ Geometric multiplicity of $\lambda = \lambda_1$ is 4

$$\Rightarrow GM(\lambda_1) = dim(N(A - \lambda_1 I))$$

$$Also,\ we\ have\ GM(\lambda_2) = 5\ GM(\lambda_3) = n - 9$$

$\Rightarrow$ The matrix A is diagonalizable

So, basically I need to solve $A(A - \lambda_1 I)x_1 = 0$ and from here I can say that I need to find out the value of this solution it means I need to find the x_1 that belongs to the null space of $(A - \lambda_1 I)$ or I can also write kernel of $(A - \lambda_1 I)$.

Now, suppose the dimension of null space corresponding to the eigenvalue λ_1 is 4 suppose I have this 4 its dimension is 4 then I say that geometric multiplicity of λ_1 is equal to 4. So, it means that the geometric multiplicity of eigenvalue λ_1 is equal to the dimension of the null space of $A - \lambda_1 I$. So, now, here I can see that the algebraic multiplicity of λ_1 was 4 and I also assume that its geometrical multiplicity is also 4.

We also assume that also we assume that geometric multiplicity of λ_2 is coming same as the algebraic multiplicity that is 5 and geometric multiplicity of λ_3 is also coming same as n -9. Now, from here you see that the algebraic multiplicity of the eigenvalue is same as the geometric multiplicity of the eigenvalues. So, in this form in this case we say that the matrix A is diagonalizable. And, how it is diagonalizable?

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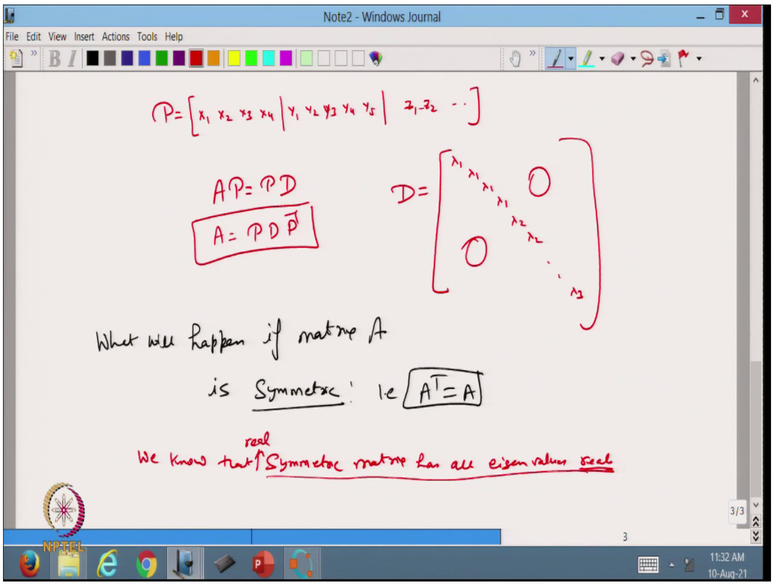

So, I can define the matrix P made up of eigenvectors. So, corresponding to the lambda 1 we have geometric multiplicity 4. So, it suppose that corresponding to lambda 1 I will get for eigenvectors they are linearly dependent.

$$P = [x_1 x_2 x_3 x_4 | y_1 y_2 y_3 y_4 y_5 | z_1 z_2 ....]$$

$$AP = PD$$

$$A = PDP^{-1}$$

What will happen if matrix A is symmetric: i.e $A^T = A$

We know that real symmetric matrix has all eigen values real.

So, suppose I have maybe I just write here maybe I can write $x_1 x_2 x_3 x_4$ and then I can write corresponding to lambda 2 I can write $y_1 y_2 y_3 y_4 y_5$ and then corresponding to $z_1 z_2$ like this one. So, this is my matrix P and all the column vectors are linearly independent.

So, I can define it is P^{-1} and from there I can define my AP will be equal to PD and I can write my PDP^{-1}, where D is my diagonal matrix which is having the eigenvalue at the diagonals. So, $l\lambda_1, \lambda_1, \lambda_1, \lambda_2$....... like this one and all other elements are 0. So, this way it is I can say that my matrix A is diagonalizable.

Now, so, up to here it is ok, now I see another thing I will see that what will happen if matrix A Matrix A is symmetric? So, in this case suppose I have the matrix symmetric that is I have $A^T = A$. So, what is going to happen in this case that when our matrix A is a symmetric matrix, then the everything is the things will change here.

Now, we know that that symmetric matrix I am talking about symmetric when I say symmetric means it is real because otherwise it will become the Hermitian matrix when it is a complex. So, I am talking about the real matrix. So, I am talking about the real symmetric matrix has all real eigenvalues. We have already proved in my earlier videos that symmetric matrix has all the eigenvalues that are real.

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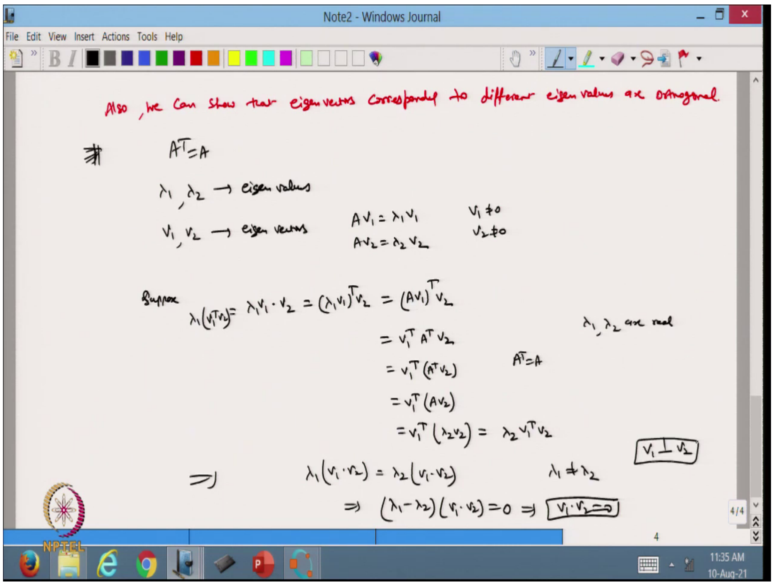

Also, We can show that Eigenvector corresponding to different Eigenvalues are orthogonal.

$$A^T = A$$

$$\lambda_1, \lambda_2 \rightarrow eigen\ value$$

$$v_1, v_2 \rightarrow eigen\ vector$$

$$Av_1 = \lambda_1 v_1 \qquad v_1 \neq 0$$

$$Av_2 = \lambda_2 v_2 \qquad v_1 \neq 0$$

Suppose

$$\lambda_1\left(v_1^T v_2\right) = \lambda_1 v_1 \cdot v_2 = \left(\lambda_1 v_1\right)^{\mathrm{T}} v_2 \quad = \left(Av_1\right)^{\mathrm{T}} v_2 \quad = v_1^{\mathrm{T}} A^{\mathrm{T}} v_2 \quad = v_1^{\mathrm{T}}\left(A^{\mathrm{T}} v_2\right) A_1, \lambda_2 \quad = v_1^{\mathrm{T}}\left(Av_2\right) \quad = v_1^{\mathrm{T}}\left(\lambda_2 v\right.$$

Also, I can show that also we can show that eigenvectors corresponding to different eigenvalues are orthogonal so, that we can show. So, how we can show that? So, let us do this one that I have a $A^T = A$ that is my matrix and let λ_1 and λ_2 are the eigenvalues ok. So, then corresponding I have suppose let v_1 and v_2 are corresponding eigenvectors.

And this is also I can write as $\lambda_1\left(v_1^T v_2\right)$. So, from here I can write that $\lambda_1 v_1 \cdot v_2$ is equal to $\left(\lambda_1 v_1\right)^{\mathrm{T}} v_2$ which implies that $\left(\lambda_1 - \lambda_2\right)\left(v_1 \cdot v_2\right)$ is 0

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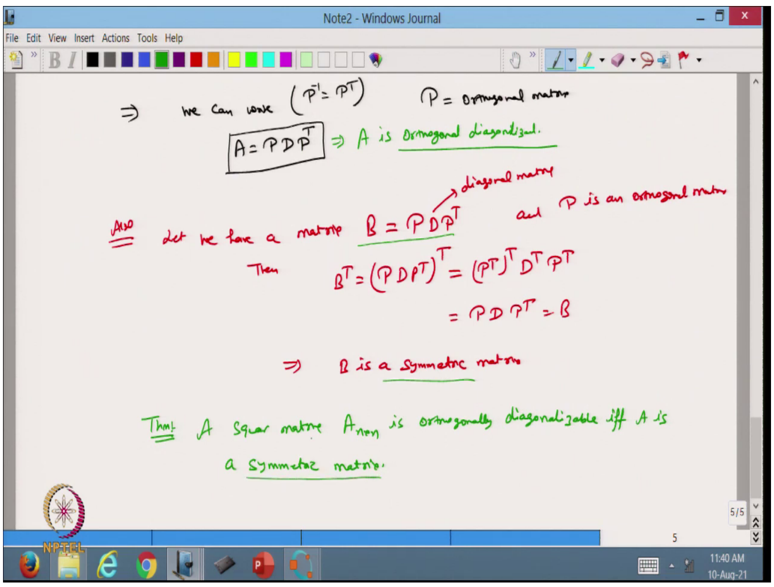

So, from here I can say that in that case we can write A as P D P transpose because instead of now what I do? I write a matrix P made up of eigenvectors and that eigenvectors are orthogonal to each other. So, in that case I will get my P inverse is equal to P transpose and the matrix P is an orthogonal matrix.

$\Rightarrow$ we can use $\left(P^{-1} = P^{T}\right) \rightarrow P$ = orthogonal matrix, $A = PDP^{\top} \Rightarrow A$ is orthogonally diagonozable.

Also Let we have a matrix $B = PDP^{\top}$ and P is an orthogonal matrix

Then

$$B^{\top} = \left(PDP^{\top}\right)^{\top} = \left(P^{\top}\right)^{\top} D^{\top} P^{\top} = PDP^{\top} = B$$

$\Rightarrow$ B is symmetric matrix.

Theorem: A square matrix A_{n*n} is orthogonal diagonalizable iff A is a symmetric matrix.

So, from here you can see that now this matrix A is diagonalizable and in this the corresponding matrix P will be orthogonal. So, from here I can write like this one in this case also let we have a matrix B which can be written as some $\left(PDP^{\top}\right)$let us write this one, then I just want to check what is $B^{\top}$.

So, it will be $(PDP)^{\top}$ and that is equal to $\left(P^{\top}\right)^{\top} D^{\top} P^{\top}$, where D is the diagonal matrix and P is an orthogonal matrix let write this. So, from here I can write P and D the diagonal matrix of transpose is same and this will be P transpose and you see from here that is equal to B so, which implies that B is a symmetric matrix.

So, now from here I can write one important observation or a theorem that an or I can write a matrix A square matrix is orthogonally diagonalizable if and only if A is a symmetric matrix, it means if A is symmetric matrix then we already know that this is diagonalizable orthogonally diagonalizable.

And here we have seen that if a matrix B is orthogonally diagonalizable then its symmetric by this way and whatever we have done here it is called the matrix A is orthogonal diagonalized. So, matrix A is a orthogonal diagonalized or orthogonally diagonalized by this way it means that the matrix P is orthogonal.

So, this is the theorem important theorem that if a matrix is orthogonally diagonalizable then definitely it should be symmetric and if a matrix is symmetric matrix then it definitely is orthogonally diagonalizable. So, this is a very important results we have.

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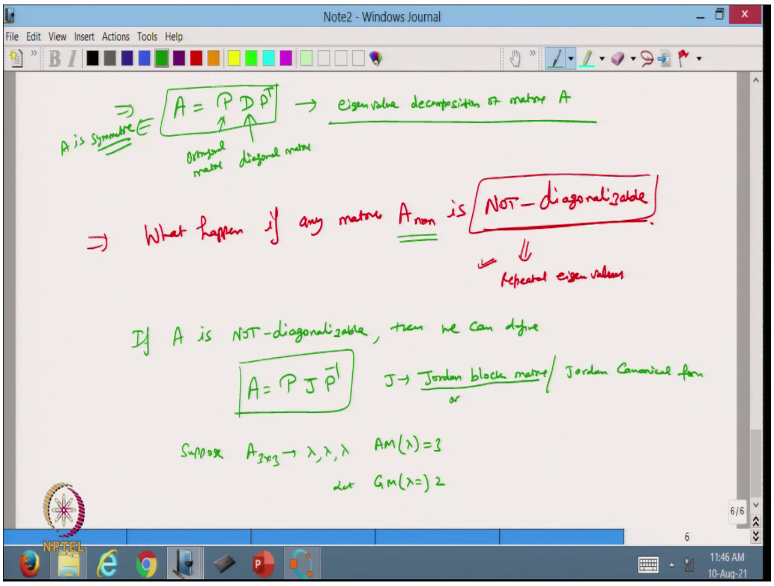

Now, so, from here we have seen that if a matrix is symmetric and if I am able to write $P\ D\ P^{T}$ then I can also write this as eigenvalue decomposition of matrix A that I am able to write the matrix A. So, A is a symmetry matrix of course, we know that. So, if I write in this form where P is orthogonal matrix, D is the diagonal matrix, having the eigenvalues at the diagonal of D.

And if you are able to write in this form then we say that the matrix A is a diagonal value decomposition of the matrix A. So, this we can do for matrix which is symmetry matrix. We have also seen in the starting that if the algebraic multiplicity of the eigenvalue is same as the geometric multiplicity then the matrix A , A is always diagonalizable that we have already seen.

Now, t what happen if any matrix A that is $n\times n$ is not diagonalizable. So, this type of things happens when we have repeated eigenvalues, because in the starting we have shown that this eigenvalue is repeating 4 times, this eigenvalue repeating 5 times and this is $n\ -\ 9$ times. So, whenever we have repeated eigenvalues then it may diagonalizable or may not. So, that is one of the very important question about the matrix.

So, when we have a distinct eigenvalue then the corresponding eigenvector will be linearly independent and that will be diagonalizable. So, now, we want to see that what is going to happen now in this case. So, from here now we can say that if A is here my A is not symmetric, but here it is any arbitrary matrix. So, this is my any arbitrary matrix A.

Now, if A is not diagonalizable then we can define my matrix A that is equal to $PJ\,P^{-1}$ that we can do, where my J is called Jordan block matrix or we also called Jordan canonical form. So, this matrix J is the Jordan block we call it. So, J is not a diagonal matrix, but J has the Jordan block of this type suppose. So, this is we can do for here. So, let us see that suppose A is just 3×3 matrix and suppose its eigenvalue is $\lambda\ \ \lambda\,\lambda$. So, this is my repeated eigenvalue here.

And. So, I can define at the algebra multiplicity of λ is 3 and let geometric multiplicity of λ is 2. So, I just let us take the all the cases. So, let us take these cases on the next one.

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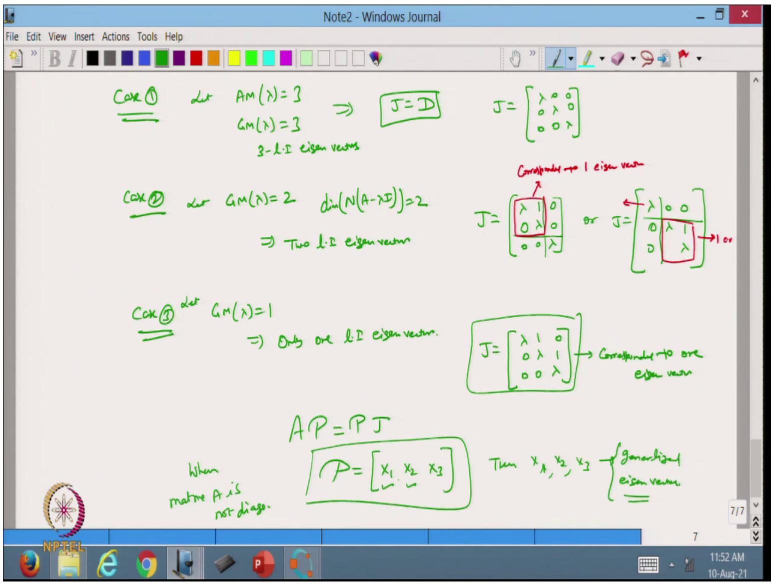

So, case 1; so, I am writing like this one. So, 3 eigenvalue repeated eigenvalue. So, case 1 let algebraic multiplicity of λ is 3 and geometric multiplicity of λ is also 3. So, in this case my J will be D it is just the diagonal matrix.

$$J = \lambda\, 0\, 0\, 0\, \lambda\, 0\, 0\, 0\, \lambda$$

Case 2, let geometric multicity of λ is 2. It means that here that the null space of $A - \lambda I$ the dimension is 2 it means we are able to find only 2 linearly independent eigenvectors. So, which implies that two linearly independent eigenvectors here, 3 linearly independent eigenvectors. So, here I am able to find only 2 linearly independent eigenvectors.

So, in this case I can write my Jacobian.

$$J = \lambda\, 1\, 0\, 0\, \lambda\, 0\, 0\, 0\, \lambda \quad or \quad J = \lambda\, 0\, 0\, 0\, \lambda\, 1\, 0\, 0\, \lambda$$

Case-3:

Now I will take the case 3 geometric multiplicity of λ is suppose 1. So, let geometric multiple of λ is equal to 1. So, in this case we have only 1 linearly independent eigenvector. So, in that case I will get my matrix as

$$J = \lambda\, 1\, 0\, 0\, \lambda\, 1\, 0\, 0\, \lambda$$

So, based on this one I can write that my matrix AP is equal to PJ, where P is made up of the vectors $[x_1\, x_2\, x_3]$ in this case. Now, you see that in this case if I take the case 1 then this $[x_1\, x_2\, x_3]$ will be linearly independent no problem, if you take the case 2 in this case only I have able to get only 2 linearly independent eigenvectors.

So, suppose this is the eigenvectors this and this and the x_3 I will find from the from a method. So, in this case I can say that when matrix A is not diagonalizable, then I have AP on this form and then $[x_1\ x_2\ x_3]$ are called generalized eigenvectors. So, these are called the generalized eigenvectors.

So, I will see that which one is generalized eigenvector and which one is not the generalized vector. So, I just write the definition first.

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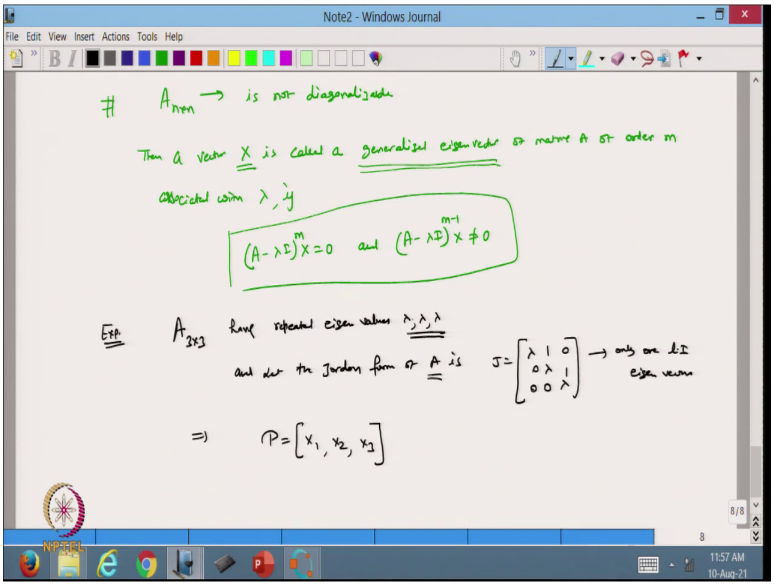

Now, suppose I have a matrix A that is $n \times n$ and this is not diagonalizable then in that case I want to find then suppose I have x. So, x suppose x then suppose A x is equal to lambda x is not diagonalizable. Then a vector x is called a generalized eigenvector of matrix A of order m if order m associated with eigenvalue λ.

If $(A - \lambda I)^m x$ is 0 and $(A - \lambda I)^{m-1} x$ is not equal to 0. So, then only I will say that this vector x is a generalized eigenvector and this we find when the matrices are not diagonalizable. So, now we can see that how we can define this one. So, now, from here I will tell you that how we can find the Jordan form of the given matrix. So, I will just take the example.

Example: $A_{3\times 3}$ hane repeated Eigen valuen λ, λ, λ

$$and\ we\ have\ Jordon\ form\ of\ \underline{A}\ is\ J = [\lambda\ 1\ 0\ 0\ \lambda\ 1\ 0\ 0\ \lambda] \quad \Rightarrow P = \left[x_1, x_2, x_3\right]$$

Now what I am going to do is that now from here suppose my get my matrix P. So, P is made up of suppose this eigenvector $[x_1, x_2, x_3]$ of course, it is a generalized eigenvector. So, let us I write like this one and from here I get my matrix.

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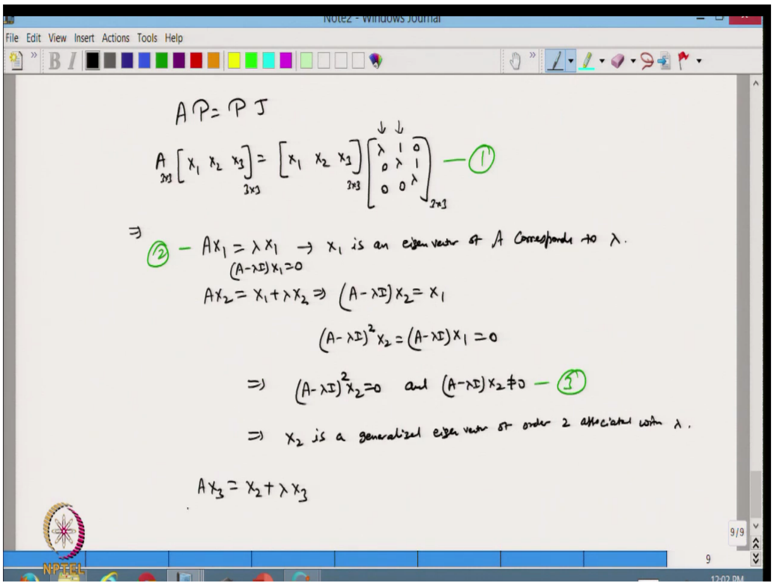

AP is equal to PJ. So, I can write from here that this is my A and this is suppose$[x_1, x_2, x_3]$ all the eigenvector written as a column vector and this I can write as $[x_1, x_2, x_3]$ and this J

$$J = [\lambda\, 1\, 0\, 0\, \lambda\, 1\, 0\, 0\, \lambda\,]$$

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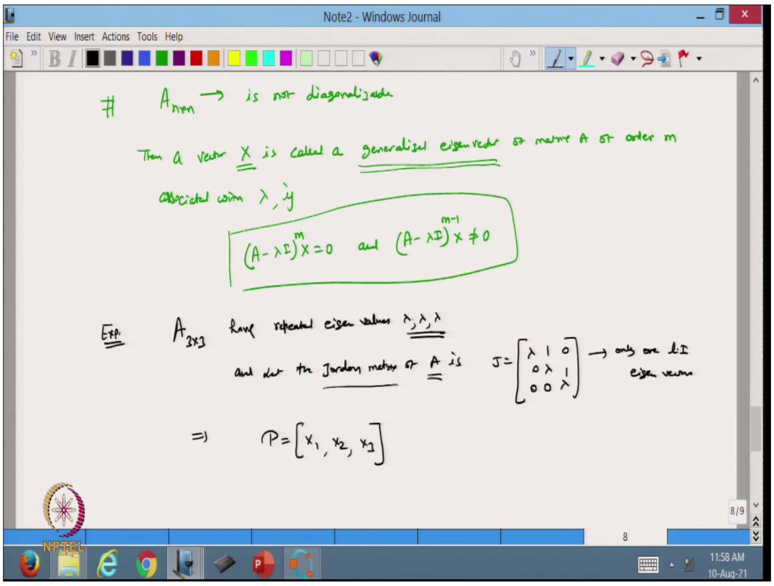

I can write the Jordan matrix here I can see from here that $A\,x_1$ now can be written as I am just taking the first column multiplied by A, again here also I am writing because A is 3*3 it will be 3*3. This is also 3 cross 3. Now, Ax_1 I can write as multiplying this column. So, that gives me lambda x 1, now from here I can say that x 1 is a is an eigenvector of A. This is the eigenvector of A corresponding to λ eigenvalue.

Now I can write Ax_2 here multiplying the second column and here also I am multiplying this second column with this matrix. So, it will give you x 1 plus lambda x 2 and from here I can write that A minus lambda I x 2 is equal to x 1. Now this is we are able to write, now what I do is that I will multiply by lambda I both side. So, it will get lambda square x 2 A minus lambda I x 1 I just multiplied with this matrix.

And, $(A - \lambda I)\ x$ this I can write from here that $(A - \lambda I)\ x_1$ is equal to 0. So, from here I can get this value 0. So, from here I can write that $(A - \lambda I)^2 x_2$ is 0 and $(A - \lambda I)^{m1} x_2$ is not 0, that we have already seen. So, from here I can say that x_2 is a generalized eigenvector of order 2 associated with the associated with λ.

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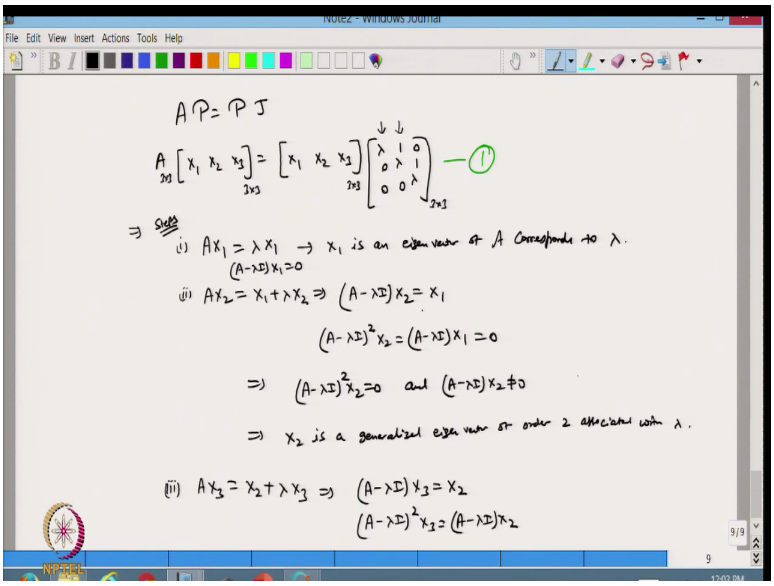

So, I can write steps first step, second step and then I can write third step. So, third step is this one, now from here you will see that I can write $(A - \lambda I)\ x_3$ is equal to x_2 . Now, so, from here you will see that $(A - \lambda I)^1 x_2$ is equal to $x_3(A - \lambda I)^1$ just multiply by both side and I will get $(A - \lambda I)^1 x_2$, because I am just multiplying by $(A - \lambda I)^1$ on the both side. Now $(A - \lambda I)^1 x_2$ is not 0. So, again I will write this one.

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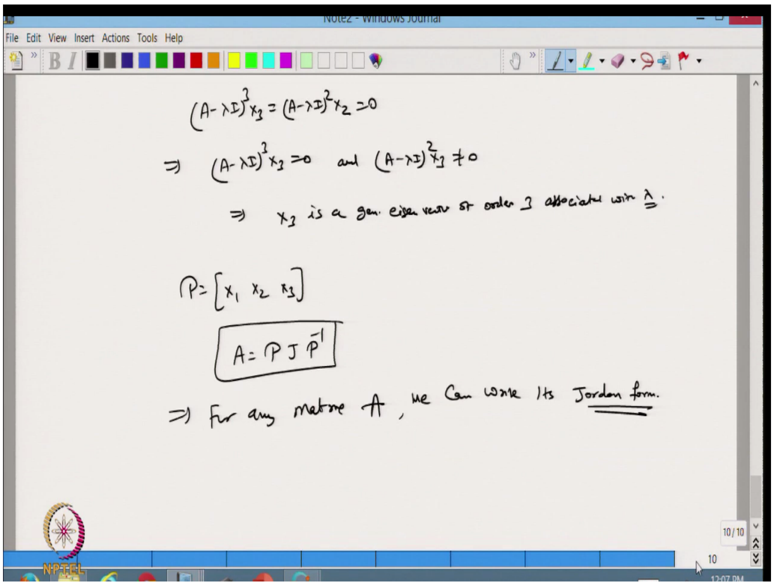

So, also I will write $(A - \lambda I)^3$ multiplying both side by this one. So, I can write $(A - \lambda I)^2 x_2$ and now $(A - \lambda I)^2 x$ is 0. So, this gives me 0. So, from here I can say that $(A - \lambda I)^3 x_3$ is 0 and $(A - \lambda I)^2 x_3 \neq 0$.

So, this way we are able to find $x_1\ x_2\ and\ x_3$ and then I can write my P will be $x_1\ x_2\ x_3$ and then based on this one I can define my matrix A that will be $P\ J\ P^{-1}$. So, in this case the thing is that the P is invertible, but P is not the orthogonal matrix. So, it means that I can write for any matrix I can write in this form whether it is diagonalizable or not, but we always can write the Jordan form of a given matrix.

So, this way we can say that for any matrix A, we can write it is we can write its Jordan form and this is my Jordan form of the given matrix A. So, it means that I start with the matrix and I showed that if it is a diagonalizable it is ok, but if it is not diagonalizable then what we can do.

Then we have seen that the things become simpler when the matrix P becomes the diagonal matrix or orthogonal matrix and that is only possible when we have the matrix A is a symmetric matrix. So, if a matrix is symmetric then we know that it is diagonally orthogonally diagonalizable and with that one we are able to find this $P\,T$ in instead of P^{-1}. So, that is the one of the benefit.

And, then we have also seen here that if the matrix is not diagonalizable then at least we can write its Jordan form and this is the way we have shown that how we can write the different type of Jordan forms for depending upon the geometric multiplicity of the eigenvalues. And this is the way this is the procedure that we can find out the generalized eigenvector for the corresponding eigenvalue lambda.

So, maybe in the next lecture we will discuss some example and let you know that how we can define the generalized eigenvector for a given matrix. So, we will stop here. So, in the lecture; today we have discussed about that which type of matrix are diagonalizable and when the eigenvalues are repeating, then we know that matrix may or may not be diagonalizable. And then we have defined another form that is called the Jordan form.

And we have shown that any matrix A can be written in the form of a Jordan blocks and so, in the next lecture we will continue with this one and we will discuss some example based on that. So, I hope that you have enjoyed these one thanks for watching. Thanks very much.