Matrix Computation and its applications
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Lecture – 05

Subspace of a vector space

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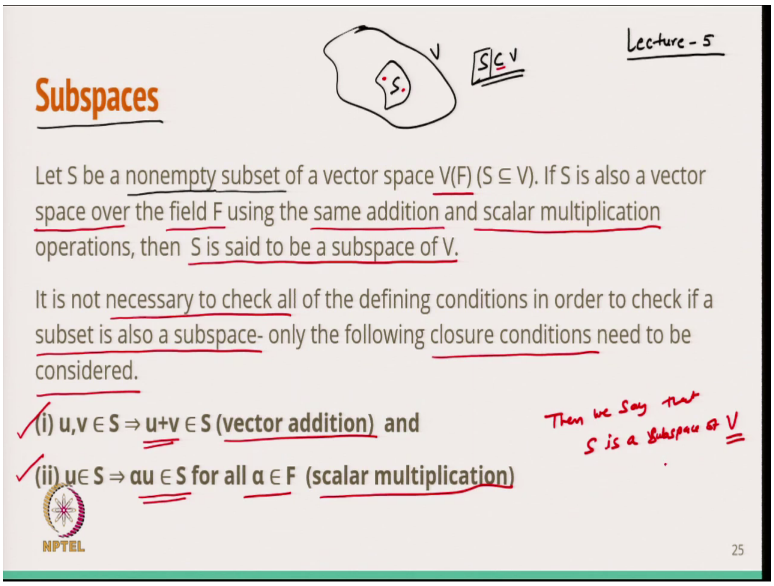

Hello, viewers welcome back to the course on Matrix Computation and its application. So, today we are continuing from the definition of vector spaces and we will discuss another definition that is Subspaces. So, what are the subspaces? So, in the previous lecture, we have seen that suppose this is my some vector space V.

So, now, what we are going to define in the vector space? it's all the properties satisfying vector addition, scalar multiplication. Now, we want to see that what will happen if I have some subset. So, suppose I take the subset S. So, S is a subset of V, now I want to see what is going to happen with this set S.

So, let us define the definition subspaces, let S be a non-empty subset of the vector space V(F). So, in this case, we are taking the vector space V and F we know that it is defined in the over the field F and S is a subset of V. So, it is a proper subset or may be equal to V also. So, that is why this equality sign is there.

Now, if S is also a vector space over the same field F. So, the same field we are taking using the same vector addition and scalar multiplication that you have to keep in mind that if S is also a vector space over the same field with the same vector addition and scalar multiplication then S is called to be a subspace of V. So, then S is also a vector space itself under the same operation and then S is called the subspace of V.

Now, the question is that if we have to check that S is a vector space or not, then whether we need to satisfy all the properties what all the eight properties we have satisfied in the case of vector spaces. So, then it becomes that it is not necessary to check all the definitions or on the defining condition in order to check if a subset is also a subspace.

So, what we need to do, we need to check only two properties, the closure property needs to be considered. It means that we need to check only the vector addition. So, this is if I take two elements, one element, and another element. And I take two elements from S and I define the same vector addition that we have defined, belongs to S then it is the vector addition and if it belongs to the same S then this property is satisfying and this one is the scalar multiplication.

So, if I take u belongs to S and any scalar α from the field and then if I define this one scalar multiplication and that also satisfy in the set S for all α then I can say that the scalar multiplication is well defined for S also. Similarly, the vector addition is well defined for S also and then if these two properties are satisfying then we say that S is a subspace of vector space V under the same operation. So, it is a subspace of V.

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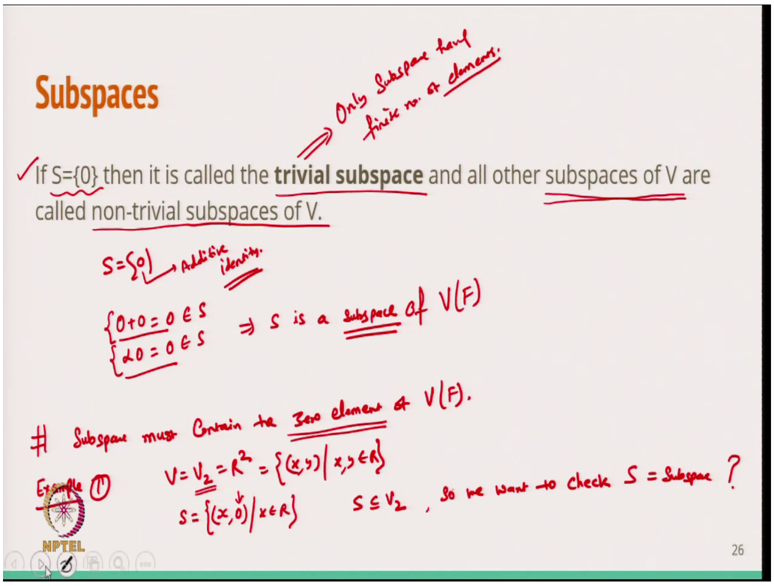

Now, let us start with the simplest subspace. So, the simplest subspace let if S is equal to 0 element. So, this is a 0 vector we are taking. It is called the trivial subspace and all the other subspaces of V are called non-trivial subspaces of V ok. So, how I want to verify that how this is the trivial subspace because it contains only one element S is equal to 0 and this 0 is an additive identity, so that is a 0 element basically we are taking.

Now you can check that it is a subspace. So, 0+0 is 0. So, that belongs to S so, vector addition is there. Now, if I take any scalar α into 0 then we have to satisfy this property that for any α.0 vector that is again 0 and that belongs to S. So, vector addition is well defined, scalar multiplication is well defined and just now we have seen that if these two properties are well defined then it is a subspace.

So, from here we can say that S is a subspace of vector space V with the field F and only one element is there that is a 0 element. So, I can say that this is the only subspace having a finite number of elements because if you take any other subspace then it will be called non-trivial and it is going to have an infinite number of elements.

So, only this subspace is the only subspace that contains only one element or a finite element and that is the trivial subspace. So, this is just the definition of the trivial subspace. Now the next thing is that how we can check that whether it is a subspace or not? So, I just want to give one property that the subspace must contain the zero elements of the vector space V, zero element means additive identity, it always contains this one if it is not contained then that is not the subspace of that vector.

So, let us do define some examples. So, let us take one example. Suppose, I take a vector space V and let us take V is equal to V_2 that is basically R^2. So, I know that this is a set of all x, y elements where x and y belong to the real number. So, that is defined and we know that it is a vector space. Now I take a set S and I choose the set S only x is there any other element I just make 0, where x belongs to the real number.

Now the question is that I know that this S is a subset of V_2 because I am just substituting one element equal to 0 that is it. Now, definitely, it is a subset of V_2. So, we want to check S as subspace or not. So, this one I want to discuss. Now I am taking the subspace from V_2. So, in V_2, we know that we have defined the vector addition as usual addition and scalar multiplication. So, I am going to do the same proper vector addition and scalar multiplication.

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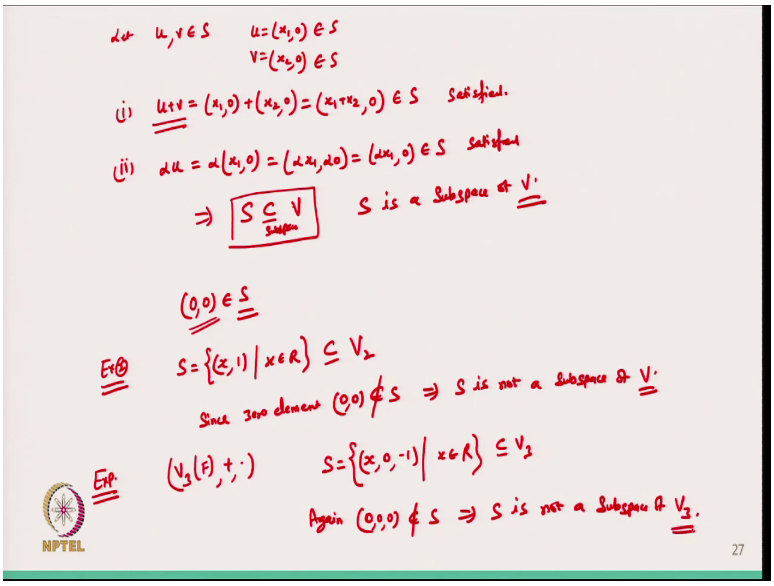

Then, let us satisfy that let I take u and v belongs to S. So, suppose u is basically I am taking $(x_1,0)$ and v, I am taking $(x_2,0)$. So, this belongs to S and this belongs to S. Now, I define u+v. So, u+v is I am defining $(x_1,0)+(x_2,0)$. So, this is going to be $(x_1 + x_2,0)$ by the usual addition. In this case and from here, I can define that this also belongs to S because it should be some real number and this should be 0.

So, that belongs to S. So, it means addition is well defined. So, it is satisfied. So, 1st property is satisfying, 2nd one is α.u. So, α and u, I am taking $(x_1, 0)$. So, from here, I will now define scalar multiplication. So, this will be equal to $(\alpha x_1, 0)$ and α.0 will be 0. So, it is $(\alpha x_1, 0)$ and that also belongs to S. So, if it also belongs to S then this is also satisfied.

So, from here I can say that S is under the same binary operation and we have to write like this one because it is a subspace. So, we should write that S is a subspace of V under the same operations or I can say that S is a subspace of V. So, it is a once for subspace we have discussed, we can have many subspaces.

So, that we do not know that how many subspaces it can have, but it can have many subspaces and so in that case I am just showing that one set I have chosen is a subspace of V and from here you can also see that $(0, 0)$ element also belongs to the.

So, I told you earlier also that $(0, 0)$ element should belong to S which is also one of the quick ways to check whether the given set is a subspace or not. If the 0 element is not there then you can say that it is not a subspace. For example, I take another example I just choose another example in the same way, I just take x here and another element I just take 1 ok and from here x belongs to R.

So, I know that this is also a subset of V_2, I have defined. So, it is a subset of V_2. Now, from here I want to check whether it is a subspace or not. So, in this case, since I can write the zero element that is $(0, 0)$ does not belong to S. So, from here I can say that S is not a subspace of V. So, directly I can say from here that this is not a subspace of V. Similarly, I can define other subspaces many subspaces.

So, let me take another example. So, in another example, I just take V_3 over the field F with the usual addition and scalar multiplication, I know that this is a vector space. So, I take S is equal to maybe I can take (x, 0, -1), where x belongs to a real number. So, it is a subset of V_3.

Now again (0, 0, 0) element does not belong to S so, which implies that S is not a subspace of V_3. So, directly we can write that it is not a subspace of V_3.

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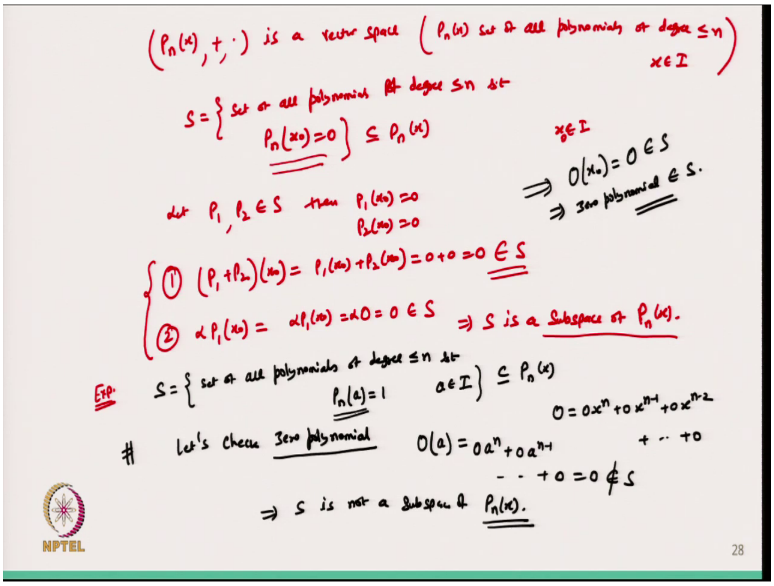

Now, let us take a few more examples. I have defined the $P_n(x)$, ok. So, $P_n(x)$, I have defined under the addition, usual addition, and scalar multiplication that is a vector space. So, I know that this is a set of all polynomial of degree less than equal to n and x belongs to my interval I. So, this one we have defined.

Now, I take a set S. So, this set, I am taking such that I can define a set of all polynomials of degree less than equal to n such that $P_n(x_0)$ is equal to 0, where x_0 belongs to I. It means I am taking all the polynomials of degree less than equal to n in which x_0 is the root of that polynomial. So, definitely, it is a subset of $P_n(x\ )$, only thing is that it should be the root the, it contains all the polynomial whose root is x_0. Now from here, you can check, that it is a subspace or not.

So, let us take P_1 and P_2 belongs to S. Then if it belongs to S then $P_1(x_0)$ is 0 and $P_2(x_0)$ is also 0, right. So, what I will do $P_1 + P_2$ this one I want to check. So, let us check what is happening at x_0. So, this is equal to $P_1(x_0) + P_2(x_0)$ and this is 0+0 and that is equal to 0, ok.

So, this is true for $P_1 + P_2$ it means that also belongs to S because in this case $P_1 + P_2$, is a polynomial that is also having x_0 as a root. If it is having x_0 as a root then it belongs to S because the degree of $P_1 + P_2$ is always less than equal to n. So, this will belong to S.

So, the 1st one is satisfied and the 2nd one is I am defining that P_1 the polynomial I am taking and multiplying by α. So, let us see what will happen at x_0. So, definitely, it will be equal to $P_1(x_0)$ and this is again it is going to be 0 because α is just multiplying I can take it outside and $P_1(x_0)$ is 0. So, I can write α.0 and α.0 is 0.

So, that is equal to 0 it means that it also belongs to S. So, vector addition and scalar multiplication are defined for the set S. So, from here I can say that S is a subspace of vector space $P_n(x)$. So, I found that this is a subspace of $P_n(x)$. So, in this example, this is a subspace.

Let us take another example. Now, in this case, I want to check whether 0 element is there or not. So, if you see that if I take a 0 polynomial and x_0. So, it is always 0 it means that belongs to S. So, from here I can say that 0 polynomial also belongs to S. Now let us take another set of S.

So, in this case, also I am taking the set of all polynomial of degree less than equal to n such thatP_n at some element a is equal to 1. So, where a belongs to the given interval this one. So, definitely, it is a subset of $P_n(x)$. Now, I am taking those polynomials in which I put the value of some element in that given interval and it gives the value equal to 1.

Now, I want to check whether it is a subspace or not. So, the main question is that I told you that first, you can check about the 0 polynomial. So, let us check zero polynomial. So, I know that the zero polynomial is the polynomial whose all coefficients are 0 and it is 0 polynomial. So, what I am taking and I am finding its value at a, ok.

So, 0 polynomial means this one $0.x^n + 0.x^{n-1} + 0.x^{n-2} + ... + 0.x^0$, that is it. So, this is my 0 polynomial, and if I am putting the value a here. So, it will be $0.a^n + 0.a^{n-1} + ... + 0$ and that will be 0, but if I take a polynomial and substitute value a then it should come 1 it means it does not belong to S. So, from here I can say that S is not a subspace of $P_n(x)$, I can say it from here that this is not a subspace of $P_n(x)$, ok. So, this way we can define the subspace.

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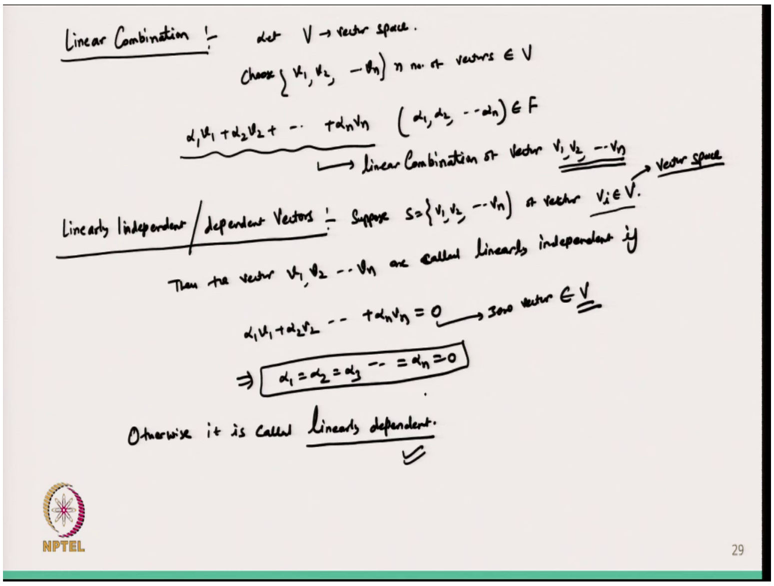

Now, so, after taking a few examples of the subspaces, I can just define one more term that is we call it linear combination. So, this term is the linear combination means that let V is the vector space ok and choose some elements $v_1, v_2, ..., v_n$. So, I am defining n number of vectors belongs to V. So, V is the vector space and I choose n number of vectors out of that and let us define their combination scalar.

So, I just define $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$. So, if we take this one then this is called a linear combination of vectors, $v_1, v_2, ..., v_n$, where $\alpha_1, \alpha_2, ..., \alpha_n$ that belongs to the given field F ok. So, this is called the linear combination.

So, from here I give you the definition of linearly independent or dependent vectors, ok. Now, suppose that the S is the set of vectors $v_1, v_2,..., v_n$ where v_i belongs to the given vector space V. So, V is some vector space there and I am choosing v_i some vectors from out of these.

Then the vectors $v_1, v_2,..., v_n$ the set of the vectors. So, these vectors are called linearly independent if I take the linear combination $\alpha_1 v_1 + \alpha_2 v_2 +... + \alpha_n v_n$. Now you see that $v_1, v_2,..., v_n$, I am taking from V and which is a vector space, and $\alpha_1, \alpha_2,..., \alpha_n$, I am taking the scalar multiplication.

So, this is well defined in this case because it is a vector space ok. So, 0 element definitely will be there. So, I am putting this equal to 0. So, if we take this linear combination equal to 0 ok. So, and these vectors $v_1, v_2,..., v_n$ are called linearly independent, if this is equal to 0 implies $\alpha_1, \alpha_2,..., \alpha_n$ all are 0 or I can just make it a little bit elaborate then $\alpha_1 = \alpha_2 =..... = \alpha_n = 0$, ok.

So, all are 0. So, in this case, 0 means the scalar is 0 and this is a 0. So, it means the zero vector I am defining belongs to the given vector space V. Otherwise, it will be called linearly dependent. If all the coefficients are coming 0, then it is linearly independent and otherwise, if any one of them is non-zero then just we can say that this is linearly dependent.

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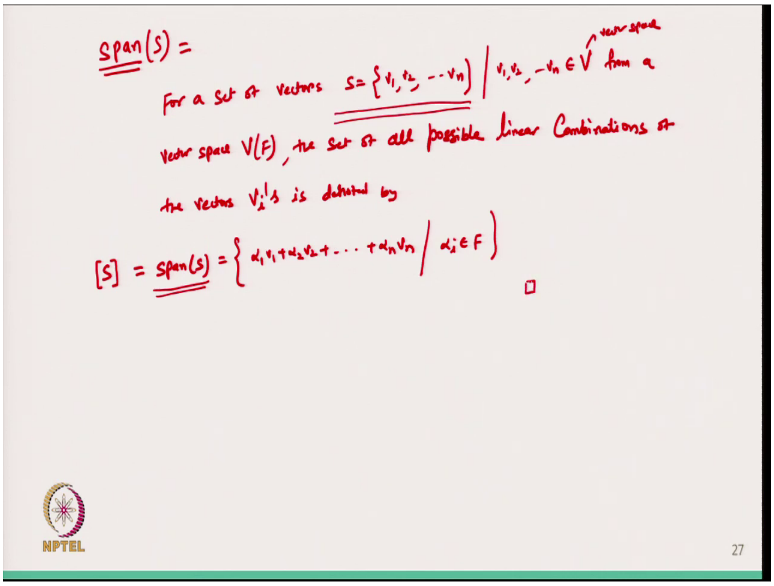

Now, we define the next definition and that definition is the span of S. So, this one I want to define ok. So, what is that for a set of vectors I define a set S as vectors $v_1, v_2,..., v_n$. So, this is I am just taking a set of vectors where $v_1, v_2,..., v_n$ belongs to the vector space V ok. So, this is a vector space.

For a set of vectors these form a vector space V over the field F, the set of all possible linear combinations of the vectors means $v_i's$. So, I am just defining all $\alpha_1 v_1 + \alpha_2 v_2 +... + \alpha_n v_n$ ok where this α_i belongs to the field F.

So, I can take all the combinations whatever the combination is there I can change my $\alpha_i's$ and then different type of combinations I can get. So, then these combinations are called span of S and this is represented by [S], which means that I am taking this as it is this set. So, of this set, I am taking all the possible linear combinations and whatever the linear combination I will take then I will get another set and that set is called the span of S ok.

So, this span is represented by this one. So, after this, let me stop here. So, today we have discussed that what do you mean by the subspaces, and then we have discussed a few examples based on the subspaces and then we have defined another definition that is the span of S. So, in the next lecture we will continue with this one.

Thanks for watching thanks very much.