Matrix Computation and its applications
Dr. Vivek Aggarwal
Prof. Mani Mehra
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture - 46
Norm of a vector

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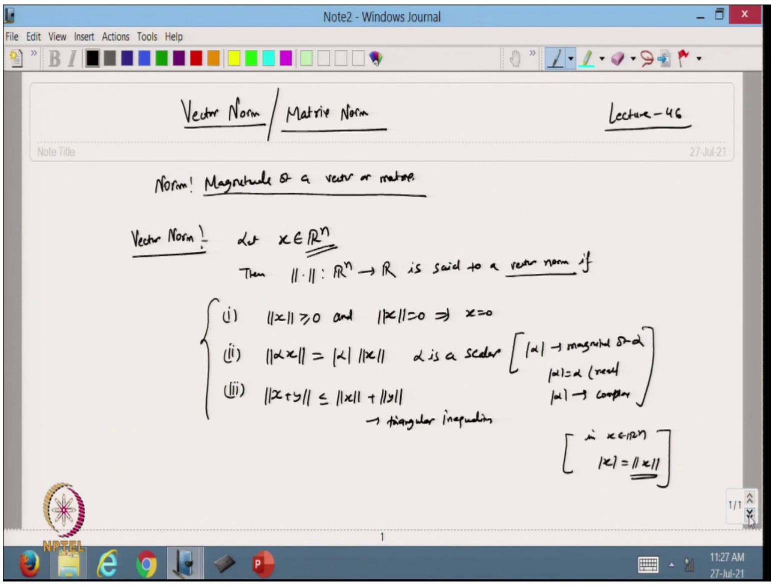

Hello viewers. So, welcome back to the course on Matrix Computation its Application after dealing with the inner product space. So, today we are going to define the vector norms and the matrix norms. So, let us do that one. So, today we are to start with the vector norm. So, norm if you see, we define the norm as a magnitude of a vector or matrix. Although we know that these matrices also can be written as a vector, in the vector species, we represent vectors and matrices differently here to define the norms.

So, let us start with the first one. How can we define the vector norm? So, let us have an $x \in R^n$. So, we are taking the n-dimensional real space, Euclidean space. Then we define the norm as defined by these two parallel lines. So, it is from R^n defining x to R the set of real numbers is said to be a vector norm if it satisfies the following condition.

The first one is that, if I take the norm of this vector x, $||x|| \geq 0$, because it is a magnitude $||x|| \geq 0, ||x|| = 0 => x = 0$.

So, this is true for any x that belongs to this one. The second one is that if I take any scalar and multiply by that vector and now take the magnitude. So, $||\alpha x|| = |\alpha|.||x||$. So, here I am writing this magnitude where α is a scalar.

So, this will become is equal to α in real and this is magnitude in the complex. So, we are talking here just the real vector space R^n. And, the third one is that, if I have two vectors taking their sum and then taking their magnitude or the norm. $||x+y|| \leq ||x|| + ||y||$. And, this property is called the triangle triangular inequality. So, all these three properties are satisfied, then we say that this is the vector norm.

Now, we also know that so in the case of the dot product. So, in R^n we have already seen that we have represented the x by the magnitude also. So, in x belongs to R n we know that this is we have represented by magnitude so this now becomes this one. So, this is the magnitude here, where x is a vector.

So, after defining this property, now let us see how we can define the norm of different types of vectors? So, let us assume that, although we can define a large number of norms of a given vector, we are going to define only a few important ones.

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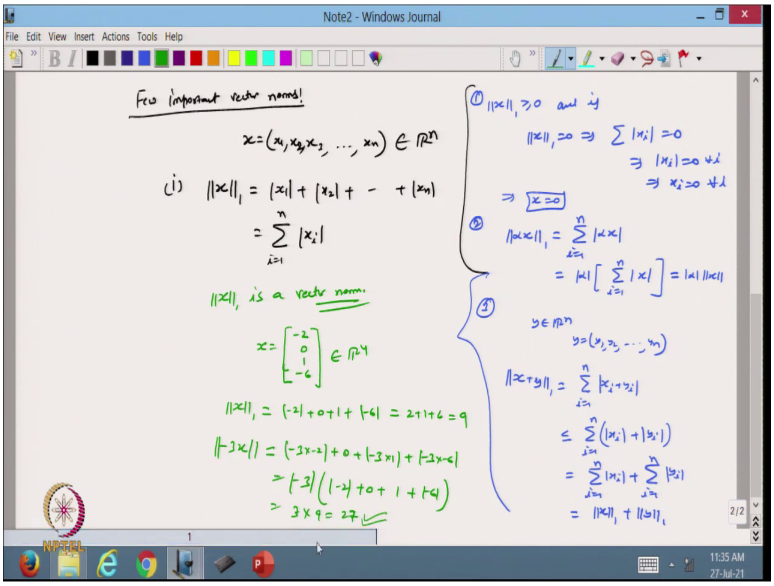

So, I can define a few important vector norms. Now, if I take x belongs to I take the x that is supposed I have (x_1, x_2,…, x_n). So, these are n components that belong to R^n. Now, I define the first norm and this is a subscript I write 1. $\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n| = \sum_{i=1}^{n} |x_i|$

So, denote the magnitude it is just the absolute value or the positive value, and this i from 1 to n. So, I am just writing this one and finding this 1 norm. Now, the thing is that first, we have decided whether a 1 norm is the norm or not. So, I know that I am taking this value. So, the first property is satisfied, because now I have to verify. So, let us verify quickly.

Now, this norm $\|x\|_1 \geq 0$, because I am just taking the absolute value and that is always greater than equal to 0. And, if I write this $\|x\|_1$ =0, which implies that the summation of x_i, that is going to be 0 and what gives me that and that is only possible when this is 0 for all i. It means that, which implies that my x=0 because x is a vector.

So, the first one is there. The second one is that suppose I take alpha x 1. $\|\alpha x\|_1 = \sum_{i=1}^{n} |\alpha x| = |\alpha| \sum_{i=1}^{n} |x| = |\alpha| . \|x\|$. Now, here when we define this type of thing, then I will tell you that, what is the meaning of magnitude here?

So, this one I can write as alpha I can take common, and then I can write this as I from 1 to n x, and this is equal to alpha norm x ok. So, this is just the magnitude I should write. Now, so, the second property is satisfied, the third one is if I write x+ y.

So, suppose I have a y_1 of another vector. So, let me define y as belonging to R^n. So, this is my y I am defining(y_1, y_2,…, y_n). So, this is another vector. I am just adding both the vectors and then defining the 1 norm. So, I am defining both the vectors as a component-wise addition. So, if you see from here, then this will become $\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i|$

$$\leq \sum_{i=1}^{n} |x_i| + |y_i|$$
$$= \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$$
$$= \|x\|_1 + \|y\|_1$$

So, this is satisfying so all the 3 conditions are satisfying. So, I can say that this is a vector norm. Now, the thing is that so, this is I can say that is a vector norm and we have verified also.

Now, I will show you that suppose I take an example I take $x = \begin{bmatrix} -2 \\ 0 \\ 1 \\ -6 \end{bmatrix} \in R^4$. Now, 1 norm if I define it here. $\|x\|_1 = |-2| + 0 + 1 + |-6| = 2 + 1 + 6 = 9$

$$\|-3x\| = |-3 \times (-2)| + 0 + |-3 \times 1| + |-3(-6)| = 3 \times 9 = 27$$

So, that is why we take the magnitude here. So, this is the magnitude. Now, this one norm is verified that this is a vector norm and satisfies all these conditions.

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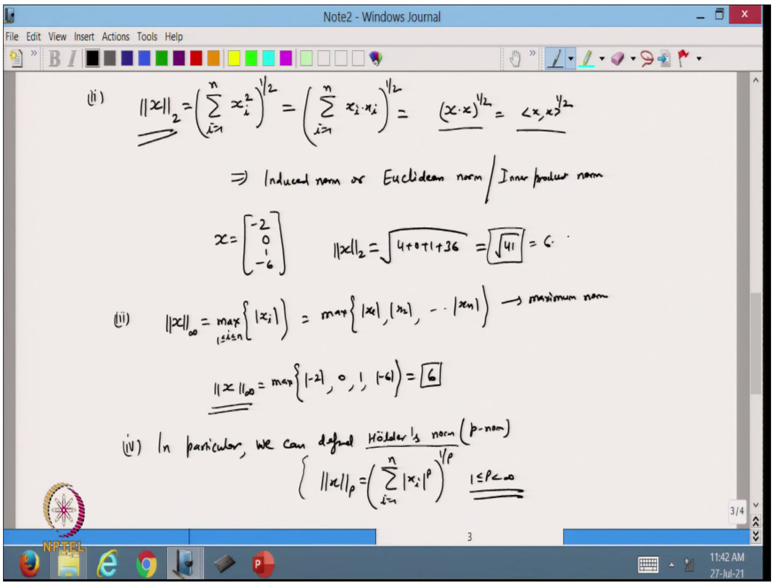

So, after this one, I take another vector norm. Let us call it 2. And, in this case, what we are going to do is I am defining again the summation, but here now I am writing $\| x \|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} = \left(\sum_{i=1}^{n} x_i x_i \right)^{1/2} = (x.x)^{1/2} = < x, x >^{1/2}$

So, this is the norm that is defined with the help of the inner product, which we have already seen in the previous lectures, whenever we deal with the inner product.

So, in this case, this norm is called. So, this norm is coming from the inner product. So, this is called induced norm or we also call it Euclidean norm or we also call it inner product norm, because it can be taken in the form of the inner product also that we have already seen. And, you can also verify that this also satisfies all the conditions of the vector space vector norm. So, this is also a norm, another type of norm, then after this, we will define the third norm.

So, if you take the same $x = \begin{bmatrix} -2 \\ 0 \\ 1 \\ -6 \end{bmatrix} \in R^4$ .So, 2 norms we already know it is $\| x \|_2 = \sqrt{4+0+1+36} = \sqrt{41}$. So, it is 6 points something this value is coming, and my x 1 norm I have defined it is 27. So, this is the norm we have taken. The third one is how we are defining and we call it and I put the subscript infinity here.

So, in this case, what are we doing? We are taking the maximum because the first condition is that it should be positive. $\|x\|_\infty = \max_{1\le i\le n}\{|x_i|\} = \max\{|x_1|,|x_2|,.....,|x_n|\}$

So, this is the maximum we are taking. So, this is called the maximum norm. So, it also satisfies all these conditions, because when the maximum is 0 then the function itself is 0. This also we can verify and the third property triangular inequality we can also verify. So, this is also one of the vector norms and it is called the maximum norm.

Now, for this example, in this case, if I want to take it as the norm, then I need to take the maximum. So $\|x\|_\infty = \max\{|-2|,0,1,|-6|\} = 6$. Now, after this one, we also define a generalization of the norms. So, we can define that in particular, we can define what is called the holder's norm.

So, that holder's norm is defined as we call it p norm basically, we can call it p norm. That is equal to $\|x\|_p = \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}, 1 \le p \le \infty$ So, this is called the p norm or the holder norm and then I can define my p as 1 here, 2 here because if I define 1 it becomes the 1 norm, simply 1 norm, 2 if I define then become the 2 norms.

And, so, from here I can say that p, in this case, is always greater than equal to 1 and so, this is my holder's norms, I just want to define this one that this way we can define any norm of this type. Now, after defining these types of norms, we see how these norms are different from each other.

One thing we have already seen, that if I take the same vector, then 1 norm gives 27, 2 norms gives root 41, and the maximum norm gives 6. But, if you see in reality this is the maximum, that x 1 is giving the maximum value and 2 norms are in between and the infinity norm is the lowest value. Because it is 6 points, but here it is 6. So, let us see how it is defined?

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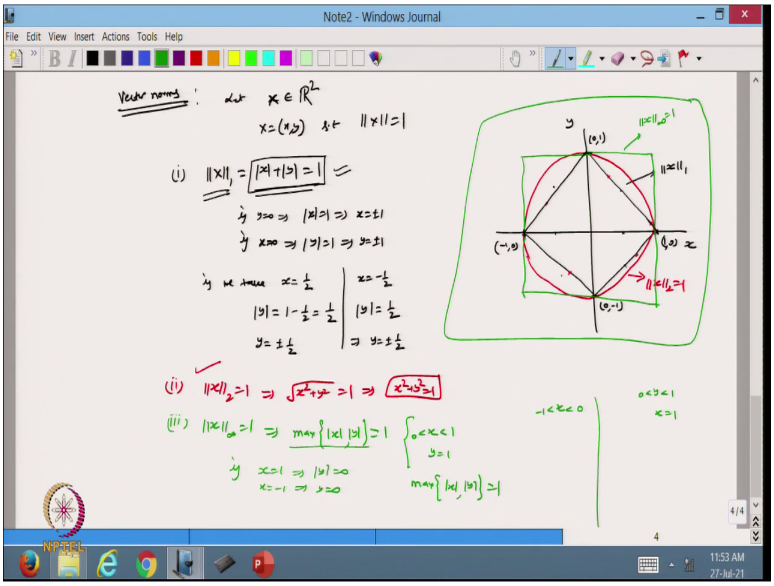

Now, vector norms, why is it called in this form? So, we want to see geometrically, how these vector norms look like? So, let we take x $\in R^2$. So, this x =(x, y). So, I am defining it from R square. So, this is my vector. Now, I take the first I define this vector $\| x \|_1 = | x | + | y | = 1$. So, let us take this one. So, I will just take a vector x from R^2 such that, the norm of x is 1 I just define this one.

Now, I am not writing which norm is 1, it is 1, it is the norm is 1. So, the first case I am taking is that let us take this norm 1 with norm 1. So, if it is equal to 1 it means that $\| x \|_1 = | x | + | y | = 1$. Now, if I take geometrically I represent x and y plane then from here if you see then a mod of |x|+|y|=1. Now, if I take y =0, then |x| = 1 it means suppose I go here it is (1, 0) and if I go here it is (1, 0).

So, if I take y =0 ,x will |x| =1, which means the x will be here, it can be here also. Similarly, if I take x =0 ,|y| =1, so if I take, if y =0 which implies |x|=1, and that gives me that x is plus-minus 1 and if x is 0, then I can define |y|=1. So, y can be plus-minus 1. Suppose, it is 1 here and it is 1 here. So, I can define where it is (0 ,-1) and this is (0, 1).

Now, all other properties do so because we have this condition. So, what will this condition look like? Suppose, I take if we take maybe I just take x is equal to half, then I can find from here mod y can be written as 1 minus half, that is half. So, it means my y can be plus-minus half. Similarly, if I take x is equal to minus half, then the same thing will happen modulus y will be again half because I am taking the modulus value and that will give me y is equal to again plus-minus half ok.

So, the same condition I can go from here, it means for x is equal to half or x is equal to minus R. So, suppose this is my half and this is minus a half. So, for both x half and minus half this and this value, x half y is giving me half plus half and minus half. So, plus half means I am going here and I am going here half, half. I am going here and I am going here.

So, this way I can take different types of things and if you see, then it is giving you a diamond-like shape with this value, this value, this value, and this value. So, if you see then this is my 1 norm. So, all the points lying on this diamond-like shape is a rhombus, we can say that if I take the 1 norm. So, all the points lying on this diamond shape satisfy this 1 norm property. So, if I take any vector, if I choose any point here or my I choose here all these vectors have the 1 norm equal to 1.

So, this is 1 norm we have taken, the second I take 2 norms is equal to 1. $\|x\|_2 = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1$ So, I can put a circle here, this is my circle. And, this circle shows you 2 norms are equal to 1.

So, all the points lying on the circle are satisfying the norm property that their norm =1, 2 norm is equal to 1. And, the third one is that I take that this infinity norm is equal to 1, which implies that the maximum of is equal to 1. Now, this is only possible if I take y =0, then x =1, and if x =0 y =1 or plus-minus 1.

So, x for x plus-minus 1, y =0 and for y plus-minus 1 =0. So, this satisfies this condition also. Now from here, I am taking the maximum value. Now, if I take a line just I choose x =1, then my y =0 or x =-1 my y =0, only then it is possible ok. So, it means that it should pass through this one.

Now, from here I suppose I take x is equal to half, then y should be I can choose any value. So, if x is equal to half, let us take x. I choose between 0 and 1. Suppose I take x between 0 and 1 and I take y =1. So, if you take these values, then it is the norm the maximum norm, in this case, will be 1. So, if I take y = 1 and x is lying from 0 to 1.

So, y =1 and x lying so I am, so I am talking about this line. Now, if I choose x this value from -1 to 0, then again this norm will be 1. So, I am talking about this line, this line, the same thing I can go for. I will choose my y from 0 to 1, and x =1. So, x =1 and y are moving from 0 to 1 and then from -1 to 0 also. So, I can take this line here. And, similarly, I can take this line here and this is mine here.

So, if you see from here, then it is a square and that represents all the points x whose infinity norm is 1. So, this way we can see that the rhombus is coming in that circle, that is a 2 norm and this circle is coming inside the given square of a dimension 1 or 1 cross 1. So, that shows how the norms look geometrically for a given vector in this case?

So, that is why this norm infinity norm is given the name infinity that does not mean that it is giving the maximum value, but by this presentation, we show that it covers the whole area under this, so all the points lying on this square satisfy this condition. So, that is the way we can represent the norms geometrically. So, after defining these norms the vector norms.

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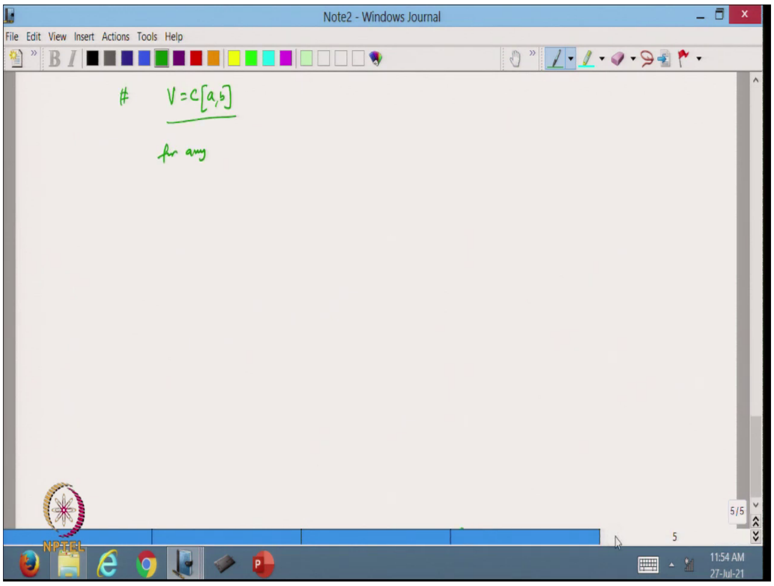

Suppose, I take the vector space I suppose I take vector space V that is all the continuous functions defined on the interval [a, b]. So, this is I know that this is an infinite-dimensional vector space. Here, we have defined whatever function we are defined here, we have defined for n-dimensional vector space. Now, we are taking the infinite dimension vector species.

Now, in this case, I want to define the norm. So, let us take for granted that, before that 1, I just want to tell you that once we can define the norm of a given vector, then we can talk about the distance between the two vectors. So, that also we can define.

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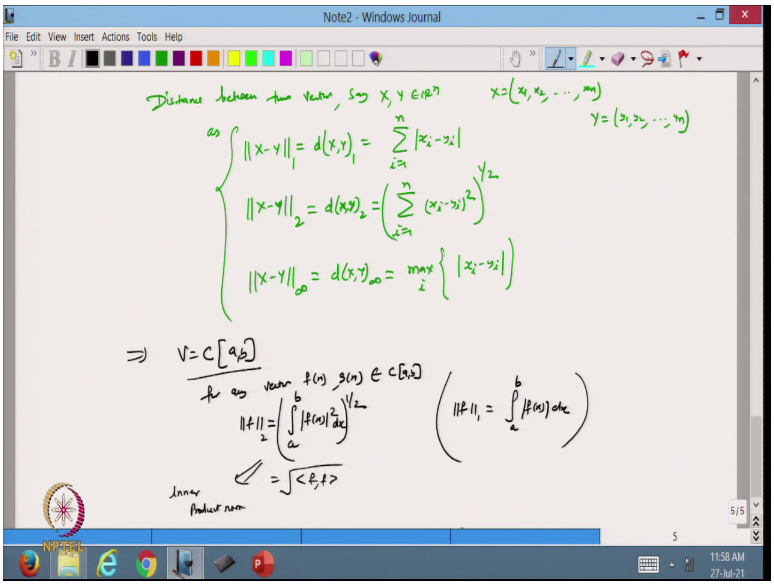

The distance between two vectors says I can just take capital X and Y, capital X or maybe x_1 and x_2 in R^n, or maybe I just take X and Y belonging to R^n . So, I can take the distance between these two so I will take X-Y defining the norm. So, this is also written as the distance between X and Y. And, how can we define this one depending upon which norm we are taking?

So, if I take the 1 norm then it will be printed by this one and this will be equal to taking from i from 1 to n. And, then so suppose my X= (x_1, x_2,…, x_n) this is a vector and I take y as another vector(y_1, y_2,…, y_n)., then I can define from here $\| X-Y \|_1 = d(X,Y)_1 = \sum_{i=1}^{n} | x_i - y_i |$ .So, this is the distance we can define with the 1 norm. Similarly, I can define the distance in 2 norms. So, that is my distance in 2 norms. $\| X-Y \|_2 = d(X,Y)_2 = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$. So, we defined infinity norm as $\| X-Y \|_{\infty} = d(X,Y)_{\infty} = \max\{| x_i - y_i |\}$ This way we can define the distance between two vectors, the given vector norm. Now, we can define another vector space v c with the continuous function.

So, in this case how can we define the norm? Now, this is an infinite-dimensional space. So, we know that so, by the way, we have defined the inner product, the same way we can use it for finding the norm. So, in this case for any vector f and g(x) that belongs to the continuous set of continuous functions [a, b] so this is the vector space. Then, here I can define the norm of the function f.

So, I will define this norm as taking the integral from a to b my function f(x), and then so suppose I take want to take 2 norms $\| f \|_2 = \left(\int_a^b | f(x) |^2 dx \right)^{1/2} = \sqrt{< f, f >}$ So, this is the integration we are taking. So, this is my 1 norm $\| f \|_1 = \int_a^b | f(x) | dx$, this is my 2 norms because this norm we have already seen.

So, if you see from here I can write this as an f taking the norm with f and then taking the under root. So, that thing we have already seen, this is the 1 norm we have defined. So, this is I can call it inner product norm also and the same way I can define here the infinity norm.

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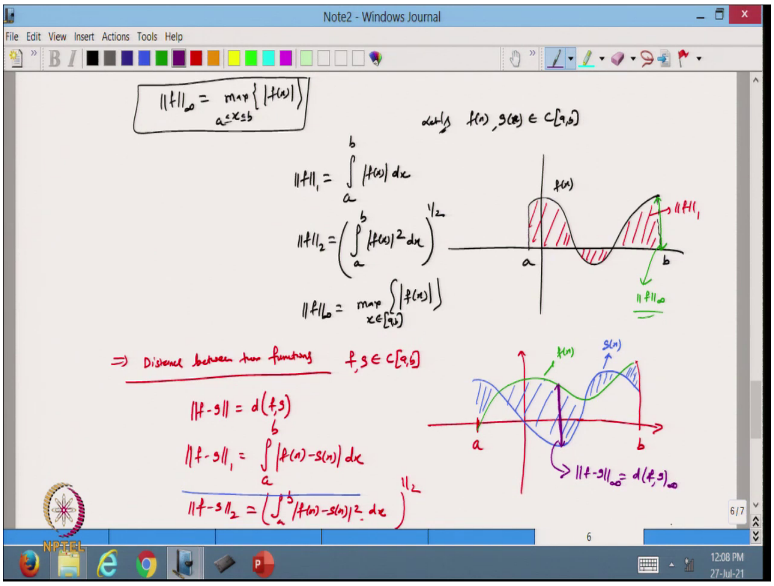

So, infinity norm we are defining just taking the maximum value of the function $\| f \|_{\infty} = \max_{a \le x \le b} \{ | f(x) | \}$. Because it is a set of continuous functions and the interval is a closed interval. So, we know that we can attain maxima in this interval. So, that maximum we can take as an infinity norm.

Now, if you geometrically see these things, then you can see from here that suppose I have a function. So, let us take a function f(x) and g(x) belonging to this one [a, b]. Now, suppose I have a function this function I there and suppose this is my a and this is my b. So, this is my function f x. So, in this case, I want to define it as the norm. So, if I want to define it as 1 norm. So, 1 norm if you see, then this is taking the interval from a to b and then f(x).

So, it means I am taking the modulus value of the magnitude and then putting the integral $\| f \|_1 = \int_a^b | f(x) | dx$. So, it is always a positive value. So, if you see then it gives the area under the curve. So, it will be equal to this part and then this part also because positive will be just coming here and then this part. So, this is my I can say that, 1 norm. Now, if I want to define the 2 norms. So, 2 norm is $\| f \|_2 = \left(\int_a^b | f(x) |^2 dx \right)^{1/2}$.

So, I am taking the square of this function and then dividing the square of this function, then integrating, and then taking the square root. So, in that case, there is. Now, if I take f infinity norm. So, it is $\|f\|_\infty = \max_{a \le x \le b}\{|f(x)|\}$. So, I have taken the magnitude of the given function and taken the maximum value.

So, wherever it is maximum, suppose this goes like this one and suppose this is the point where it is maximum or maybe I can just this is the point I think it is showing the maximum value here. So, in this case, this value is coming up. So, this green part will be the infinity norm. That is my function taking the modulus value and then finding the maximum of this value, so this is the maximum value here so, this is you can say that this is an infinity norm it is giving.

So, once we define this norm geometrically also, we can define another property and that is the distance between two functions. So, any f and g, I am taking belongs to the vector space that is a continuous function. So, from here I can define $\|f-g\| = d(f,g)$. So, d(f, g) I can define. So, maybe the same way we can define now, so let us draw this one.

Now, suppose I have one function, this is one of the functions and another function is like this function. Suppose, I write this way and suppose this is my a and it is going up to b. So, now, if I want to define the distance between these two so, if I define like maybe 1 norm.

$$\|f-g\|_1 = \int_a^b |f(x)-g(x)|\,dx$$

. So, what I am going to define, I am taking every point f(x) -g(x). What are the values there and then magnitude? So, if you see from here then it is giving you this part. So, this is the distance I can define with the help of 1 norm, or maybe I can define the 2 norms also, I am giving the distance here $\|f-g\|_2 = \left(\int_a^b |f(x)-g(x)|^2 dx\right)^{1/2}$.

So, again this is a function f another is g(x) taking the difference between these two value squares and then right. So, it is giving the areas in between the given functions. In the same way, I can define, now I can define another way.

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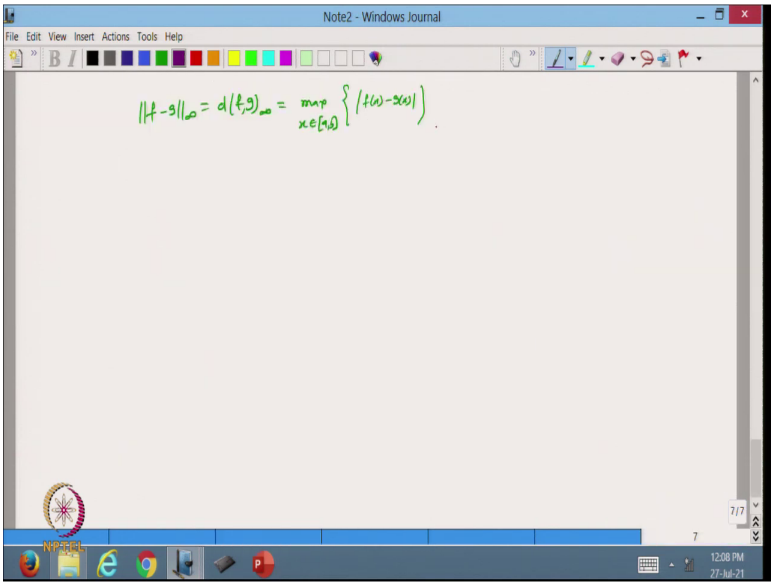

Let us see infinity, which is the distance between f and g infinity form. So, it will $\| f-g\|_\infty = d(f,g)_\infty = \max\{| f(x)-g(x)|\}$. Now, we use another colour. So, I am taking the difference of this one and if you see from here then it looks like that the difference is maximum here this part.

So, I am taking the difference of this function everywhere, I am I will make the difference, and then I will choose the maximum of all this distance. So, this is the distance between these, this is a distance and then I chose this one. So, if you see from here this is a maximum. So, this part whatever it is there that I can highlight.

So, this will be f minus g infinity norm or I can write that this will be the distance between f and g in the infinity norm. So, that will be my infinity norm. So, geometrically this is the way we can represent the different norms and we can show them, what how they look like for a given vector? So, we will stop here. So, in this lecture, we have discussed the vector norm.

And, then we have defined how they look geometrically for a given function or a given vector in the R^2 space, or if in the space of continuous functions defined on the interval [a, b]. So, in the next lecture, we will continue with this one and discuss some other properties. So, thanks for watching.

Thanks very much.