Matrix Computation and its applications
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Lecture - 51
Continued…

Hello, viewers. So, welcome back to the course on Matrix Computation and its Application. So, in the previous lecture, we have discussed about the Least Square Approximation and in this lecture, we also continue with that one.

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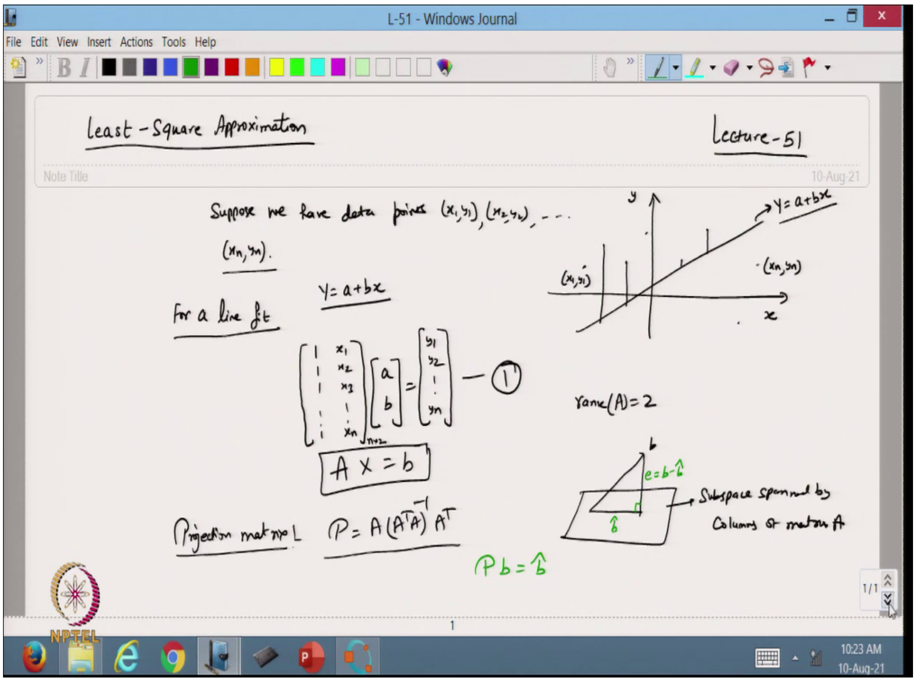

So, in the previous lecture we have discussed that if we have some data points so, suppose we have some data points and these data points are spread over the x – y plane. So, we have taken n number of data points starting from here x_1, y_1 and this is x_n, y_n and then we have seen that if we want to fit a line. So, we need a line that is closest to all these points and we have found the error and minimizing this error.

Least-Square Aproximaton

Suppose we have data points $(x_1, y_2), (x_2, y_n), ...(x_n, y_n)$ for a line fo $y = a + bx$

$$\left[1\ x_1\ 1\ x_2\ 1\ x_2\ \vdots\ 1\ 1\ x_n\right][a\ b]_{n*2} = \left[y_1\ y_2\ .\ y_n\right] - (1)$$

$$Ax = b$$

$$rank(A) = 2$$

Projection matrix L : $$P = A\left(A^{\top}A\right)^{-1}A^{\top}$$

$$Pb = b\hat{}$$

So, this is the line I have taken and then we came across the matrix A and this is x and this is I can write as y. Now, also we have seen that based on this one we found that this system is inconsistent and then we have talked about the projection matrix. Now, so, based on this we have discussed about the projection matrix that is P and this is $A\left(A^{\top}A\right)^{-1}A^{\top}$. So, this is my projection matrix which project the or maybe I should take this as a b.

So, we have seen that now P is of $n * 2$. Now, we have seen that the rank of A is 2 because they are linearly independent both the columns and then we have taken the projections. So, suppose this is the plane suppose I take a subspace basically just showing like this one. So, it is a subspace spanned by columns of matrix A. So, this is the columns and then and let we have a vector.

So, suppose this vector is not in the linear combination or the column space of A. So, suppose this is my vector b and then we have shown that we are taking this projection of this on this subspace spanned by the columns of the matrix A. So, in this case it is dimension is 2 basically, then we have taken this projection.

So, this is I have taken as a $b\hat{}$ and this is the error that is $b - b\hat{}$and this is right angle triangle. So, and then we have seen that if I take projection matrix P applying on b I get b hat. So, this one we have se

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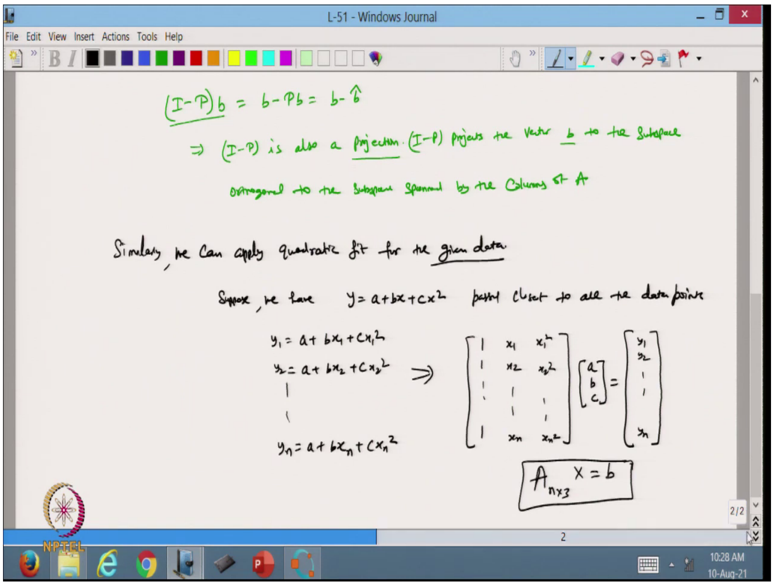

Now, I want to see what will happen if I want to take $I - P$. So, if I take $I - P$ and apply on the vector b which is the b vector I have taken on the right hand side. Then I can write this as $b - Pb$ and this can be written as b minus b hat. So, if you see from here it is $b - b\hat{}$hat.

$$(I - P)b = b - Pb = b - b\hat{}$$

⇒$(I - P)$ is also a Projection (I-P) Projects on the vector b to the subspace orthogonal to subspace spanned by the column of A
Similarly, we can apply quadratic fit for the given data,
Suppose, we have $y = a + bx + cx^2$ passing close to the date points

$$y_1 = a + bx_1 + cx_1^2\ y_2 = a + bx_2 + cx_2^2\ 1\ 1\ y_n = a + bx_n + cx_n^2 \Rightarrow \left[1\ x_1\ x_1^2\ 1\ x_2\ x_2^2\ 1\ 1\ 1\ 1\ x_n\ x_2\right][a\ b\ c\ ] =$$

$$A_{n*3}X = b$$

So, from here I can say that $I - P$ is also a projection and so, $I - P$ projects the vector b to the subspace orthogonal to the subspace spanned by the columns of a because this is the subspace spanned by the columns of a and I take the projection over this subspace. So, that is a $b\hat{}$.

And, then we take a and then we take the error terms $b - b\hat{}$. So, it means this is also a projection and it project the vector b to the subspace that is orthogonal to this subspace that is spanned by the columns of A. Now, so, let us a we talk about. So, we have seen this things for the linear fit. Now, similarly we can apply quadratic fit for the given data.

In this case, I will assume that that suppose we have y_1 is equal to $a + bx_1 + cx_1^2$ and the data points are this one. So, this is suppose I take this quadratic equation and I assume that passing through passing closest to all the data point.

So, in this case I need to find the value of a b c and on the right hand side I have the values of y s that is given to me this way. So, also I can say that this in this case I have a matrix that is of order n*3 and this on the right hand side. So, in this case I get the same system only condition is that the matrix now have the 3 columns made up of one x_i and x_i^2 .

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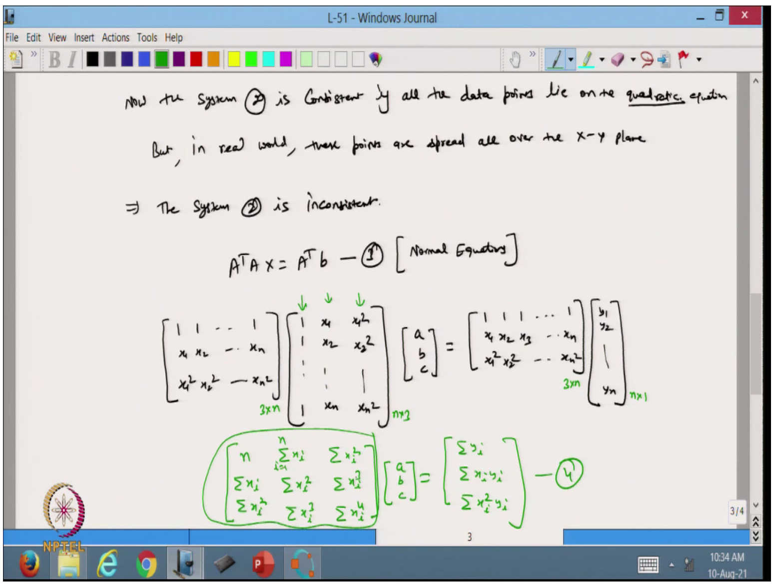

Now, so, this is my equation I have. Now, I know that this system. So, I can say that now just I give the this 2. Now, the system 2 is consistent if all the data points lie on the quadratic. So, if all the data point lies on this quadratic equation this one then the system will be consistent and we get the solution unique solution. But, so, if the point lies on the quadratic equation, but in real world these points are spread all over the $X - Y$ plane. So, this points are as we have seen that this is spread over the $X - Y$ plane.

Now the system 2 is inconsistent

$$A^{T}Ax = A^{T}b \; - \; (1) \text{ [Normal Equation]}$$

$$\left[1\,1\,\cdots\,1\,x_1\,x_2\,\cdots\,x_n\,x_1^2\,x_2^2\,\cdots\,-x_n^2\right]_{3*n}\left[\downarrow\;\downarrow\;\downarrow\;1\,x_1\,x_1^2\,1\,x_2\,x_2^2\,.\,.\,.\,1\,x_n\,x_n^2\right]_{n*3}\left[a\,b\,c\;\right] = \left[1\,1\,1\;...\;1\,x_1\,x_2\,x_3\right.$$

$$(\left[n\,\Sigma_{i=1}^{n} n_i\,\Sigma x_i^2\,\Sigma x_i\,\Sigma x_i^2\,\Sigma x_i^3\,\Sigma x_i^2\,\Sigma x_i^3\,\Sigma x_i^4\right][a\,b\,c] = \left[\Sigma y_i\,\Sigma x_i y_i\,\Sigma x_i^2 y_i\right] - \;(4)$$

So, in that case we say that the system 2 is inconsistent and then we will find out the same way that we multiply by $A^{T}A$ and I will write $A^{T}Ax$. Now, if you see from here so, this is the equation number 3 and it is called the normal equations.

So, I can write this equation third 3 in this form. Now, if you start doing the multiplication of this matrix then you will see that it will multiply. So, from here I will get a matrix. So, this matrix is basically 3 cross n and this is n cross 3. So, I will get the 3 by 3 matrix. So, if it is I am taking this one. So, you will see that it is n number of times taking the summation. So, it will be n.

Then I am taking first column and this row then I am multiplying this first column with the second row and I will get summation of all $x_i's$ and then this one I will get the summation of all x_i^2. So, it will be summation $x_i\,\Sigma x_i^2$.

Similarly, I can take the summation. So, now, I am taking the third column. So, it would be x i square, x i cube and this will be summation x i power 4 and this is a b c. And on the right hand side this is again 3 cross n and this is n cross 1. So, I will get a vector of order 3 cross 1. So, I can write from here. So, I am taking this summation it. So, it will be summation of y i's because multiplying this one, it will be the summation x i's, y i's and summation x i square into y i and, this equation 2 I can write this as equation number 4.

So, if you see from here then I can write from here I multiply by this. So, n and you know that if you see from here then this matrix is a symmetric matrix this and this value a symmetric, this value is also symmetric.

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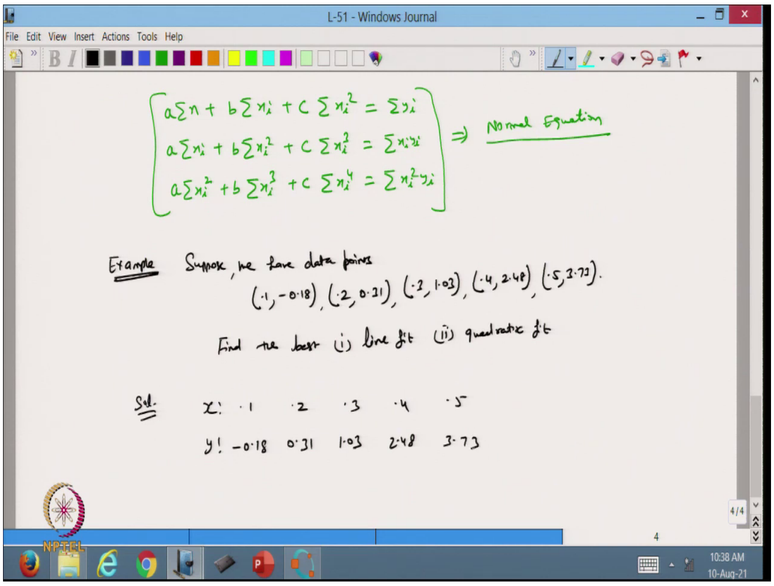

So, from here I can write

$$\left[a\sum n + b\sum x_i + c\sum x_i^2 = \sum y_i \; a\sum x_i + b\sum x_i^2 + c\sum x_i^2 = \sum x_i y \; a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4 = \sum x_i^2 y \right.$$

Example: Suppose, we have data points
$(0.1, -0.18), (0.2, 0.31), (0.3, 1.03), (0.4, 2.48), (0.5, 3.72).$

Find the best (i) line fit (ii) quadratic fit

$Sol.$ $\quad x{:}.1\;.2\;.3\;.4\;.5 \quad y: -0.18\; 0.31\; 1.03\; 2.48\; 3.73$

So, this is if you remember from the least square method what we can be drawn draw from the scientific computing or numeric analysis way, then if you remember then this is equal to the normal equation. So, these are the normal equation and then we can solve this one to find the value of a b c. So, this is the way we can write the normal equation through the linear algebra. Now, let us take one example.

So, in this I am taking one example and suppose, we have the data point that is given to me like this one 0.1 and this is minus 0.18, 0.2, 0.31, 0.3, 1.3, 0.4, 2.48 and the fifth point is 0.5, 3.73. So, these are the data points given to us. So, then a question is find the best first one line fit and the second one is quadratic fit. So, this is the question. Now, so, I can. So, my x is 0.1, 0.2, 0.3, 0.4, 0.5 and my y is minus 1.03, 2.48 and 3.73.

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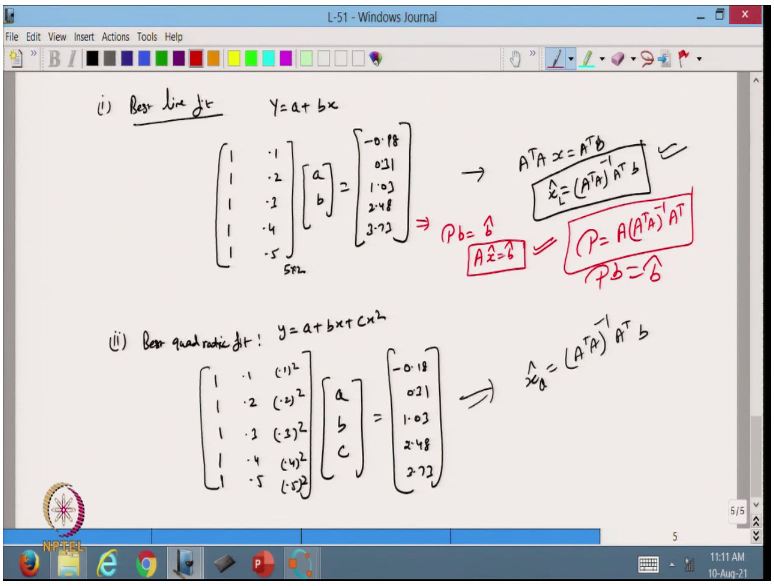

So, now if I take the line fit in the case 1 best line fit. So, in the because this is the least we have the error and that error is the least one. So, that is why it is the best fit and in this case I will choose the line Y is equal to a plus bx and then based on this one I will get the matrix that is 1 1 1 1 1 because five points are given to me and these x are 0.1, 0.2, 0.3, 0.4, 0.5 this is my a b and in the right hand side we have 0.18, 0.31, 1.03, 2.48 and 3.73.

(i) Best line fit $y = a + bx$

$[1\ .1\ 1\ .2\ 1\ .3\ 1\ .4\ 1\ .5\ ][a\ b\ ] = [-\ 0.18\ 0.31\ 1.03\ 2.48\ 3.72\ ] \Rightarrow\ \rightarrow A^{\top}Ax = A^{\top}D^{-1}A^{\top}B = \left(A^{\top}A\right)^{AB}$

$$Pb = Ax\hat{} = b\hat{}\hat{}$$

$$P = A\left(A^{\top}A\right)^{-1}A^{\top}\underbrace{}$$

(ii) Best quadratic fit : $y = a + bx + cx^2$

$\left[1\ .1\ (.1)^2\ 1\ 2\ (.2)^2\ 1\ .3\ (.3)^2\ 1\ .4\ (.4)^2\ 1\ .5\ (.5)^2\ \right][a\ b\ c\ ] = [-\ 0.18\ 021\ 1.03\ 2.48\ 2.73\ ] \Rightarrow\ x\hat{}_0 = \left(A^{\top}A\hat{}\right)^{-1}B$

So, if we solve it manually then it is going to take much time from here. So, we can solve this with the help of MATLAB or Octave.

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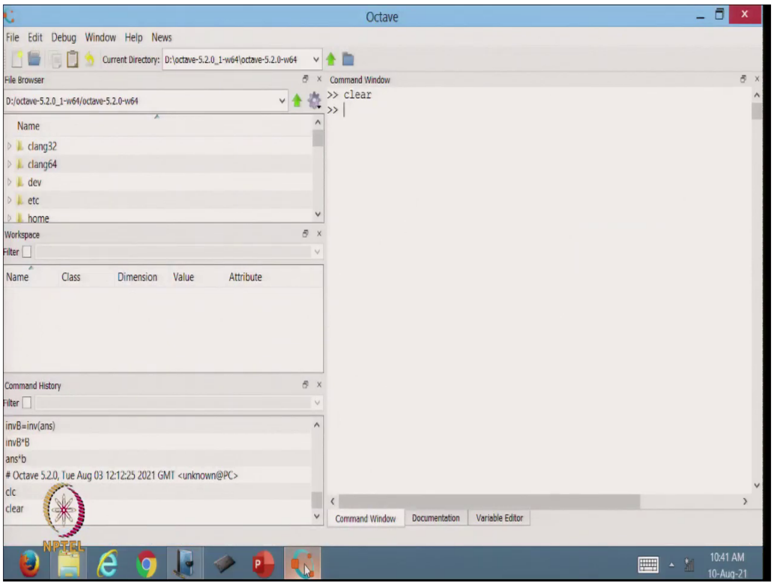

So, let us do with that one. So, first I will write a script for this.

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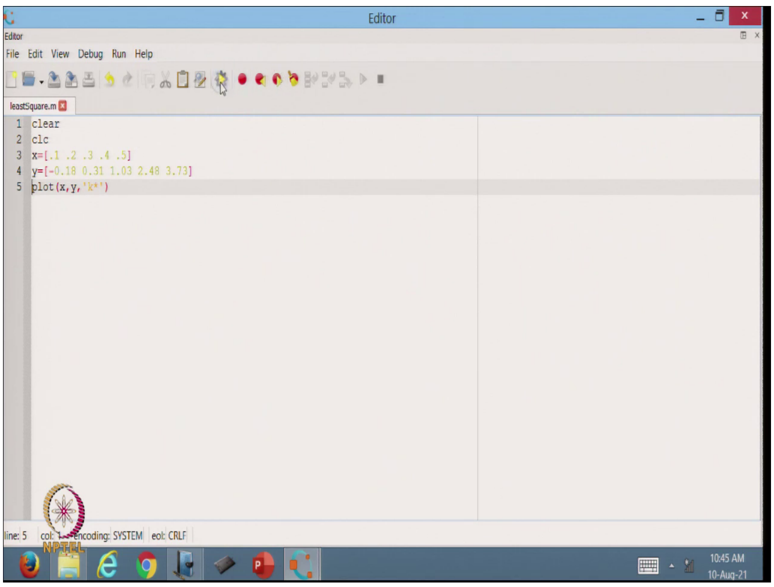

And, I will save this script as maybe I can put it on the desktop and I will write it as leastSquare. So, this is my script file. So, in this case so, first of all I want to write that I want to clear all this variable and then clc. Now, I need to define all my x points. So, I will just first take the x coordinates. So, x I can write as $0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$. This value now the y is given to us that $is\ -0.18, 0.31,\ 1.03,\ 2.48\ and\ 3.73$. So, these are the y coordinates we have taken. Now, whatever the points are given to us, now, I want to plot these points.

So, I will write the plot $x,\ y$. So, I just want to plot this one with points. So, I just write k means black star and this one. So, this is now if I suppose I just stop here and I run this one ok so, this is the change directory.

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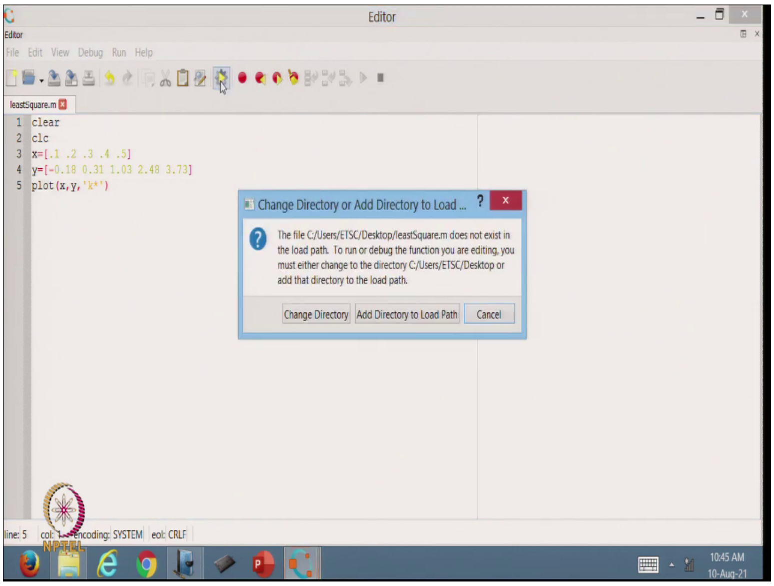

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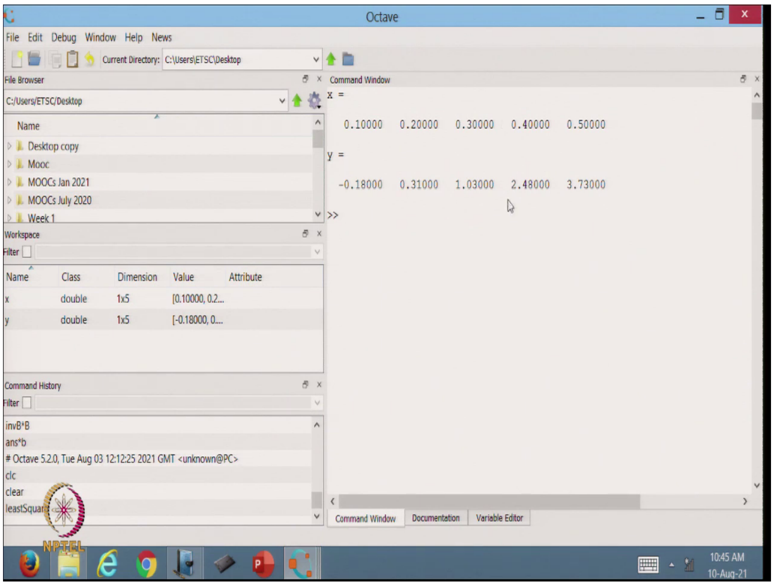

So, this is my x coordinates and this is the y coordinates.

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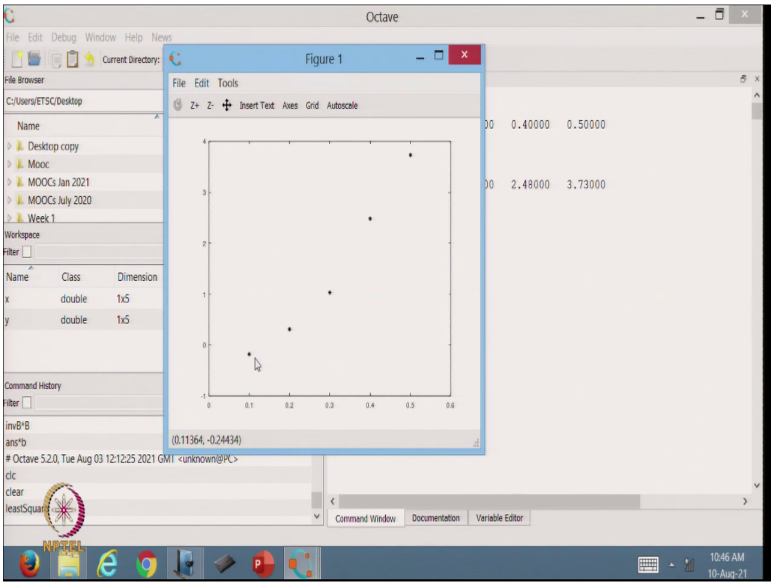

And, this is my plot. So, you can see that we have a five points. So, this is the points is given to me 1, 2, 3, 4, 5 and all this point if you see that does not lie. So, this point they do not lie in a line. So, if I take a best line fit, then definitely the system will be inconsistent we have to take the line which is closest to all this point.

Similarly, so, one line I can take like this one and the another quadratic will be the another type of parabola which is also not passing through all this point, but going very close to this one. So, this is the figure I have just taken.

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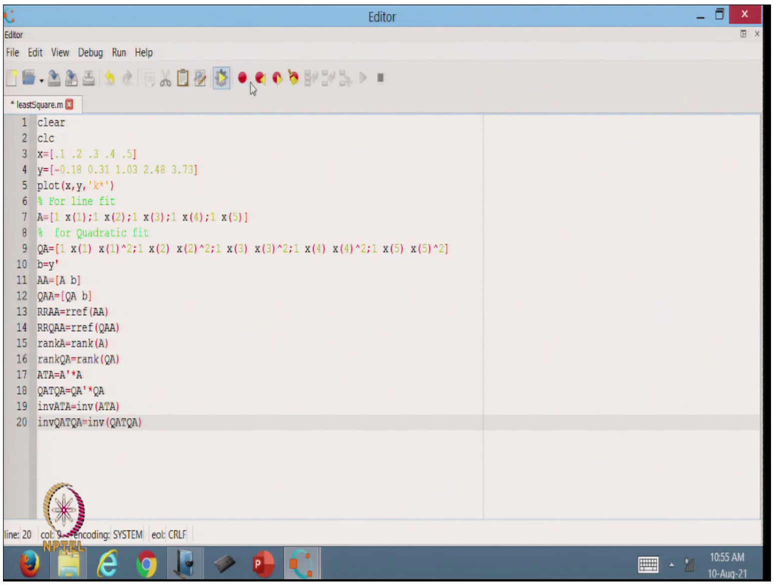

Now, so, I can write this one as for the line fit and similarly, I can write for quadratic fit. So, in the quadratic fit I will again write the matrix. So, let us say I write this quadratic means I just should write QA. So, this is my matrix and if you see from here I can write the same matrix here and I will just copy this. So, control C. So, here I can just take this values.

Now, if you see from here this is 1x1; now I have to put here x 1 square so that only the change I have to do x_2^2, x_3^2 $square$, x_4^2 and this is x_5^2. So, this is my quadratic fit, then I write the right hand side vector. So, I just write b is equal to so that is my basically y. So, y I have written in the row form. So, I will just take the transpose of that one. So, this will be my y.

$AA = [Ab]$
$QAA = [QAb]$
RRAA=rref (AA)
RRQAA=rref (QAA)
rankA=rank (A)
rankQR=rank (QA)
ATA=A'*A

$$QATQA = QA' \times QA$$

InvATA=inv (ATA)
InvQATQA =inv (QATQA)

Now, so, after this one I just want to check whether this line fit or the quadratic fit are passing through all this point. So, for that one I take a I write a matrix AA. So, just write first A and then b. So, this is my augmented matrix I am writing here and similarly, I am writing the augmented matrix QAA for quadratic. So, that is I am writing here QA and then b because everything is same except the matrix on the left hand side.

Now, first of all I want to check the rank. Now, after augmented matrix I just want to find out the row echelon form. So, row reduce echelon form. So, I write just right RR row reduce of A just I am writing. So, it is the command that row reduce echelon form of. So, it will give you the row reduced echelon form. Similarly, I can write row reduce of QAA; so, that is also I can write this one.

So, from here I will get the row reduce form and if you want to check you can check the rank of also. So, I can just check the rank of A, I can write this of the matrix A. Similarly, I can find the rank of QA and that is rank of matrix QA. Now, from there you will see that the rank will be different so, this will be inconsistent. Now, after this one I will define the matrix A transpose A.

Now, I want to define the matrix A transpose A. So, A transpose A I just write short form A transpose A. So, this matrix I am writing as a variable. So, from here I am writing the matrix this matrix. So, now I am writing from here matrix A transpose A. So, this one I am writing. So, that is I am writing like this one ATA. So, what I am going to do here I am just taking the matrix A taking its transpose and then multiply by k.

So, this is the matrix I am getting that is A transpose A similarly, I can get QA matrix, then the transpose QA. So, this matrix I am writing here. So, that is QA multiplied by QA, ok. So, we are able to find out the A transpose A matrix here. Now, after doing this one so, I am writing from here this is x is equal to A transpose B and from here I will get A transpose A inverse then A transpose B. So, I will get this answer.

So, this is my x hat. So, I will write this here. Now, I can write here inverse A transpose A. So, I am writing here inverse of ATA because that matrix will be invertible, no problem. Similarly, I can define inverse of QATQA just I am writing this name and that is also I am finding the inverse of QATQA this one and here I am writing just I am giving the name this name I am giving. So, let us try to run this one and see that if we have made any mistake.

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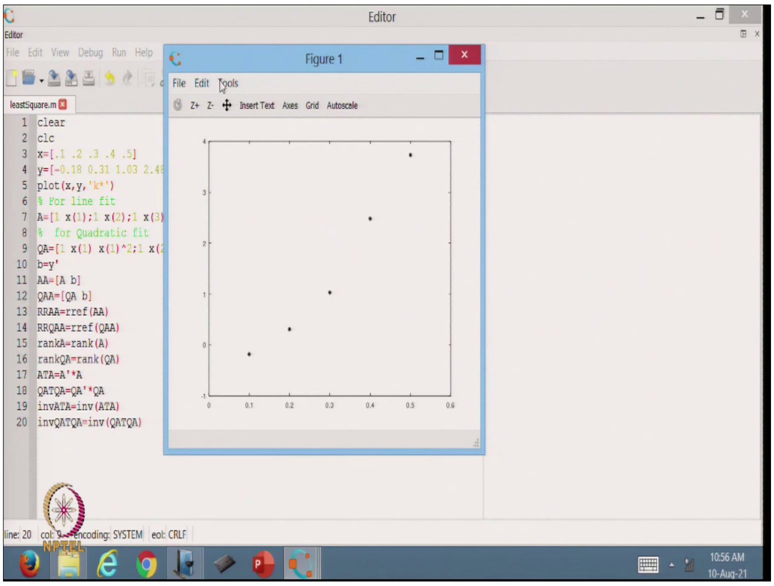

So, I just run this one. So, everything is ok. So, this plot is ok and if you see from here. So, I am getting this matrix.

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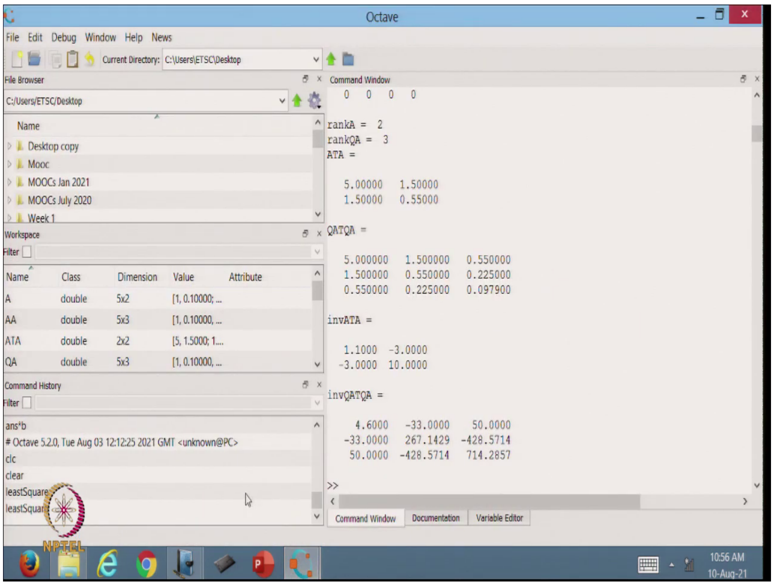

Now, we have started from here.

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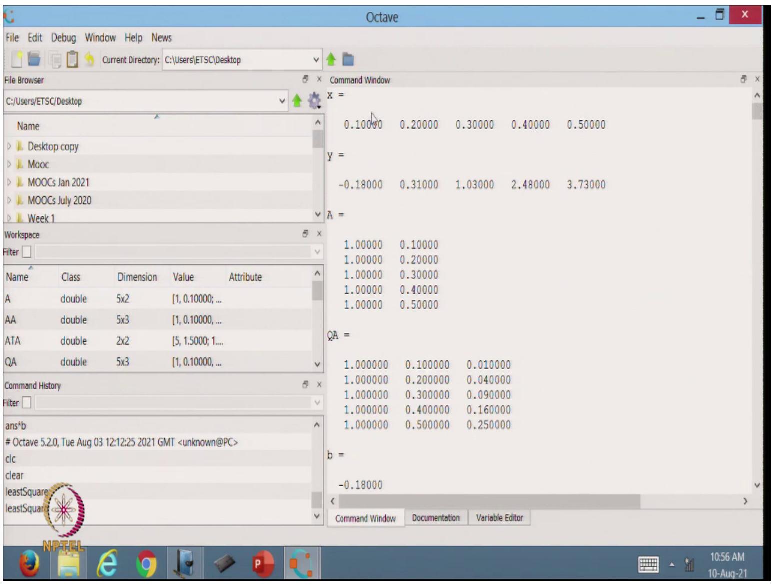

So, I have this x coordinate, this y coordinate and this is my matrix A and this is my matrix QA that is for the quadratic I am taking.

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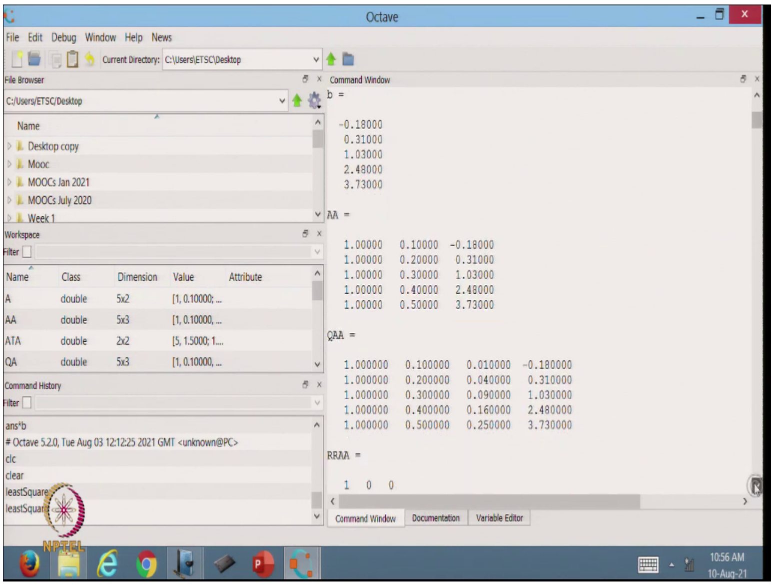

And, then I am I have defined from here this is my right hand side b, now you can see from here I have defined the AA matrix. So, AA just I have taken the augmented matrix and QA is the I have taken the augmented matrix.

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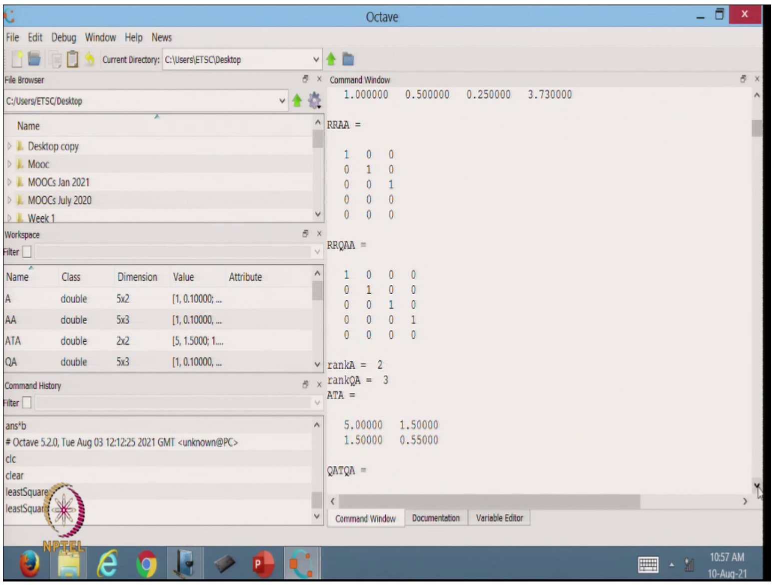

And, then I found the rank. Rank row reduce echelon form; so, this is my row reduce echelon form of AA matrix. And you can see from here that it is rank is 3, but the rank of matrix AA is 2. So, this is inconsistent similarly the rank of this matrix is 4 and the rank of this matrix the matrix we are taking is 3. So, this is inconsistent. So, you can also check that the rank of matrix A is 2 and rank of matrix QA is 3.So, this is the inconsistent matrix.

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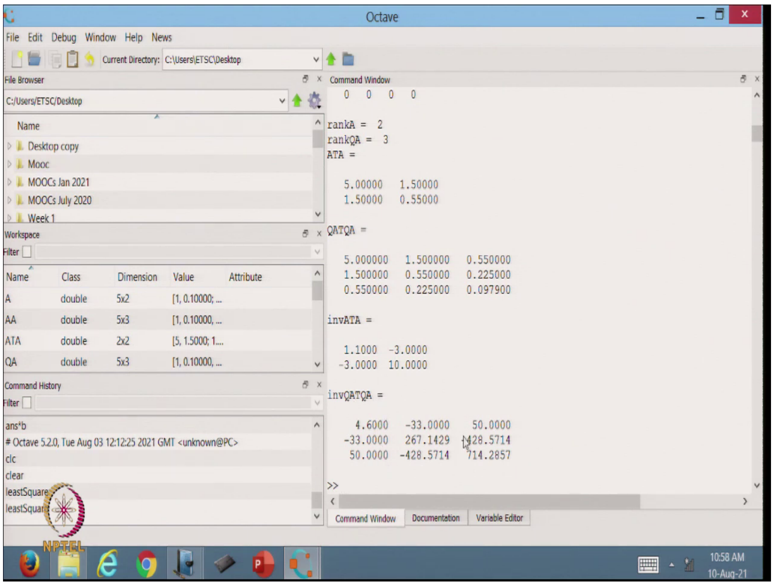

Now, after that I have defined ATA. So, A transpose A matrix I have defined. So, this is 2 cross 2 matrix and this is the values. Similarly, I have taken the QA transpose QA. So, this is the matrix 3 by 3 matrix I will get and you can check that this is a symmetric matrix positive definite. So, this is my matrix then I have defined the inverse of this ATA.

So, this is the inverse it is coming and here it is the inverse of QA. So, that is the quadratic corresponding quadratic I have taken the inverse this value is coming. So, up to now it is ok.

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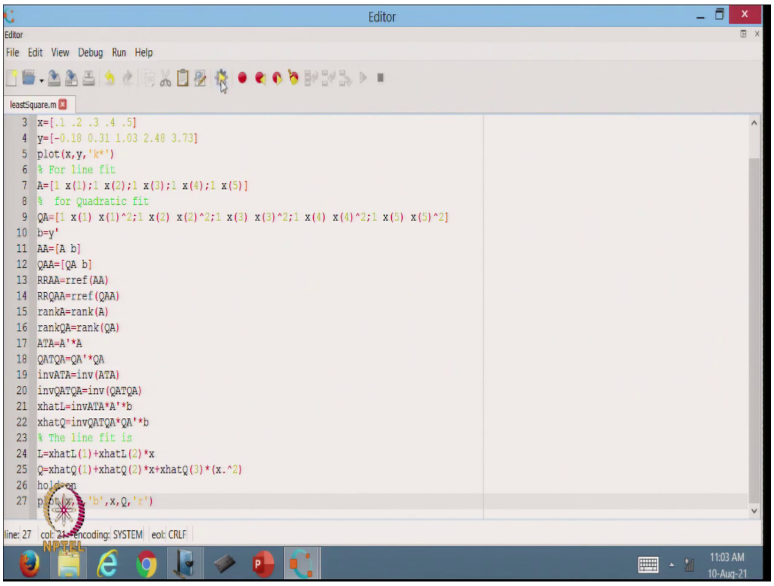

So, I can define the next. So, now, I want to find the so, solution of this one. So, after defining I want to find the x hat for A and then x hat for QA. So, let us define the x hat. So, this is I am going to define the x hat and this is for the corresponding line. So, I just take L and that will be equal to the inverse I am defined ATA. So, this is the inverse we have defining multiplied by the transpose of A because that we have to define star the b on the right hand side. So, that will be my x hat in the terms of the line fit.

Similarly, I can define the x hat in the terms of quadratic fit and that is also I have to define the same way. So, it is inverse star QA taking the this value. So, I have taken the inverse of A transpose A, A transpose b. So, this is what we are defining A transpose A inverse A transpose B here also I am taking. So, I am defining this by L the line and I am defining here by the quadratic x hat. So, this is also I am defining A transpose b.

So, now, this is my x hat L and this is my x hat Q. Now, after defining this one so, I am able to find the solution. Now, I have to define the line and the quadratic. So, let us define the line. So, I define because y points are given to me. So, I just define the line here now. So, I just write the line fit is so, the line fit is so here I am defining y or maybe I defining L that is on my line.

So, line means I am I have to define a plus bx. So, x hat L so, I can take from here x hat x hat L first coordinate so, that will be my A plus x hat L 2 times multiple by x. So, this is my line L and similarly I can define Q the quadratic. So, that will be x hat Q first coordinate plus x hat Q second coordinate multiplied by x and plus x hat Q 3 multiplied by x square. So, this one I can just write x.

So, this is my quadratic fit and that is my line fit and now, I can plot these things together. So, what I am going to do is now I will take hold on; hold on means it will hold the plot in the above and then I can write plot. So, I can plot x corresponding L with I just defined by blue color and x with Q and then I defined by red color. So, this is I am doing here.

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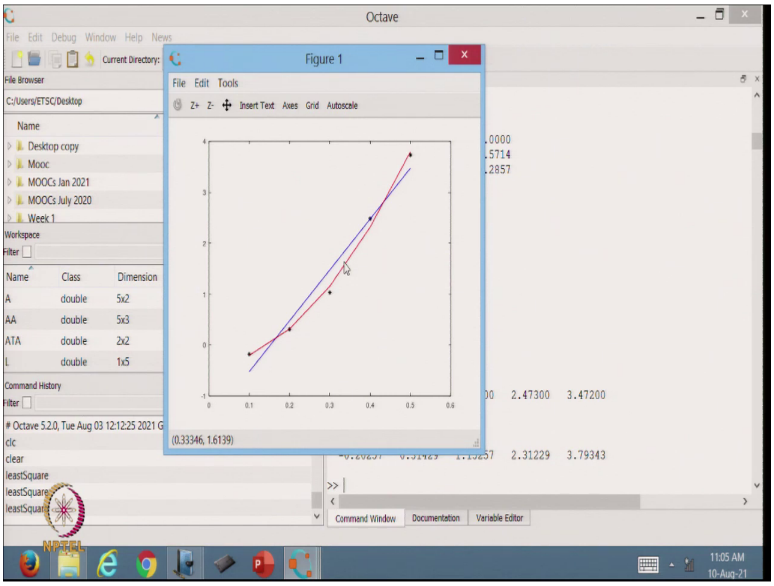

So, let us save this one and then run this. So, just first I will check these things.

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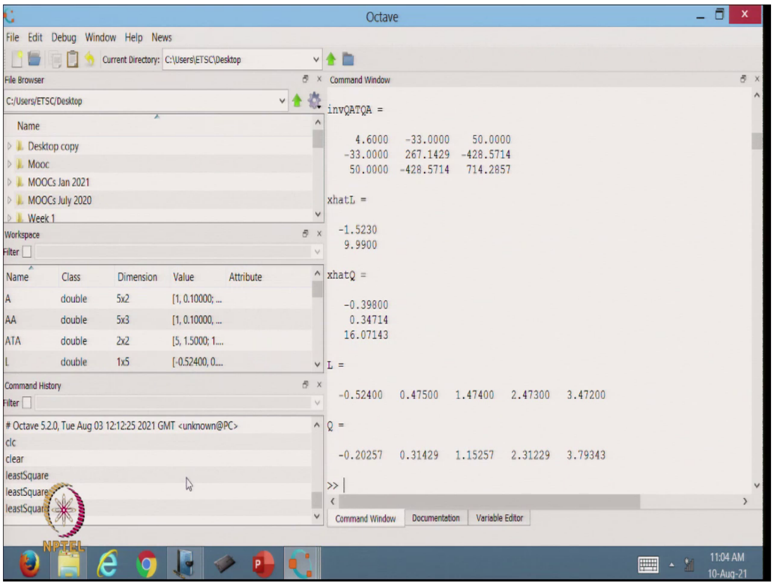

So, now up to now we have seen that. So, this is my x L hat. So, the value is A is minus 1.52 and this is b is 9.99 and for the quadratic the value of A is point 9 3980, b value this one and c value this one. So, my the line is passing. So, this is my coordinates of the line that is passing through the given point and this is the coordinates of the quadratic passing through the whatever the coordinates we found from the given quadratic form.

And, if we see this one now this is the corresponding plot. Now, you can see that this star points are there. So, if I take a line is blue color. So, that is the best fit and this is going close to all these five points and the red one is the quadratic. So, you can see that in the case of quadratic it is passing exactly from three points and going little bit away from these two points, but the line is passing through only one point and going away from these four points.

So, you can see from here that in this case the quadratic is I think it seems better than the linear fit because these points are going very close to the quadratic equation. So, this way we can plot the all the three plots together in one plot by the command of hold on. So, this way we can define the corresponding or we can write the code for finding the least square method.

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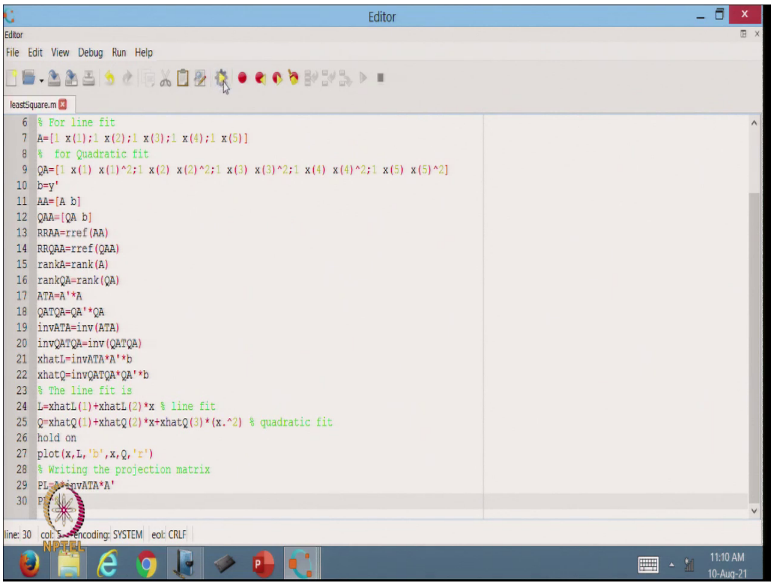

So, after doing this one you can also check about the projection matrix because we also define the projection matrix. So, I will just show you that we can also I have solved this one, but these things can also be solved with like this one here what we do we define projection matrix P taking b. So, it will get b hat and then I can solve it A x hat is equal to b hat, this way we can solve.

So, by this way also we can solve the given system and after that. So, projection matrix may be just I can write the projection matrix. Let us define that one also. So, writing the projection matrix here. So, I can define my P is equal to. So, I am writing the projection matrix first for line L because it depends upon the matrix. So, projection matrix for the line.

Now, if somebody is becomes confused, then they can also write here the some commands and like this one. So, I can write here that line fit; here I can here I can define as quadratic fit. So, similarly I can define here the projection matrix. So, projection matrix you know that for the line is I just take A matrix putting star here and then I am writing the inverse of A transpose A. So, it is I am writing here inverse of A transpose A star then A transpose star. So, this is my projection matrix.

So, projection matrix is AA transpose A inverse A transpose. So, this is my projection matrix and I am writing this one here. And, if you want to check, then you will see that the projection matrix is a symmetric matrix and if you see then it is square and its square will be also the same. So, this one I can just save and I just run this one.

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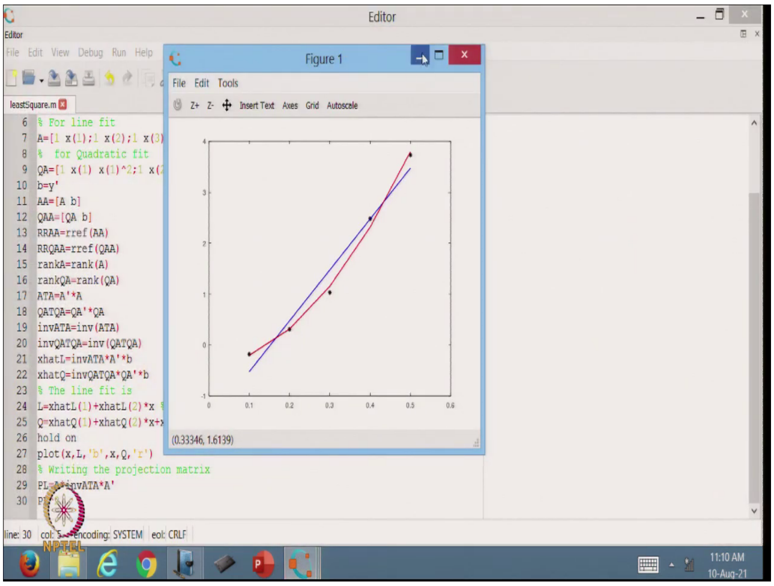

So, I will get the similar things here.

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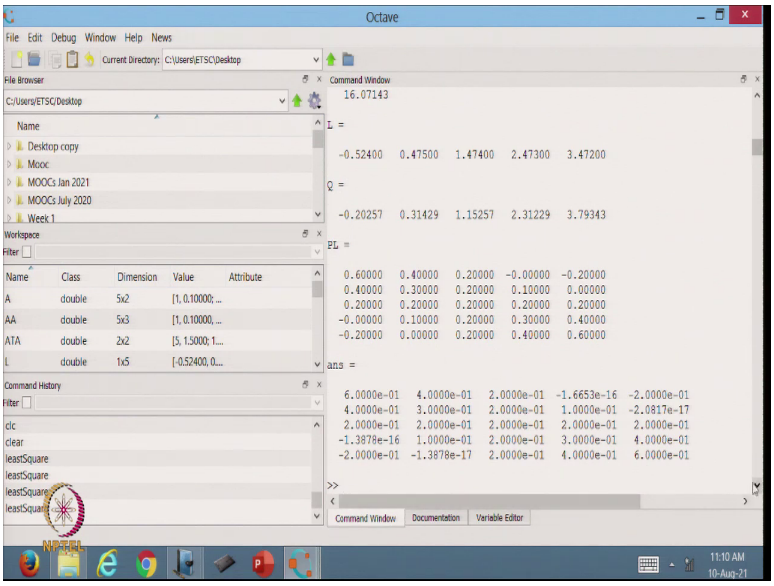

So, you now I this is my the projection matrix in the case of five points. So, it is 5 cross 5 matrix, and then you can see that this is just the symmetric matrix you can see from here and if I taking it is square then this is P L square. So, P L square is also equal to P L you can see that it is 6 point this 10 raise to power minus 1. So, that will be 0.6, it is 0.4, 0.2 it is e raised to the power minus 16 means it is 0 and here it is 0.2. So, you can also verify that that P L square is equal to P L. So, this is my basically the projection matrix.

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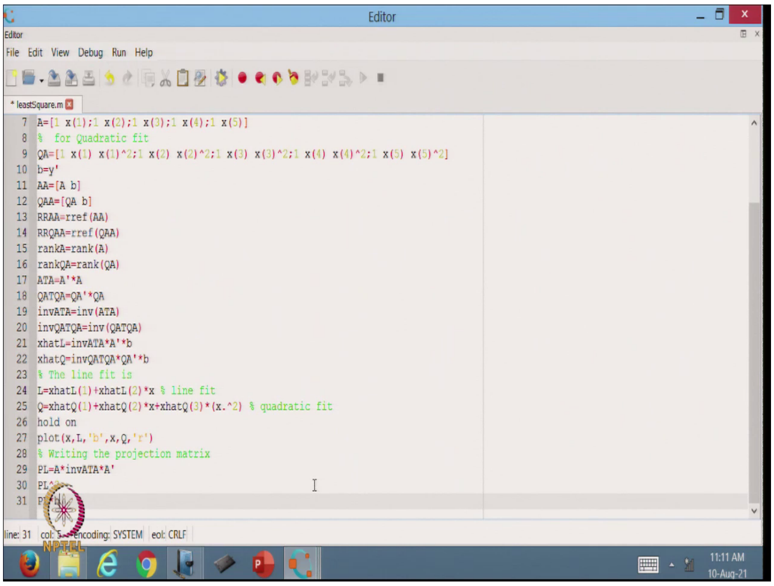

And, now I can even check from here that my P L star projection star b. So, that will be my b hat basically maybe I can just write here.

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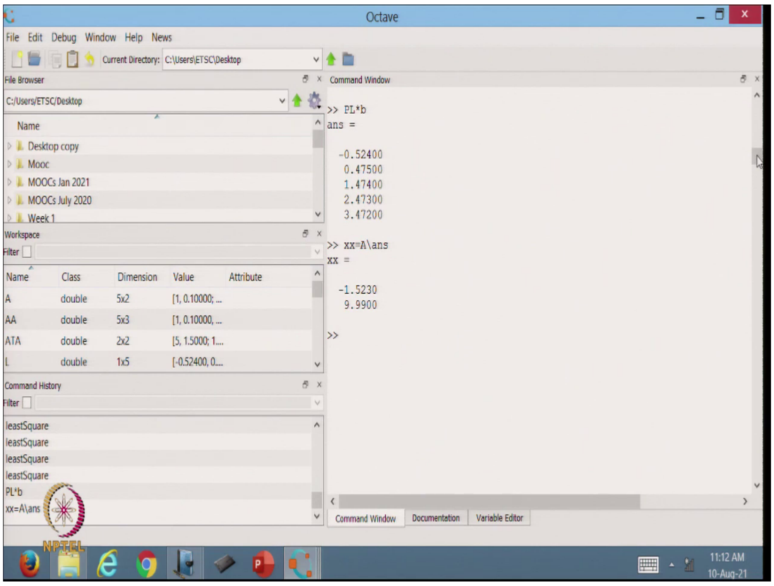

And, now if I write $PL * b$ so, this is my b hat and now, from here you can see that $\hat{b}$ hat is this value. So, now, if I apply this one Pb so, this is my b hat and now, I can solve this $A x$ is equal to b hat and then I will get the solution. So, you will get the same solution at this one; maybe I can write here the some value I just get x, so that I can write A backslash and the answer. So, this is the answer what I am getting and this is x hat.

So, $\hat{x}$ is -1.52 and this is 9.9 and if you see this is also the same this value -1.5 and this value. So, you can verify that this both are coming same. So, either you can go directly by the normal equation or you can go by the projection matrix and then you solve the system this one. So, both the ways you can find the same solution and this way we are able to do this.

So, the same way we can go for cubic for fourth order fit. In that case only the calculation will be more and then you can use this type of code and then based on this one you can find out any fit maybe it is so, you can find the nth degree polynomial fit using the same code. So, it is just the small code we have made that how we can extend this code for the other type of polynomial fit also. So, I think now we will stop here.

So, in today's lecture we have discussed about the least square method and we have shown that for a given data how we can define the line fit and the quadratic fit. And, then we have written a code in the octave, so using that code we can find out a line fit or a quadratic fit or maybe we can extend this one to any polynomial fit of degree n. So, I hope that you have enjoyed this one. Thanks for watching.

Thanks very much.