Matrix Computation and its applications
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Lecture - 49
Orthogonality of the four subspaces associated with a matrix

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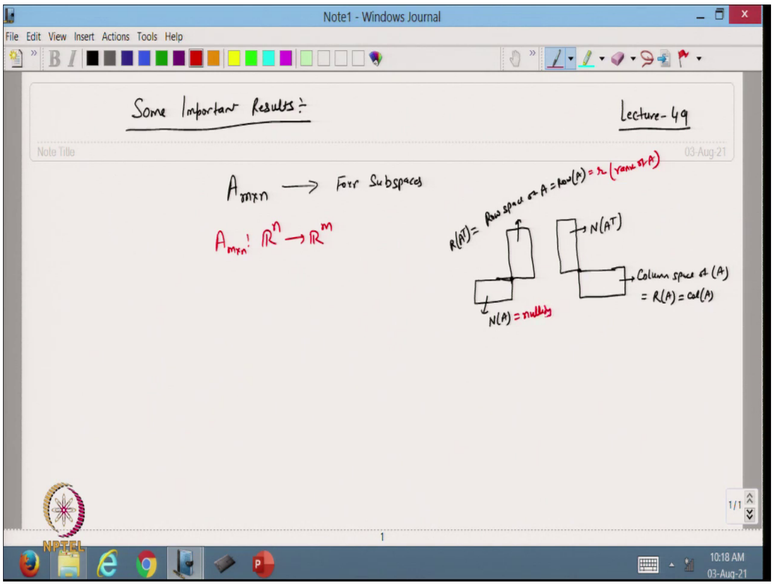

Hello viewers. Welcome back to the course on Matrix Computation and its application. So, today in this lecture, we are going to discuss very important results based on linear algebra.

So, in the previous lectures, we have seen that for a given matrix A of m*n dimension, then we have seen that this matrix is associated with four subspaces and these subspaces we can make like this one. Suppose, this is I will let you know why we are making this type. And the common point is that just one point is intersecting and another one I can represent the same way. Maybe I can draw like this one. So, these are the four spaces that are associated with this one.

Now, I know matrix A can be written as a linear transformation. So, it is the, so this I write column space of A and this is also we write as a range space of A. So, I can write the range space of A as a column space of A. Now, this is I know that the null space of A transpose.

Now, this one I write the null space of A and this is I am writing the row space of A, then this is also equal to I can write as null space of sorry, the range space of, the range of A transpose. So, I can write this as a in the short form row of A, so it is a row space of A.

And the point of intersection is this one. So, we are going to show that these subspaces are orthogonal to each other. And we are also showing here because this is a matrix is from of dimension m*n, so it goes m cross n is a linear transformation from R^n to R^m. So, if I know that if its rows be the range is r, so that the rank of A, and this is nullity. So, suppose this is nullity, then I know the dimension; so, maybe I can write something not like this one.

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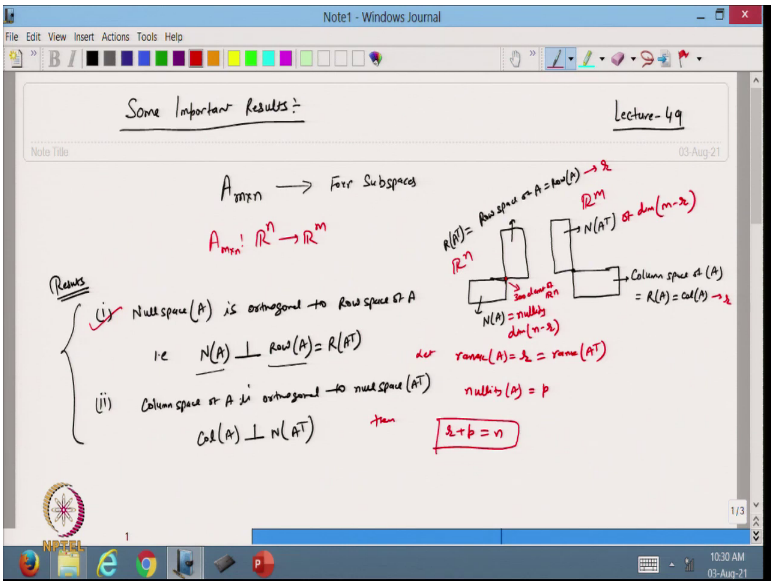

So, I can write here that the rank of A is equal to r and that is also equal to rank of A transpose. And suppose nullity of A is some p, then I know that r + p, r +p = n that is a rank nullity theorem, ok.

So, if I can say that if the dimension of this one is this one, so its rank is r, then its rank is r, then this is of dim(m-r), and nullity will be of dim(m-r); because this is a. So, this is basically R^m and this is basically R^n. So, that thing we know from this transformation.

Now, from here we want to show a very important result is that; so, the first result I want to, the first one is the null space of A is perpendicular is orthogonal to row space of A. So, we can write as that is the null space of A is perpendicular to the row space of A, and the row space of A I can write as the range space of A transformation. So, this is the first one.

Second one is the column space of A that I can write, so the column space of A is orthogonal to the null space of A transpose here. So, it means that column space of A is perpendicular to the null space of A transformation. So, these are the two results we want to discuss here.

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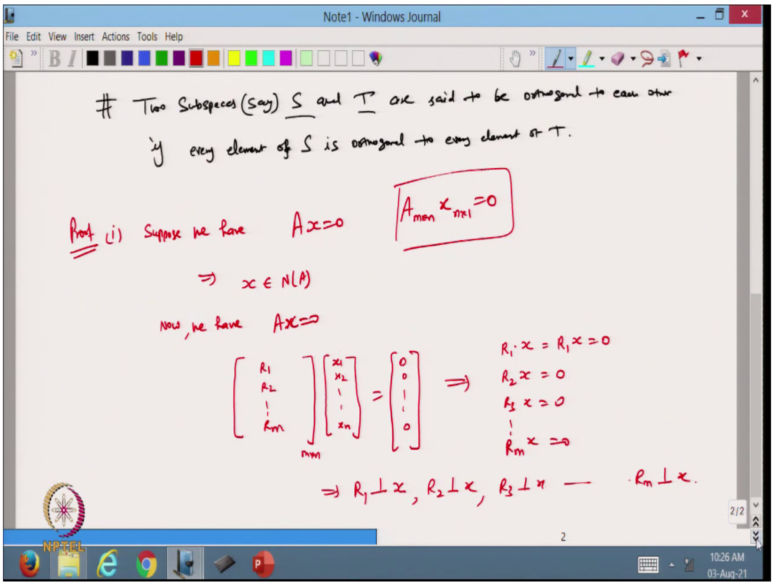

Now, so, for this one, I just want to write a definition. Two subspaces say S and T. So, two subspaces that are S and T are said to be orthogonal to each other if every element of S is orthogonal to every element of T, then only we can say that these are orthogonal. So, in this case, we know that these are the subspaces because null space is a subspace, row space is a subspace.

So, this subspace is orthogonal to each other, meaning each and every element of null space is orthogonal to each and every element of row space. Similar case is here. So, let us prove this one proof of the previous first. So, this is, first I want to discuss that the null space of A is orthogonal to the row space of A.

Now, suppose we have Ax=0. So, this is my A, that is m*n, ok. So, this is I know that this is A is m* n and x is n*1, so that is equal to 0. So, if this is equal to 0, it implies that x belongs to the null space of A because it is going here. Now, we have Ax=0. This is what we have.

So, A is basically, if I write the A, if you see from here then this is R_1, this is R_2,… R_n and this is my x, so it is x_1, x_2, …,x_n because x is coming from n*1, so nth dimension. And this is m*n. So, m number of rows and n number of columns is equal to the vector (0, 0, 0).

Now, if you see from here then what I am doing is that we are multiplying this vector with the first row of the matrix and we are getting 0. So, from here if you see that I can write from here that $R_1.x = 0$ because this is my row vector and this is my column vector, and I am taking the dot product, that is equal to 0. Or maybe I can write this as directly the matrix form. So, I can write R x = 0 because it is a row and a column, so it is just the matrix form I can write.

Also, R_2 x = 0, R_3 x = 0, and R_n the last one, so this is my m because we have a n number of rows, so it goes up to m. So, that is equal to 0. So, from here, you can see from here that I can say that the R_1 is orthogonal to x, R_2 is orthogonal to x, R_3 is orthogonal to x and so on, R_m is orthogonal to x. So, all these rows are orthogonal to the vector and this vector lies in the null space of A.

Now, we are able to see this one, but as we have seen there, we are able to tell that two subspaces are orthogonal to each other, only if each element is orthogonal to each element of the other subspace.

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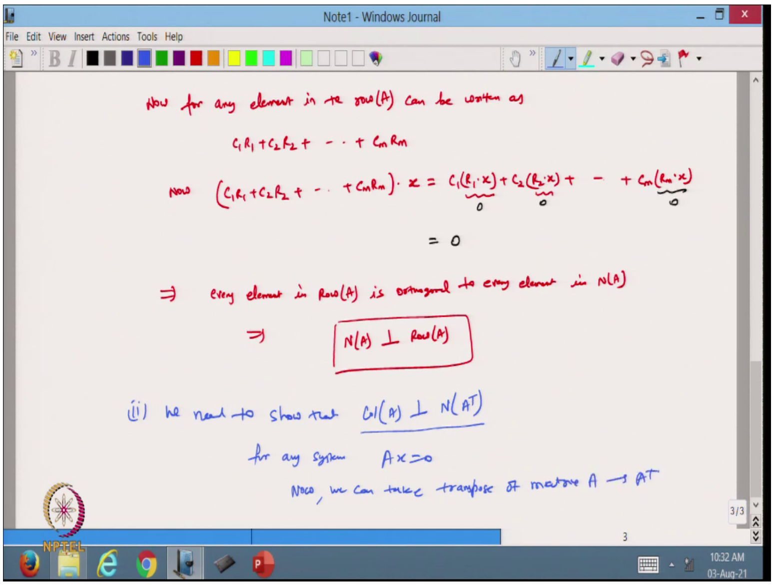

Now; so, let us take any element. Now, for any element in the row space of A, so if I take any element in the row space of A that element will be spanned by the rows. So, that element will be a linear combination of the rows. So, if any, now for any element in the row A is written as, can be written as C_1 R_1 + C_2 R_2+ ….+C_m R_m because it will be the linear combination of this rows.

Now, I take this element. It can be any element, and I am taking the dot product with x. Now, from the definition of the dot product I know that this will be equal to $C_1(R_1.x)$ + $C_2(R_2.x)$+…..+$C_m(R_m .x)$. And this value we already know is equal to 0, 0, 0, so that is equal to 0.

So, from here I am able to see that any element from the row space taking the dot product with any element x from the null space is giving this value 0. So, which implies that every element in row space of matrix A is orthogonal to every element in null space of A. So, which implies that the null space of A is orthogonal to a row space of A. So, this is one of the important results.

And from here you can see that the common element will be only the 0 element, that will be lying in both the subspaces because it is the subspaces, so 0 element will also lie in the N(A) and 0 element is also lying in row space of A. So, that is why in here we are writing this connecting with the point, so this is the 0 element, 0 element of R^n. So, that is the 0 element of R^n.

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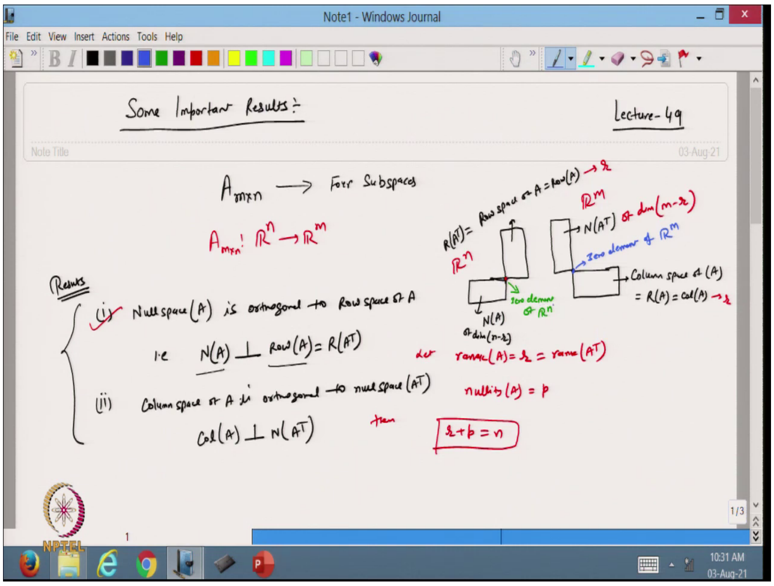

Or maybe I can write it with some other color. Maybe I will. So, N(A) of dim(n-r) and this point is 0 element of R^n. Similarly, this is also 0 element of R^m. So, they have only one point common to each other that is just the 0 element of this one. So, we are able to prove the first part of this one.

The second part is the column space of A is orthogonal to the null space of A transpose. So, second one, we need to show that the column space of A is perpendicular to the null space of A transpose because both are lying in the R^m. They are the subspaces of R^m. And N(A) and row space are the subspaces of R^n. So, this one we want to show.

Now, it can be clearly proved that now for any system Ax= 0, so that was just now we have proved. Now, we can take the transpose of matrix A. So, that is A transpose. So, I will not take this system.

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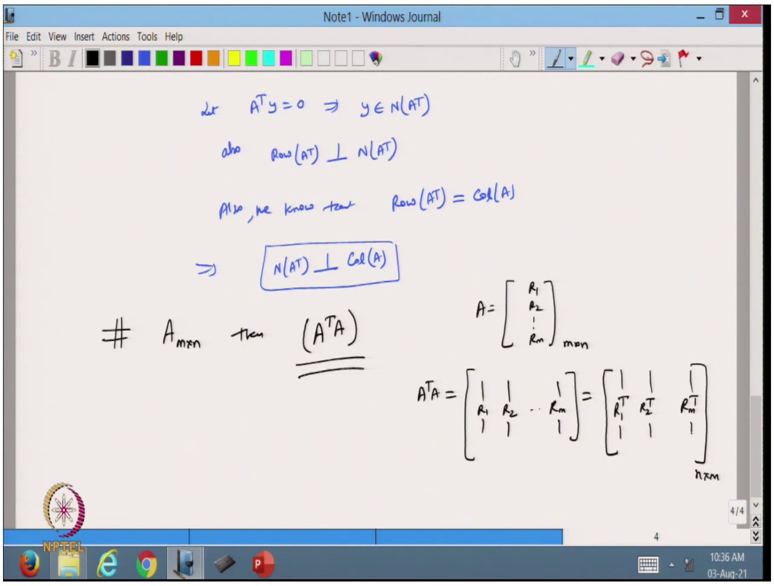

Now, we take the transpose of the matrix, then $A^T y$, then let I take the system $A^T y = 0$. Now, from here it implies that $y \in$ null space of A transpose. Just now we have showed that in the previous one that the $R(A^T) \perp N(A^T)$.

So, the null space of A transpose is perpendicular to row space of A transpose. And also, we know that $\mathrm{Row}(A^T) = \mathrm{col}(A)$, because just taking the rows as the column in the transpose. So, the row space of A^T that will be the same as the column space of A. So, which implies that null space of A transpose is orthogonal to column space of A. So, they are also orthogonal to each other. And then the point of common intersection is only the 0 element. So, this is the result we want to show.

Now, one more thing we want to discuss. So, now, we want to discuss the matrix. Suppose, we have A m cross n dimension, then we want to discuss about A transpose A, this matrix. Because in this case we what we are doing that suppose I have a matrix A, so A matrix is

$$\begin{bmatrix} R_1 \\ R_2 \\ : \\ R_m \end{bmatrix}$$

Then if you see from here $A^T A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ R_1 & R_2 & \dots & R_m \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ R_1^T & R_2^T & \dots & R_m^T \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$ .So, if it is of m*n dimension, this will be n*m dimension.

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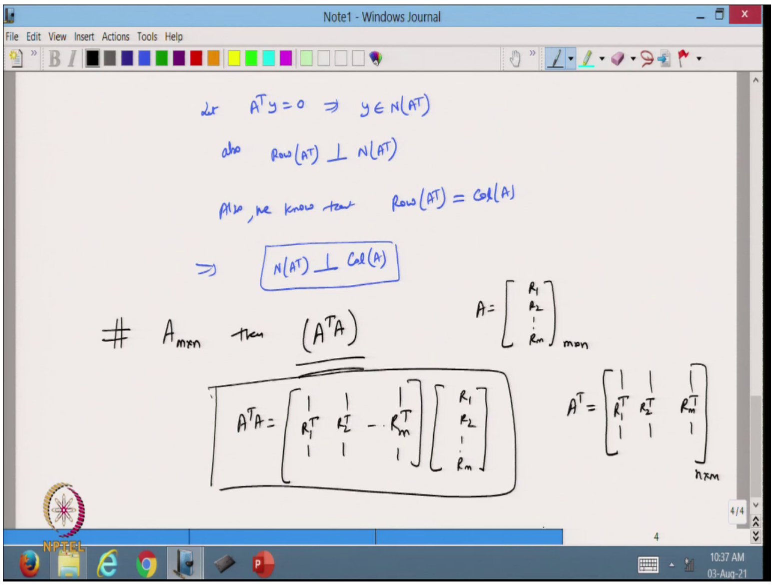

So, basically this will be A^T. So, if A, this one the A^T will be this one. So, basically, I can write this directly from here. No need to write this. So, this is my A^T. So, from here if I want to write A^TA, so what we are doing here is that I am taking my $\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$.

So, this is my A^TA . It means that I am multiplying. I am taking the dot product of each and every row of the given matrix. So, now, I want to discuss some properties of A^TA.

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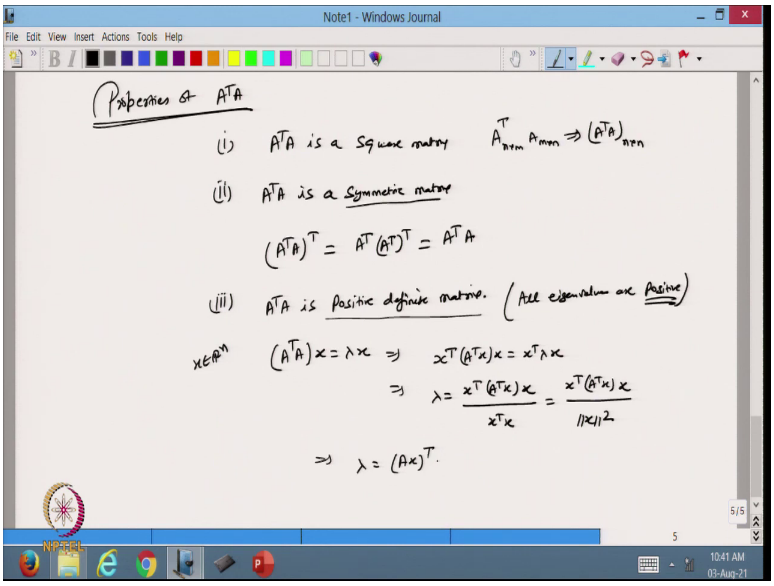

So, properties of A^TA. So, first thing is that A^TA is a square matrix always, because if A is, A^T will be n*m and A is m*n, so it means that A transpose A will be n*n. So, that is always a square matrix.

Second thing is that A transpose A is a symmetric matrix. Here we are taking all the real valued spaces, so that is why we are dealing with the transpose only. So, it is a symmetric matrix because $(A^TA)^T = A^T(A^T)^T = A^TA$

And third one is we have already discussed that A^TA is a positive definite matrix. So, it is very difficult to construct a positive definite matrix, but if any matrix is there, we can write A^TA is a positive definite matrix, and we have proved this earlier also. So, a positive definite matrix means all eigenvalues are positive.

Now, how can we show this one? So, this one I can write, that I write the, so $(A^TA)x = \lambda x$

⇨ $x^T(A^TA)x = x^T\lambda x$

⇨ $$\lambda = \frac{x^T(A^TA)x}{x^Tx} = \frac{x^T(A^TA)x}{\|x\|^2}$$

x $\in$ R^n $\lambda = (Ax)^T$

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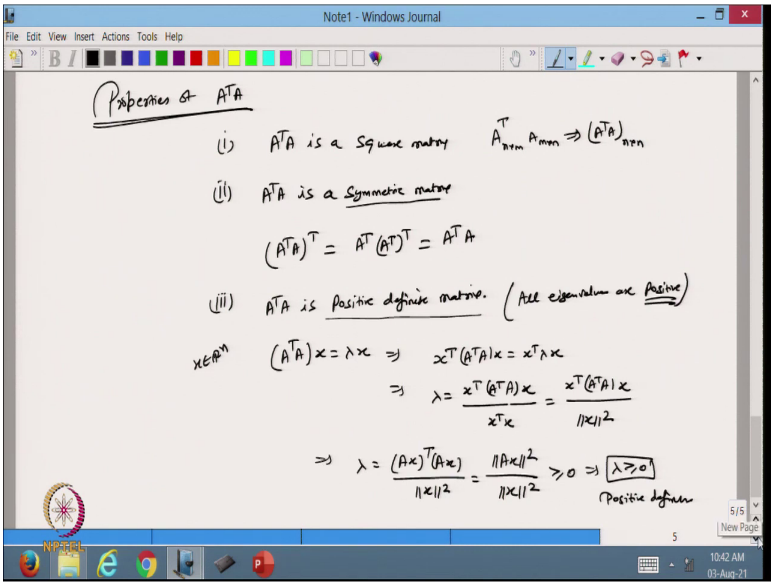

Now, we know that

$$\lambda = \frac{(Ax)^T (Ax)}{\| x \|^2} = \frac{\| Ax \|^2}{\| x \|^2} \geq 0$$

$$\lambda \geq 0$$

So, it is positive definite. And this will be 0 only when this part Ax = 0, otherwise this will never be equal to 0. So, after showing this is a positive definite, we want to discuss another important result. So, this is the third property.

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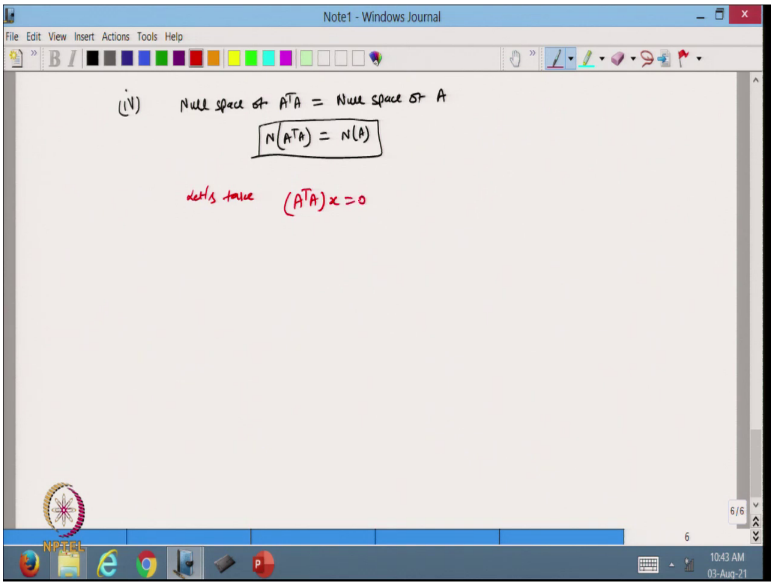

So, the fourth property I want to discuss is that, the null space of A^TA = null space of A that is I want to discuss this. So, this one I want to discuss because null space of A is the subspace of R^n and this is also A^TA is a n*n matrix. So, the null space of A is also of the dimension subspace of R^n. So, I want to discuss that this is the equal. So, let us do that one.

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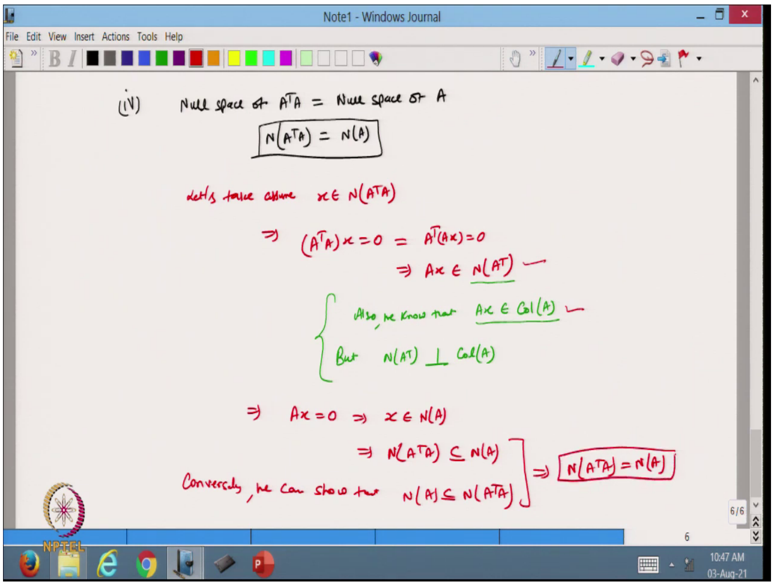

Now, let me take, so let us take $(A^TA)x = 0$ or maybe I just take, let us assume $x \in$ null space of A^TA which implies that $(A^TA)x = 0$. This one I can write as $(A^TA)x = 0$, and from here I can write that if $(A^TA)x = 0$ which implies that $Ax \in N(A^T)$.

Also, we know that also we know that $Ax \in col(A)$. Because Ax is, what is that? It is the span of the columns of A, so it belongs to the columns space of A. But so, this also belongs to column space A and this also belongs to the null space of A transpose.

Just now we have seen that it is orthogonal to the column space of A. So, which implies that Ax also belongs to the null space, Ax also belongs to this one, but this is orthogonal to each other, so orthogonal to each other and the common point is only 0, which implies that my $Ax = 0$ because only intersection point is the 0 of that space. So, that is what we have seen.

So, which implies it should be 0 and from here I can write that the $x \in$ null space of A. So, that implies that I have taken the element from the null space of A^TA and I showed that this belongs to the null space of A. So, it is a subspace of this one, a subset of the null space of A.

Now, we can go directly to the converse also. So, conversely very easily we can do that, we take the element from N(A) and go the same way. So, conversely, we can show that N(A) is a subspace of $N(A^TA)$. The same way, I will take the element from this, I will reach here, then from here I will reach here, and will go back from here. So, from these two we can show that the $N(A^TA)=N(A)$.

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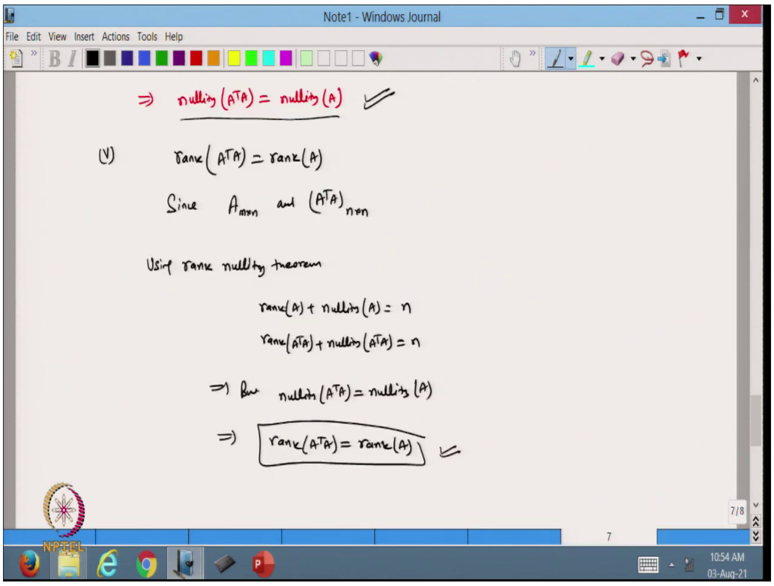

So from here I can say that nullity(A^TA)= nullity(A). This is what we are able to make from this. Now, after doing this one I want to, so this is the fourth property. So, I want to discuss the fifth property. So, the fifth property is that the rank (A^TA)= rank (A), this one I want to discuss.

Now, we know that since now we know that this A is m cross n, and A^TA is n cross n. Now, using the rank nullity theorem, I can say that the rank(A) + nullity(A) = n. Also, rank(A^TA)+ nullity(A^TA) =n because A transpose n and cross n dimension. So, it is a subspace of R^n.

Now, from here, from these two, but nullity(A^TA) = nullity(A), that we have seen here. So, if I have this things, and this is all n*n, so which implies that a rank(A^TA)=rank(A). So, this one can be proved this way. So, it means the nullity(A^TA) is the same and the rank(A^TA)is the same.

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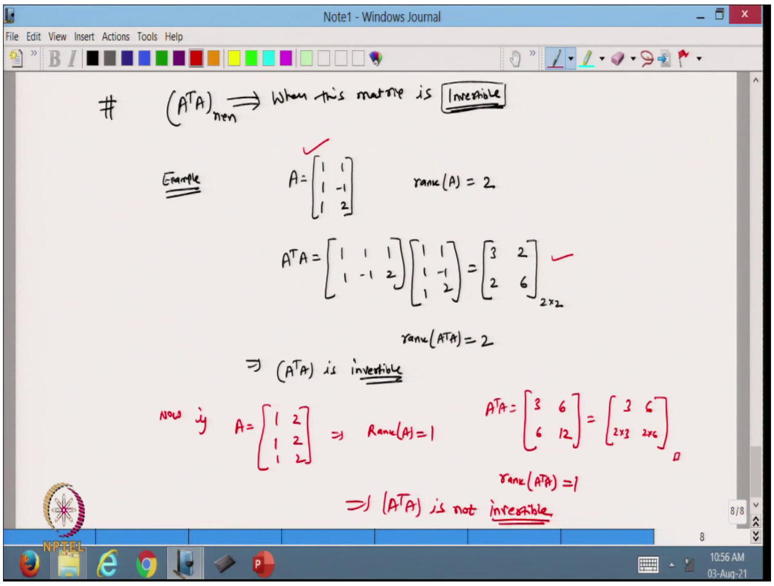

Now, after doing this one, one thing we also discuss is that now we want to discuss that this A^TA is a square matrix, then we want to discuss that when this matrix is invertible. So, this one we want to discuss. Because it is a square matrix, and we found that the rank of A transpose is equal to the rank of A, then how can we say that this matrix is invertible or not.

So, this one we can find out when we can say that this matrix is invertible. Now, let us take an example. So, this will be seen with the help of an example. Let us take the example.

Now, suppose I take a matrix A having suppose I it having two columns. So, let us take the example I take $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$. So, this one I am taking. Now, if you see from here then these vectors, so first columns the second columns are linearly independent to each other.

So, I can say from here that rank(A) = 2, in this case, because the first column and the second column are linearly independent to each other that we can verify very easily, that this vector cannot be written as a scalar multiple of this factor. So, in this case the rank of A is 2.

Now, if I take A^TA, so this will be $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$.

So, if I multiply by 3 here, I cannot get the second row with the first row, and also, from the previous one I know that the rank of this is equal to this. So, it is a full rank. So, now, from here I can say that this is a 2 * 2 matrices and rank 2. So, from here I can say that the matrix A^TA is invertible. So, I can take the inverse of this one.

Now, if I take the matrix A= $\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$. So, in this case, I know that the first the second column is 2 times the first column, and from here if I solve this one in the echelon form, then I can find that the rank of matrix A is just 1. So, in this case, it is just rank 1. And now if I take the A^TA, so this will become $A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2\times 3 & 2\times 6 \end{bmatrix}$

And now the rank of A^TA in this case is 1. So, from here I can say that A^TA is not invertible. So, when the given columns are linearly independent then A^TA is invertible, and when they are linearly dependent, then the corresponding A^TA is not invertible. So, those are things we have to keep in mind.

So, let us stop here. So, in today's lecture, we discussed the four subspaces associated with the given matrix. They are orthogonal, in the sense that we have shown that the null space of matrix A is orthogonal to the row space of the matrix A, and the null space of A transpose is orthogonal to the column space of the matrix A. And then we have discussed the another important matrix that is A transpose A, and discuss its few properties.

In the next lecture, we are going to use these properties to discuss a Least Square Solution. So, thanks for watching.

Thanks very much.