Matrix Computation and its applications
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Lecture - 45
Inner product on complex vector spaces and Cauchy – Schwarz inequality

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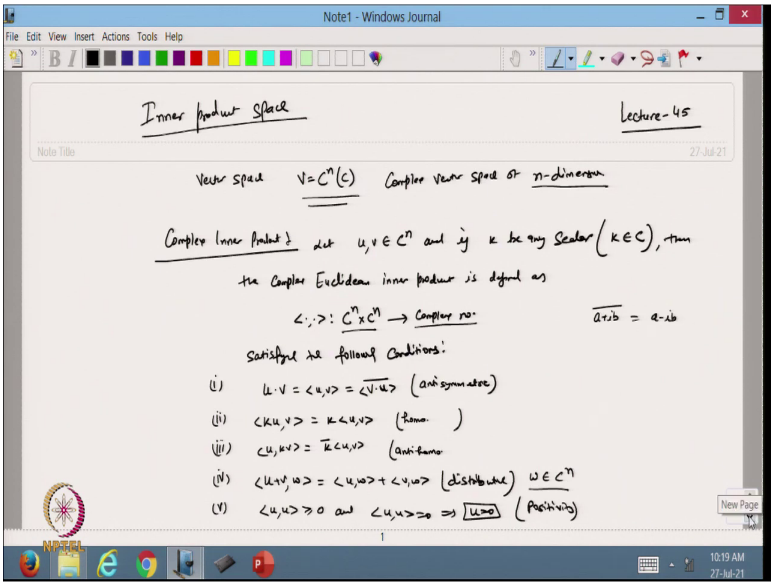

Hello viewers. Welcome back to the course on Matrix Computation and its application. So, in the previous lecture, we have just discussed the complex vector space. So, today we are going to use the inner product for how we can define the inner product for the defined inner product on the complex vector space.

So, let us do that one. So, in the previous lecture we have discussed that suppose we have a vector space that is v. So, we are talking about the nth dimensional vector space that we are defining on the complex number. So, it is a complex vector space of n dimension.

Now, we want to define the inner product over this vector space. So, how we can define it is so, let us define a complex inner product. Now, let me take 2 vectors u and v that belong to C^n. So, it is a C^n means I am just taking the field as a complex number. So, we can remove this one. So, we are talking about the field that is a complex number. So, u and v belong to

C^n, then and if k be any scalar, so, k be any scalar belongs to the complex number C, then the complex Euclidean inner product.

So, the complex Euclidean inner product is defined as. Now, we define the inner product as u, v by this sign, now this is the inner product if it is satisfied. So, we define the inner product as this one. So, you can see from here that maybe I can write like this. So, I will define this inner product. So, that is from C^n * C^n to real numbers or maybe I can say complex numbers set of a complex number.

Now, how we are going to define it because in the real vector space whatever we are defining was the real numbers now we are talking about the complex number. So, we define it like this one and so, this is the complex Euclidean inner product satisfying the following conditions, conditions, or axioms. So, the first one is the inner product I am defining. So, u dot v because I am talking about the C^n. So, I am just defining u.v So, you can write this one as the same as I am defining the inner product.

$u.v = < u, v > = < \overline{v.u} >$ So, we are taking the conjugate here and this property is called antisymmetric in the real vector space it is symmetric, but here we are defining the anti-symmetric.

The second one is that we take a scalar $< ku, v > = k < u, v >$, this is we know that additive or distributive, the third one is if I take $< u, kv > = \bar{k} < u, v >$. So, this is or maybe I can write this one as homogeneity and this is anti-homogeneity, the fourth one is $< u + v, w > = < u, w > + < v, w >$

So, this is called distributive and the last one is that if you take $< u, u > \geq 0, < u, u > = 0 \Rightarrow u = 0$ .So, this is called positivity. So, the condition here is changing only concerning the conjugate, because we are dealing with complex numbers. So, here I can write that w belongs to C^n and this is true for all u v.

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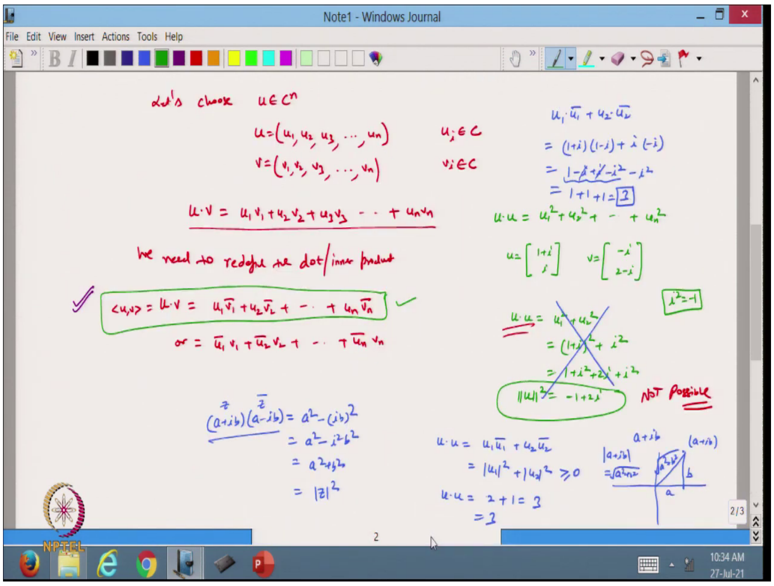

Now, let us define the inner product. So, suppose I take a vector. So, let us take so, let us choose a vector u belonging to C n then definitely u is equal to u_1, u_2, u_3,… u_n, where each u_i is a complex number then i take v as v_1, v_2, v_3, …v_n. So, each v_i is also a complex number then we define u dot product v.

So, in u dot product v what we are going to do is that I will apply $u_1 v_1 + u_2 v_2 + u_3 v_3 + u_n v_n$. So, let us apply the same way we are applying for the real vector spaces now the problem, in this case, is that suppose I take u. u then this will be $u_1^2 + u_2^2 + u_n^2$.

So, for example, so, let us take example let us take u from maybe I take it I just take it 1 plus i and then I suppose I take this one and I take $u = \begin{bmatrix} 1+i \\ i \end{bmatrix}, v = \begin{bmatrix} -i \\ 2-i \end{bmatrix}$. Now if I take the dot product of this one that is the way we are defining. Now, if I take u dot product u and if I write like this one then it is giving us that I should have $u_1^2 + u_2^2$ this way.

Now from here if you see it will be $u.u = u_1^2 + u_2^2 = (1+i)^2 + i^2 = 1 + i^2 + 2i + i^2 = -1 + 2i$

which means this is if you represent this is giving you the length and that length is coming from a complex number which is not possible because the complex length should be a real number.

So, for this one what we need to do is that we need to redefine the dot product dot or inner product. So, what I do, I will define u dot v or maybe u v here I am defining because we are dealing with C^n. So, now, we will define like this one $<u,v>= u.v = u_1 \bar{v}_1 + u_2 \bar{v}_2 + u_3 \bar{v}_3 + ... + u_n \bar{v}_n$ or $= \bar{u}_1 v_1 + \bar{u}_2 v_2 + ... + \bar{u}_n v_n$ because we have to take the conjugate of the other one to satisfy this condition. So, generally, we can choose anyone, but generally, we choose this definition. So, this definition we are going to deal with, but both are the same.

Now, this can be verified from here. Now if I use this one then so, this is the wrong way now I define u dot u by this way. So, what I am going to do is that I am going to take $u.u = u_1 \bar{u}_1 + u_2 \bar{u}_2$. So, this will be I know that this will $= |u_1|^2 + |u_2|^2 \geq 0$ and from here you know that this is equal to.

So, real number plus real number, this is always greater than equal to 0 and then we can find out its value. So, if I take the value of u. So, u dot u is I am taking that modulus value of u 1. So, the modulus value I am just defining here. So, suppose I know the value a plus i b is my complex number. So, this is my supposed a and b.

So, let my complex number a+i b be this value, then I know that this is a b. So, it says a square plus b square under the root and I know that the modulus value of a+ib is this one under the root. So, and also I can also verify this as here also that a+ib is multiplied by a-ib. So, this one I can write as $a^2 +(-ib)^2$, and this will be $a^2-i^2 b^2$ and that will be $a^2 +b^2$.

So, from here and I am taking this value as a^2 here. So, now, this can be written as. So, this can be written as supposing this is my complex number z and this is $\bar{z}$. So, I can write from here that this is $\bar{z}^2$, because $\bar{z}$ is a square root of this one. So, now, we can use this one. So, it is 1+i. So, I can write from here 1 is i, a is 1 and i is also the coefficient of i is 1. So, it will be 2, now 1 plus 1 2 and so, u_2 = i. So, if it is 1, it will be 3. So, my u and u dot u is 3, and this one I can write like this also.

So $u.u = u_1 \bar{u}_1 + u_2 \bar{u}_2$ = (1+i)(1-i)+i(-i) = 1-i+i-i^2 -i^2 =1+1+1=3 .So, from here you can find this dot product in this way, and that works fine. So, this is the change we have done in the inner product.

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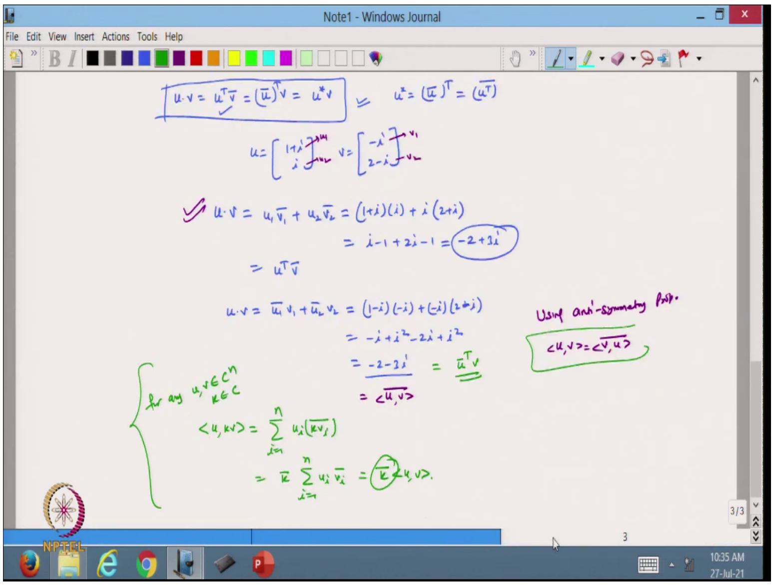

So, it means that for u.v I can define as so, now if I want to define in the matrix form. So, I can define as $u.v = u^T \bar{v} = (\bar{u})^T = u^* v$ I know that this is equal to conjugate transpose of transpose conjugate they are same.

So, I can define any of this one all will be giving the same values because you can define the same way here and you get the same value. So, this one is our inner product in this case and I have my u and v. So, if I take 1 +i and i and v are -i ,2 -i now I define u.v.

So, this is I am just defining as transpose or maybe I just take it u dot v as; so, this is my $u.v = u_1 \bar{v}_1 + u_2 \bar{v}_2 = (1+i)(i) + i(2+i) = i - 1 + 2i + i^2$,this one is the dot product of u v.

So, and this one I can write as u^Tv bar. So, we have taken this value or maybe I can define this as

$$u.v = v_1 \bar{u}_1 + v_2 \bar{u}_2 = (1-i)(-i) + (-i)(2-i) = -i + i^2 - 2i + i^2 = -2-3i$$
$$= \overline{<u, v>} = (\bar{u})^T v$$

So, if you see from here I have taken in this form now it is coming like this one, but using anti-symmetry property that shows that u v if you take the dot product that is equal to v u this one. So, u dot v I have taken with the definition and we have taken this is the definition. So, that is our definition.

So, one definition we have to take and that definition is this one, the same way I can define here maybe we can check with the. So, I can define it from here.

So, it becomes this can be written as u conjugate transpose v this one. So, I am using the anti-symmetric this will be the same. So, this way we can define the inner product in the set of the complex in the complex vector space. Now, so, after doing this one now we want to check whether it is satisfying all these properties or not.

So, all the other properties are ok and anti-symmetric we have just seen, this is what we have already seen in the case of real numbers also. The only thing we have changed is this one, so it is anti-homogeneity. So, this property we can check in this case also. Now we have defined that u and k v. So, if I take this one for any u and v belongs to C^n and k belongs to the complex number. So, then I can write this as my $<u, kv> = \sum_{i=1}^{n} u_i (\overline{kv_i}) = \bar{k} <u, v>$

So, that is why this conjugate of the scalar will come out in this one. So, that is the only property we need to satisfy in the case of complex vector spaces and the anti-symmetric we have already done, all other properties are the same and we can verify ourselves.

So, after doing this one, I will define the inner product. So, let us take the help of the inner product to check whether two matrices are orthogonal to each other or not. Now, before that one, I will define the inner product.

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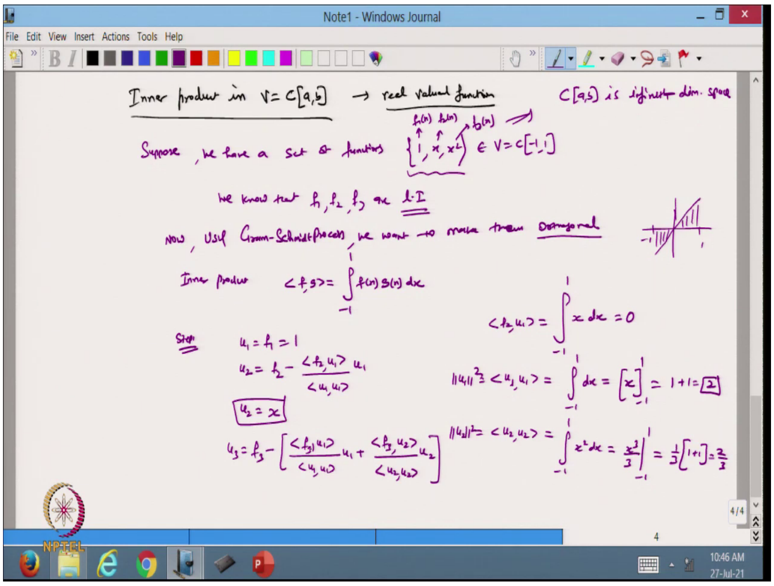

So, let us define the inner product in the vector space V that is a set of continuous functions from a to b. So, here we are taking a real-valued function. So, we are dealing with only real numbers. Now we have already defined the inner product in the case of polynomials, yeah.

So, here we have defined the inner product in the case of the complex in the case of continuous function and then we have defined this one. So, we have already defined what we are going to do. I will use this to find out how we can say that two continuous functions are orthogonal to each other or not.

So, let us do this one. So, this is an inner product and we have already defined, now suppose we have a set of functions I just take the standard basis. So, let us take $\{1, x, x^2\}$ So, these are the bases I have taken for three functions belonging to this one. So, they belong to the vector space V.

So, which is C, and let us define from a to b. So, let us take this one as in this case. So, we have to define the interval. So, I just defined this as from minus 1 to 1 or we can define anything, but let us just take it from minus 1 to 1. Now, these are sets of vectors so we know that. So, I can take this as an f 1. So, this one I can write as a f 1 x, this is my f 2 x and this is f 3 x.

Now, we know that f1, f2, and f3 are linearly independent. So, they are linearly independent because if I saw that we compare the coefficients of the same power of x and then we can define then this is linearly independent then. So, and also this function, this vector space is infinite-dimensional.

So, it is not the basis it is just I am taking the set of functions. So, C[a, b] is an infinite-dimensional infinite-dimensional space and this is a set of vectors, it is not the basis of a set of vectors and they are linearly independent. So, that we have already seen. Now, using the Gram Schmidt process we want to make them orthogonal. So, this one we want to make orthogonal. So, this is a Gram Schmidt process.

So, let us see how we can find out. So, we are making them orthogonal, not orthonormal. So, let us do that one. So, step 1 now first before that one we need to define the inner product. So, the inner product we are taking for any $< f, g >= \int_{-1}^{1} f(x)g(x)dx$. So, this is the inner product we are defining, and that we already know that this is an inner product defined on the set of all the continuous functions from a to b.

So, let us take step 1 first. So, this is I am just taking $u_1 = f_1 = 1$. So, this is a vector I am defining.. Now I take u2. So, I know that $u_2 = f_2 - \frac{< f_2, u_1 >}{< u_1, u_1 >} u_1$ So, by the Gram Schmidt process, we already know that this is the way we can define u 2.

Now from here first I need to define what is my $< f_2, u_1 >= \int_{-1}^{1} xdx = 0$

So, this dot product is 0 it means that both f2 and u1 are already orthogonal to each other, and also if I want to see $< u_1, u_1 >= \int_{-1}^{1} dx = [x]_{-1}^{1} = 1 + 1 = 2$. So, from here, but this thing is already 0.

So, it means from here I can write my $u_2 = f_2 = x$ and this thing is already 0. So, this is my u2 only. So, $< u_2, u_2 > = \int_{-1}^{1} x^2 dx = [\frac{x^3}{3}]_{-1}^{1} = \frac{2}{3}$ and from here I know that this is equal to u2 norms u1 norm square and that is 2/3.

Now, I want to define what my u3 is. So, $u_3 = f_3 - \left[\frac{< f_3, u_1 >}{< u_1, u_1 >} u_1 + \frac{< f_3, u_2 >}{< u_1, u_2 >} u_2 \right]$

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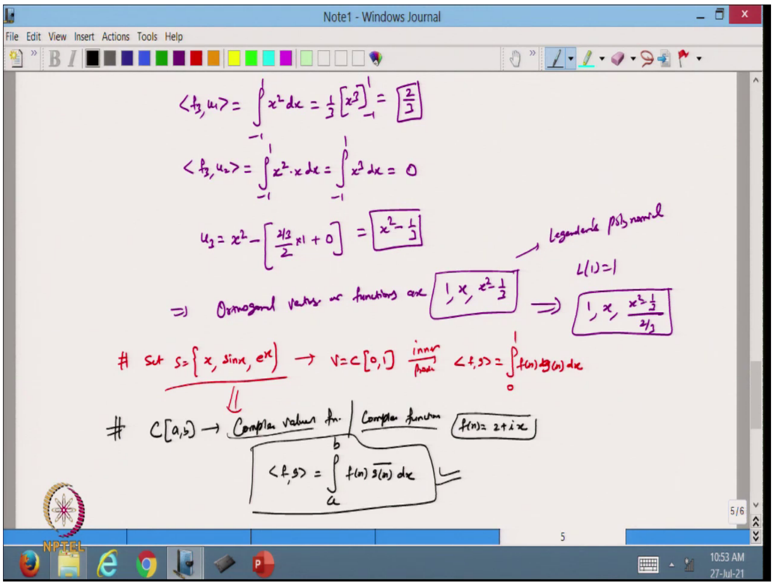

So, I need to define first what is f3 with u1. So, it is from $< f_3, u_1 > = \int_{-1}^{1} x^2 dx = [\frac{x^3}{3}]_{-1}^{1} = \frac{2}{3}$.

$$< f_3, u_2 > = \int_{-1}^{1} x^2 . x dx = [\frac{x^4}{4}]_{-1}^{1} = 0$$

$u_3 = x^2 - \frac{1}{3}$ .So, this is what I get. So, from here I get orthogonal vectors or functions in this case are. So, this is 1, x, and $x^2 - \frac{1}{3}$ and if you see from here then this will look like the

Legendre polynomial Legendre's polynomials, the only condition is that they satisfy one condition that Legendre polynomial satisfies this condition.

So, from here we can write this as it is satisfying this one only the problem is coming here 1,1/3. So, it is 2/3. So, if I want to write from Legendre polynomials I can write as 1, x, and x^2 -1/3 / 2 / 3. So, this becomes the Legendre polynomial. So, this way we can even define the same polynomials in the interval from 0 to 1, and then you can make them orthogonal.

So, everything depends upon which type of inner product we are defining, what is the range over the integration. So, everything we can do with the help of the Gram Schmidt process can make them orthogonal. So, this is the way we can define what the functions are.

Even you can take another set of functions I can just define the set of functions like I take a set as maybe I can take x, sin x and exponential x3 function I can take and then I can from there I can take this belongs to a vector space C from maybe I can take 0 to 1, then these functions are linearly independent.

So, I can make them orthogonal by defining the inner product. So, in this case, this is the inner product here I can define the inner product as $< f, g >= \int_0^1 f(x)g(x)dx$. So, then the inner product will change the integration from 0 to 1 it will be and the same procedure you can follow. So, this way we can define.

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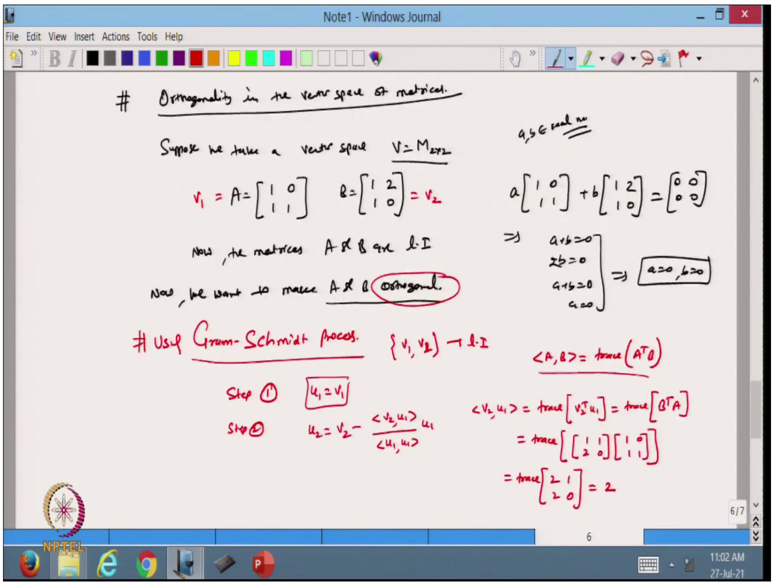

Now, after doing these functions the next thing we want to discuss is how we can say the orthogonality in the space of in the vector space of matrices. Now before doing this one, these things we have defined in the real-valued function or the real function means the functions are real.

The same way I can define the set of a continuous function over the interval a, b complex-valued function or complex functions means I can define the function f(x) as maybe I can define it as 2 +ix. So, this is one of the functions I have defined. So, in this case, if you see the inner product I will define that will be $< f, g >= \int_0^1 f(x)\,\overline{g(x)}\,dx$ So, this way we can define.

So, this will be you know the product we can define when we are dealing with complex functions. Now we define the orthogonality in the case of vector spaces of matrices that we have already seen. Now suppose we have a vector space V of M 2 cross 2. So, I am taking the 2 crosses 2 matrices just for the example.

Now let us take I take a matrix $V_1 = A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = V_2$ Suppose I am taking this matrix. So, these 2 matrices belong to this set. Now the matrices A and B are linearly independent. So, that we already know or maybe we can verify by just taking the linear combination $a\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}+b\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$.

So, these are the scalars and that I put equal to 0 matrices because we are taking the linear combination and from here you can see that I can define my $a+b=0, 2b=0 => a=0, b=0$. So, they are linearly independent.

Now, we want to make A and B orthogonal. So, this one I want to do, I am just taking the 2 matrices and then the same procedure we can extend for 3 matrices 4 matrices n-dimensional any n number of matrices. So, this one we can do. So, now, we are going to use the Gram Schmidt process. So, what we are going to do. So, let this be the matrix so, I call it maybe I can call it v1. So, I just call these matrices vector v1 and I can call it v 2.

So, I know that the set v1 and v2 is l I. So, let us make them orthogonal. So, step 1 so, but before that, I know that the inner product between the matrices A and B is a trace of A transpose B that we already know. So, I am using this inner product. So, step 1 is that I take u1 as v1. So, I am mating making them orthogonal not orthonormal, so it just does not matter whether they are normalized or not. So, I take u1 =v1.

Now, in step 2 I can take $u_2 = v_2 - \frac{< v_2, u_1 >}{< u_1, u_1 >} u_1$. So, this one we need to define. Now from here, I want to find what is this, v2 taking the inner product with u1 so, this one we need to define. $< v_2, u_1 >= trace[v_2^T u_1] = trace[B^T A]$. So, I am finding this value.

So, $= trace\left[\begin{bmatrix}1 & 1\\2 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\right] = trace\begin{bmatrix}2 & 0\\2 & 2\end{bmatrix} = 2$ .So, this is my trace that is coming 2 in this case ok. So, after doing this one I want to take what is my u1 with u1.

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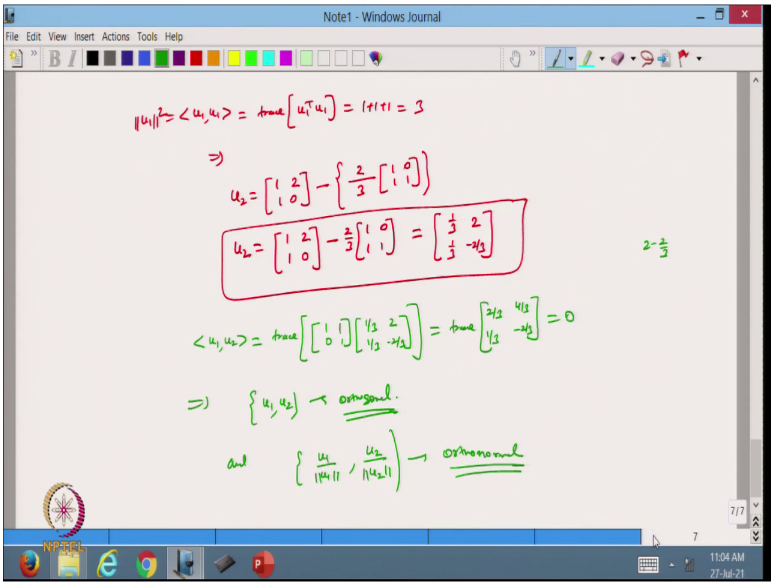

So, I want to define. So, I am taking the square or norm of this one. So, this is $< u_1, u_1 >= trace[u_2^T u_1] = 1+1+1 = 3$.

$$u_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - \left\{ \frac{2}{3} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 2 \\ 1/3 & -2/3 \end{bmatrix}$$

. So, this is my u 2. So, I have already seen that u1 is my v1 and then u 2 is my this one and you can also verify from here that if I take u1 u2 dot product.

$$< u_1, u_2 > = trace\left[\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 2 \\ 1/3 & -2/3 \end{bmatrix} \right] = trace \begin{bmatrix} 2/3 & 4/3 \\ 1/3 & -2/3 \end{bmatrix} = 0$$

So, the value is coming to 0. So, they are becoming orthogonal to each other. So, now, from here I can say that the set u1 and u 2 is orthogonal and if I take this set u 1 divided by its magnitude, u 2 divided by it means I am dividing by its norm. So, that becomes orthonormal.

So, using the concept of inner product and the Gram Schmidt process we can take any vector; it may be a function or it may be a matrix. Using the inner product we can define them, and then we can use the Gram Schmidt process to orthogonalize them. So, the same way we are doing here.

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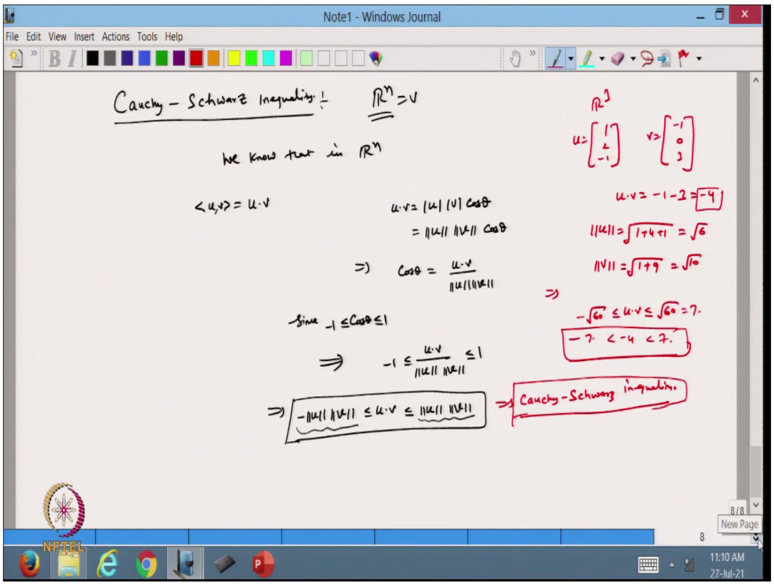

So, after doing this one I want to define a very important property taken from the dot product. So, this is Cauchy - Schwarz inequality. So, this is what we are defining in the R^n. So, in the R^n we know that we know that in R^n. So, this is my vector space V if I take the Euclidean. So, because in this case, we are defining the Euclidean inner product that is similar to a dot product. So, u dot v is what we are defining.

Now, in this case, we already know that u.v =|u| |v| cos θ that we have already know about the dot product of vectors in Rn and I can maybe write this as a norm does not matter, because ultimately it is a length of the vector u and v. So, from here we know that I can take

$$\cos\theta = \frac{u.v}{||u||.||v||}.$$

Now, since cos θ is always lying from -1 to 1 so, which implies that I can write that u dot v the dot product this can be written as lying from minus 1 to 1, and from here I can say that

$$-1 \le \frac{u.v}{||u||.||v||} \le 1.$$

So, it gives you the bounds on the dot product, it means if somebody asks that if you take the dot product of two vectors then how big and how small that can be. So, it is always if you take the $-\|u\|.\|v\| \leq u.v \leq \|u\|.\|v\|$. So, this property is called the Cauchy- Schwarz inequality.

So, this is Cauchy - Schwarz inequality. So, for example, let us take one vector. So, I just take in R3 I take a vector $u = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$. suppose I take this one and if I take the dot product u and v. So, this is $u.v = -1 - 3 = -4$, and now if I take the norm of u. So, the norm of u $\|u\| = \sqrt{1+4+1} = \sqrt{6}$.

Now, if I take the dot product of this one. So, it will be $\|v\| = \sqrt{1+9} = \sqrt{10}$

$-\sqrt{60} \leq u.v \leq \sqrt{60} = 7$. So, it is approximately 7 point something and u and v is -7 point something. So, u and v are coming minus 4.

So, it is satisfying that this is in this case it is less than -4 greater than -7, and less than 7. So, this is satisfying. So, from here we know that this inequality is satisfying. So, this one we can find from any vector with the help of this Cauchy - Schwarz inequality.

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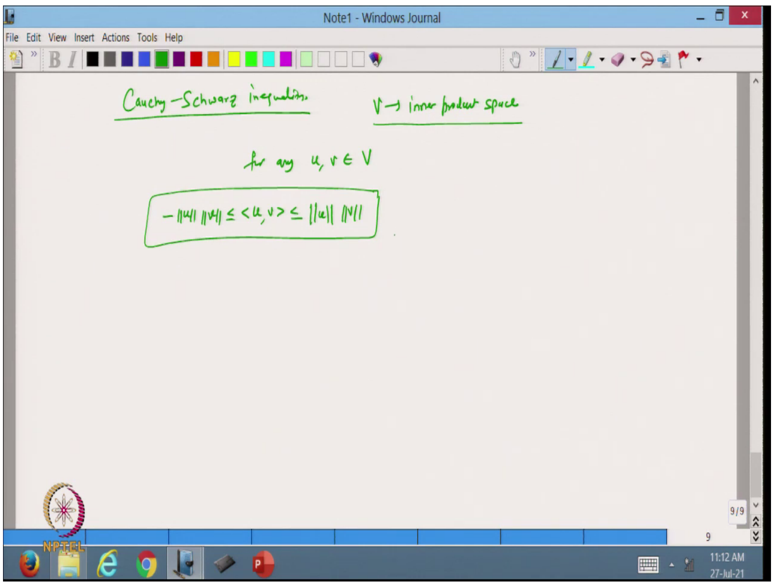

So, this is a very important inequality in the case of inner product space. So, based on this one I can now define the same Cauchy - Schwarz inequality. So, now, I define any inner product space. So, V is an inner product space then for any u and v belongs to the vector space V; I can define u, v inner product. So, this is my inner product u and v can be matrix also, it can be a function also, it can be polynomial anything.

So, then this is always I can write as minus the norm of u into the norm of v norm of u and norm of v. So, this is the same way we can define. So, this is the Cauchy - Schwarz inequality in the vector space that is inner product space having an inner product. So, this is also satisfied or applicable in any of the inner products. So, this is where we can define the Cauchy - Schwarz inequality.

Now, we will stop here. So, in today’s lecture, we have discussed that how we can define two functions or maybe two matrices orthogonal, and with the help of the Gram Schmidt process, we have shown that if we have a set of linearly independent vectors in terms of functions or matrices then we can make them orthogonal using the Gram Schmidt process.

So, and also we have defined another important property of the inner product space that is a Cauchy - Schwarz inequality. So, in the next lecture, we will discuss the norms and the other things of a given vector in a vector space. So, thanks for watching.

Thanks very much.