Matrix Computation and its applications
Dr. Vivek Aggarwal
Prof. Mani Mehra
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture - 44
Inner product on different real vector spaces and basics of complex vector space

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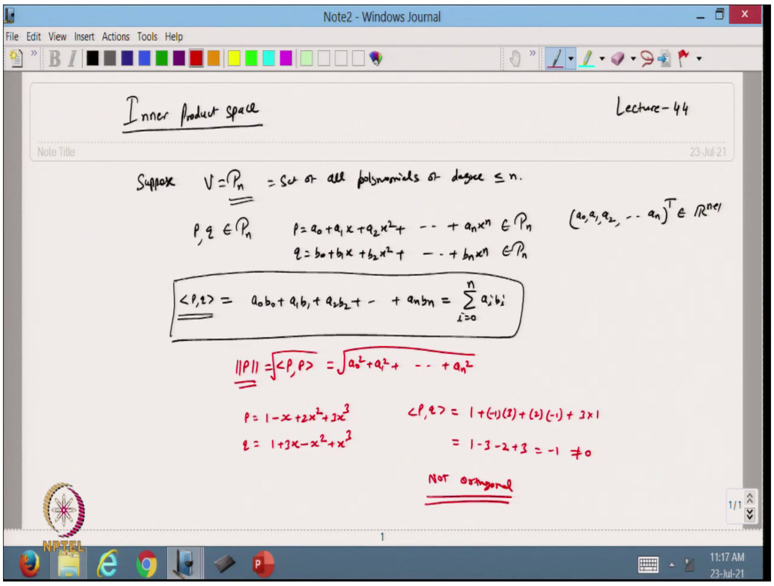

Hello viewers, welcome back to the course on Matrix Computation and its application. So, in the previous lecture, we have introduced the concept of inner product space. So, in this lecture also, we are going to discuss inner products in different vector spaces.

So, let us start with the inner product in the vector space of polynomials. So, suppose, we have a vector space V that is set of all the polynomials of degree n. So, I can say that the set of all polynomials of degree less than equal to n.

Now, I know that this is a vector space of dimension n plus 1. Now, from here, now I take two polynomials, suppose I take p and q that belong to the polynomial set of polynomials P, this one. So, suppose my p is basically I can call it

$$
\begin{aligned}
p \quad &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in P_n \\
q \quad &= b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n \in P_n \\
\langle p, \mathrm{q} \rangle \quad &= a_0 b_0 + a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=0}^{n} a_i bi
\end{aligned}
$$

So, for this one, I need to define the inner product in the polynomials. So, the inner product we are defining here is that because you can see that if we take the polynomial here of nth degree. Then, its coefficient a_0 , a_1, a_2 …., $a_n \in R^{n+1}$.

Similarly, I have a q that also the coefficient of this one makes the vector in R^{n+1}. So, I can say from here that a_0 , a_1, a_2 …., a_n if I take it as a vector, it is a column vector. So, this will belong to R^{n+1}. So, this is generally, what we are doing? We are taking a simple dot product of the coefficients of the given polynomial.

So, this way, we are defining this inner product in the space of matrices or space of a polynomial P_n. Now, the thing is that now we are able to define this in a product. So, we need to check or verify whether it is satisfying all the four conditions or not.

So, if you see from here, then this is just the dot product and this dot product we already know satisfies all the four properties of the inner product. So, this one, we can verify yourself that this is an inner product defined on the set of all polynomials. So, now from here, I will define the norm on the p.

So, this one will be equal to taking the inner product of p itself and then, taking the square root. So, from here, you can say that this is equal

$$
\| P \| = \sqrt{\langle P, P \rangle} = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}
$$

So, this is I can define the length of the polynomial or the norm of the polynomial.

So, from here if I take suppose I take a polynomial p and then, suppose I take this as maybe

$$
p = 1 - x + 2x^2 + 3x^3
$$

$$
q = 1 + 3x - x^2 + x^3
$$

So, let us take these two matrices or two of these cubic polynomials, then I want to check whether these polynomials are orthogonal to each other. So, I will just take the inner product of this.

So, inner product if I am taking, so this is 1 taking the constant and then, I will get $<p,q>= 1-3-2+3= -1\neq 0$,so, not orthogonal. But now, suppose I take the basis; standard basis.

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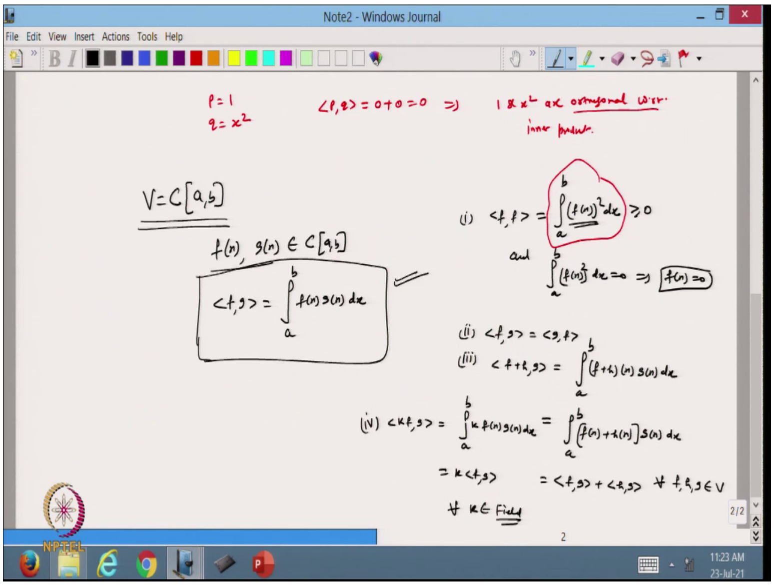

So, let us take the polynomial p = 1 and $q = x^2$, then I know that if I take the inner product of these two, then this will be 1 and then, coefficient of this one 0 and coefficient of $x^2 = 0$ here. So, that is equal to 0. So, I can say that the 1 and x squares are orthogonal with respect to the inner product.

So, whatever the inner product, we have defined this one, they are orthogonal to each other. So, this is the corresponding inner product. It may happen that we may define a different type of inner product also.

So, with this inner product, I can say that these two polynomials are orthogonal to each other. Similarly, we can define now that I take the vector space V as a set of all the continuous functions from [a , b] in the set in the close interval [a ,b].

So, this is a vector space, we know that it is an infinity dimension vector space, then I just take that two elements for this one or two vectors from this one. So, let us take f(x) and g(x) that belong to the vector space V. Now, I want to define the inner product of this one.

So, for this one, we take the inner product as

$$< f,g >= \int_0^1 x.x^2 dx = \left(\frac{x^4}{4}\right)_0^1 = \frac{1}{4} k \int_a^b f(x).g(x)dx = k < f,g >$$

$$|| f ||= \sqrt{< f,f >} = \sqrt{\int_0^1 [f(x)]^2 dx} = \sqrt{\int_0^1 x^2 dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

$$\frac{x}{1/\sqrt{3}} = \sqrt{3}x$$

$$\hat{f}$$

$$\sqrt{2} = 1 \times \sqrt{2}$$

So, actually this integration is coming from the dot product because the dot product is defined on the discrete vector and when we take the function, then this dot product changes into the integral. So, this is the inner product, we are defining on the space of all the continuous functions defined over the interval [a, b].

So, now, we can verify from here whether it is an inner product or not. So, we can verify all the four conditions. Now, the first one is that taking the inner product

$$< f,f >= \int_a^b [f(x)]^2 dx \geq 0$$

$$and \int_a^b [f(x)]^2 dx = 0 => f(x) = 0$$

because it is a positive function and positive function and we know that it is the area under this function $[f(x)]^2$. So, the area is always positive. So, it is always greater than equal to 0 and if this area is 0, then only I can say that the function itself is 0. So, I am satisfied.

The second one is that, if I take <f,g> = <g,f > So, it is symmetric also. Third one is that if I take <f + h ,g >, three functions I am taking define the inner product.

$$< f+h, g > = \int_a^b (f+h)(x) g(x) dx$$

$$= \int_a^b f(x) g(x) dx + \int_a^b h(x) g(x) dx$$

$$= < f, g > + < h, g >$$

So, this is true for all f ,h, g ∈ V, what we have defined here. So, the third property is also satisfied.

And the fourth property is then I can take any scalar, so we are defining this one.

$$< kf, g > = \int_a^b kf(x).g(x) dx = k\int_a^b f(x).g(x) dx = k < f, g >$$

So, this is true for all k belongs to the field. So, here the field is a real number. So, all these properties are satisfied.

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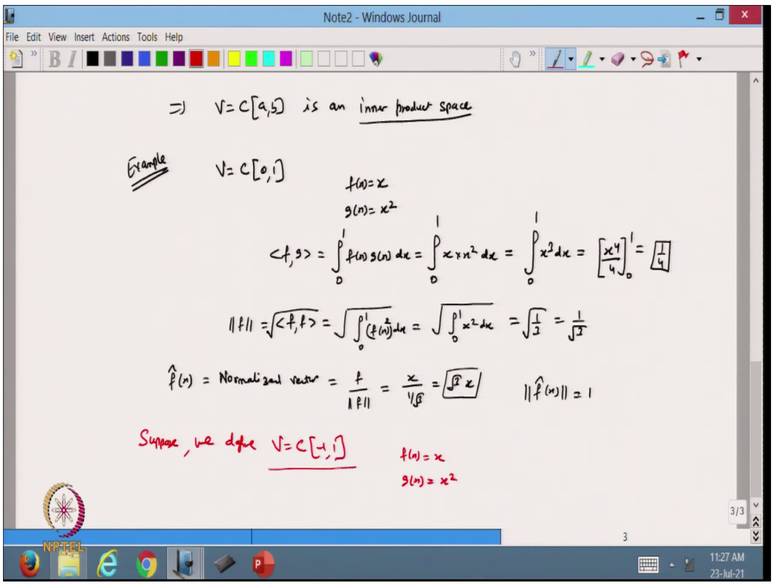

So, based on this one, we can say that this is an inner product and I can say that from here that V the vector space with a set of all continuous functions is an inner product space. Now, based on this one, suppose I take the vector space for example I take a vector space of all continuous functions defined from maybe [0, 1] and suppose, I take a function f (x)= x ,I take g(x)= x^2.

Then, I define the inner product of <f ,g>. So, this will be taking the

$$< f, g >= \int_0^1 x.x^2 dx = \left(\frac{x^4}{4}\right)_0^1 = \frac{1}{4}$$

Now, suppose I want to find its magnitude or its norm. So, this one, I can define as norm and I take the under root.

$$\| f \| = \sqrt{< f, f >} = \sqrt{\int_0^1 [f(x)]^2 dx} = \sqrt{\int_0^1 x^2 dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

So, this is the norm of this vector. So, from here, I can write the unit vector or normalized vector, we can write as f here divided by its magnitude. So, from here, I can write this is equal to $\frac{x}{1/\sqrt{3}} = \sqrt{3}x$. So, it is a normalized vector I can represent this one by $\hat{f}$; that means, its magnitude is 1.

So, now, from here, you can take this one and you can verify that if I take its norm, then it will be 1. So, we have to keep in mind that weight is always taking the square root of this. So, this way we are able to find the normalized vector. Now, suppose I take, suppose we define another vector space V taking from [-1 , 1].

Now, in this case, I am taking the set of all the continuous functions defined from [-1,1]. So, in this case, let us take one function f(x)=x and suppose I take g(x)=x^2 and now, I define the inner product between these two. So, I define the inner product and just to check whether these two vectors are orthogonal to each other or not.

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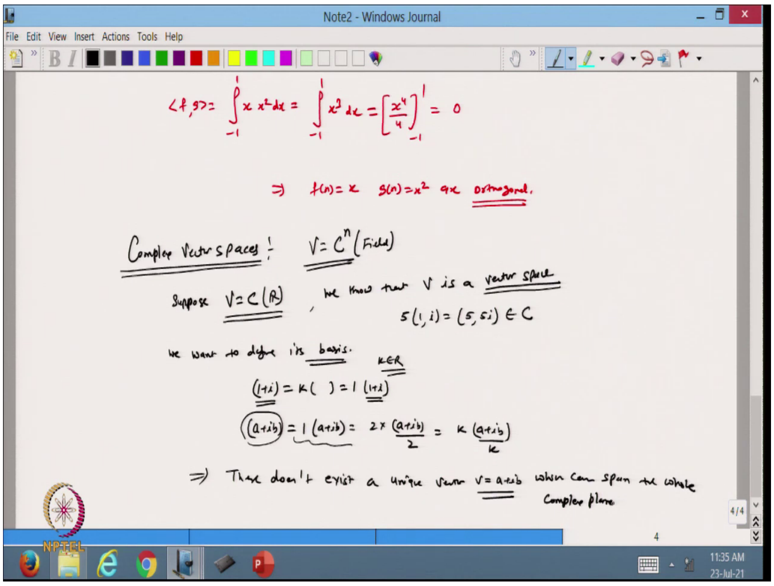

$$\langle f, g \rangle = \int_{-1}^{1} x x^2 dx = \int_{-1}^{1} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$$

Then, I can say that the function f(x) =x and g(x)= x^2. So, you can see from here that the same set of vectors, we have taken x and x^2 on this interval, in that case if I take the inner product that was not equal to 0, it was 1 by 4. So, these vectors or this function was not orthogonal with respect to this in a trans inner product that is from 0 to 1.

But if I change the interval, then the same vectors become orthogonal to each other. So, everything depends upon how we define the inner product. So, from here, you can say that these two vectors are orthogonal to each other under the inner product from integration taking from minus 1 to 1. So, this is the way we can define. Now, after dealing all these things in the real vector space, let us see what is going to happen when we deal with complex vector spaces.

So, in this lecture, I am going to give you a quick review of how the complex vector spaces are defined. As we already saw that in the complex vector spaces like suppose I take the vector space V and we call it C^n. So, in this case, I am taking the field F. Now, I start with the very first vector space I am taking.

So, let us say I suppose we take the vector space V over C that is just the complex line defined on the field R because R is also a field. So, I am defining it as this one. So, if you see from here, then the vectors are coming from the complex line or complex plane and these are the scalars coming from R, the real number.

Now, let us see what is going to happen in this case. So, if I know that the vector space V is a vector space because it satisfies all the conditions and we know that if I take 5(1, i) = (5,5i) ∈ C ok. So, an addition is also well defined.

So, everything we can check that this is a vector space. Here I am taking the field as a real number. Now, I want to define; so, we want to define its basis. Now, basis means I want to find a complex number such that by using that complex number, I should be able to create all the complex numbers.

Now, you can see that for example, I take a complex number

$$(1+i) = k() = 1(1+i)$$

$$\underline{(a+ib)} = 1(a+i) = 2 \times \frac{(a+ib)}{2} = k\frac{(a+ib)}{k}$$

It means if I choose any vector from the vector from the complex plane that can be written in this form because the scalars are coming from the real line. So, I cannot take those as a complex, it is coming only from the real line k that belongs to R.

So, now from here, we found that if I take any complex number that can be created in this way. So, from here, I cannot say that there is so which implies that there does not exist a unique complex number, a unique vector V that is a complex number which can span, so which can span the whole complex plane.

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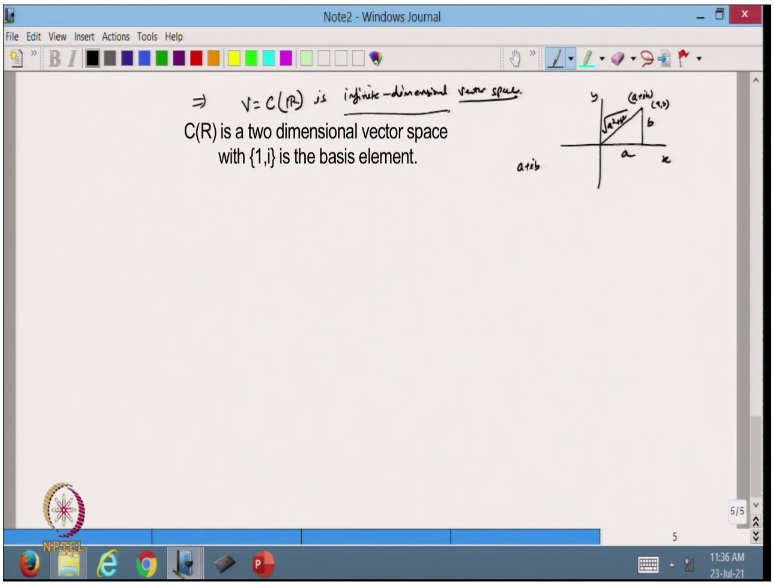

So, from here, we can say that the vector space V with the set of complex numbers defined on the field of real number is infinitely dimension is infinite, infinite-dimensional vector space and complex plane, we you know that (a + ib) can be written as so I take the a on this direction, b on this direction. So, suppose this is my (a + ib) and we also represent by a b, this one. So, it is my a and this is my b and this is a square b square under root.

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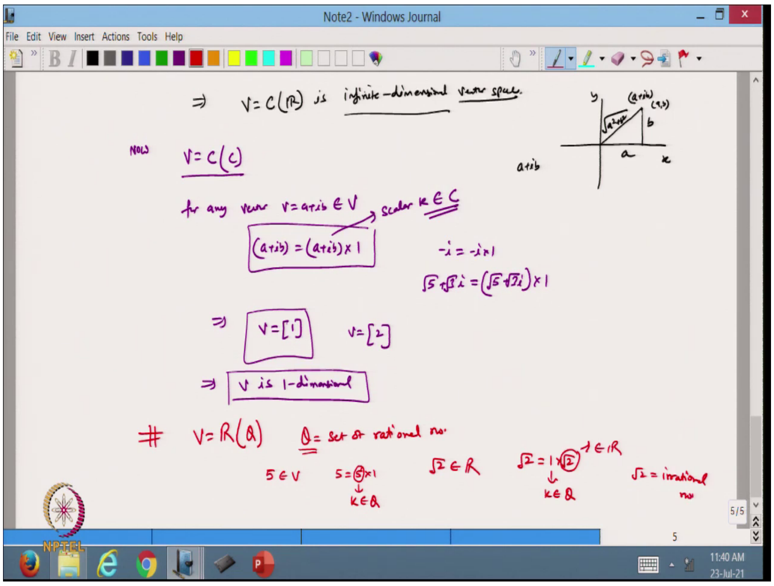

Now, suppose I take the vector space V with the same complex. Now, I am taking the field as a complex. So, in this case, for any vector V that is (a + ib)that belongs to the vector space V, here, we can write (a + ib) as taking (a + ib) and the vector I can take 1, here or I can write as (a + ib) by 2 into 2.

So, this way I can also write. It means that if I, so this is the same way. So, I just removed this one and I took this here. So, I choose any vector suppose you take -i. So, I can write this as a -i * 1, you just take

$\sqrt{5} + \sqrt{3}i = (\sqrt{5} + \sqrt{2i}) \times 1$

So, from here, I can say that this whole vector space V is spanned by 1 because all the elements are spanned by 1 by taking all this one. So, here it is the scalar k that is coming from the set of complex numbers and from here, we can say that the vector space V is one-dimensional or I can say V =[2] by this way divided by 2 into 2, but it will have a basis which contain only 1 element.

So, from here, I can say that V is one-dimensional vector space. So, everything here, we can say that a lot of things change when we change the field. So, this is where we can define the complex numbers. For example, I just take the vector space of real numbers; I just take the vector space of real numbers and I am taking the field as a set of rational numbers, where Q is a set of rational numbers.

So, in this case I also suppose I want to find maybe 5. So, 5 belongs to the vector space V. So, I can write 5 = 5 * 1. So, this 5 is coming from k coming from the set of rational numbers. So, it is a rational number and 1 so, no problem. But suppose I want to write a vector I take maybe $\sqrt{2}$. So, $\sqrt{2}$ ∈ R. So, I can write $\sqrt{2} = 1 \times \sqrt{2}$.

Because I cannot take here $1 \times \sqrt{2}$ because $\sqrt{2}$ is not the set of rational numbers. It does not belong to the rational numbers ok. So, in this case, I can write $\sqrt{2}$ like this one. So, it means that I am taking a vector that belongs to the real line and that I am taking a k belonging to rationals. I just want to tell that the $\sqrt{2}$ is an irrational number. So, it means that all the irrational numbers, we have to define in this way.

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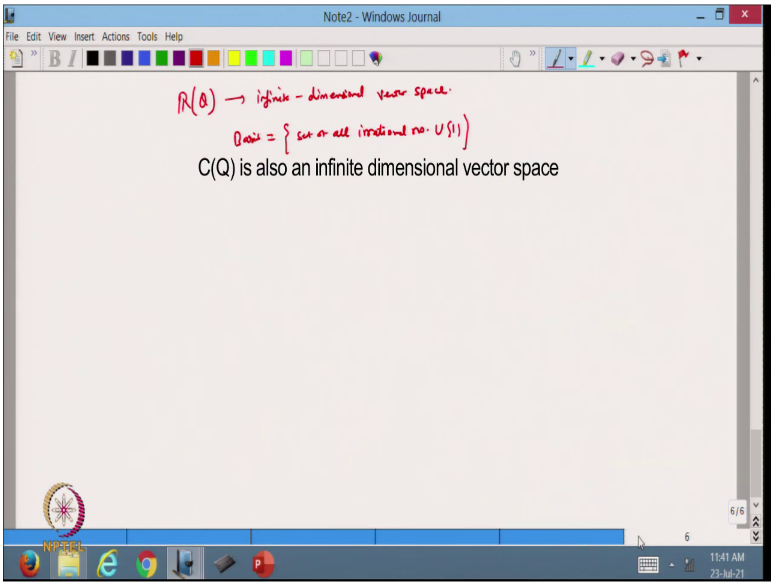

So, this way also I can say that if I take the set of real numbers on the set of rationals, the field, so it is an infinite dimensional vector space and what is the basis? So, I can say that in this case, the basis is I can take all the sets of all irrational numbers and then, I just take union 1; maybe I can take 1 because all irrational numbers are generated in this way and if the rational number is there, we can generate it this way. So, this contains an infinite number of elements and so, it is an infinitely dimensional vector space.

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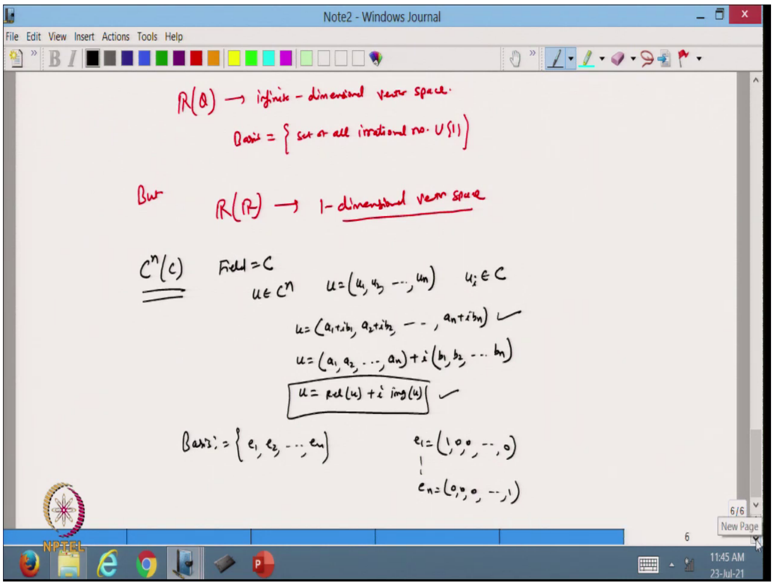

But if I take R(R), then we know that this is one-dimensional vector space. So, till now, we have taken such types of things that we are taking the vector space R(R) or $R^n(R)$ and in that case, we know that this is well-defined. But if we change the field, then these things may happen.

Now, we talk about the complex vector space just because we want to define the inner product in that one. So, let us define this one. So, let us take the C^n. So, I am taking the C^n (C) , which means that the field I am taking is a complex number. And this is. So, if I take any vector. So, let us take u belongs to C^n,

$$u \in C^n, u = (u_1, u_2, \dots, u_n) \quad u_i \in C$$
$$u = (a_1 + ib_1, a_2 + ib_2, \dots, a_n + ib_n)$$
$$u = (a_1, a_2, \dots, a_n) + i(b_1, b_2, \dots, b_n)$$

So, this is my u basically. So, u is always of this form because all the components are also complex. So, this will be of this form. So, this is my one of the vectors in u.

So, any vector u can be written as; so, this is if you see that it is a real component of the vector u. So, I can write from here that this is u=real(u) +i img(u).

So, I can define any vector from C^n in this way. Now, we can define its basis. So, I know that if I want to define the basis of this one. So, its basis will be again; so, I will just talk about the standard basis. So, it is n-dimensional.

So, n-dimensional means the basis will be standard will be $\{e_1, e_2, \dots, e_n\}$; where e_1 =(1, 0, 0, 0) so like this one. So, e_n =(0, 0, 0, 1) because I am taking the field as a complex number.

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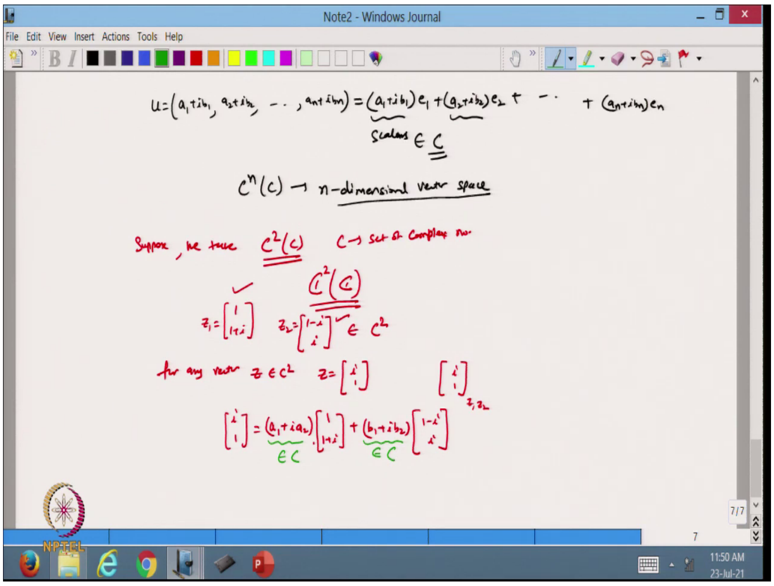

So, if you take any complex number, I know that I can write

$u = (a_1+ib_1,\ a_2+ib_2,\ldots\ldots,\ a_n+ib_n) = (a_1+ib_1)e_1 + (a_2+ib_2)e_2 + \ldots\ldots + (a_n+ib_n)e_n$. So, these are all our scalars coming from the set of complex numbers. So, this is we know that the C^n over the C is n-dimensional vector space. So, it is a n-dimensional vector space.

Now, from here, we can now suppose I just want to see how we can define the linear combination in the terms of a vector space. So, let us take one example. Let us suppose we take $C^2(C)$. So, C is a set of complex numbers; some books also write like this one.

So, this is also sometimes written like C(C). So, that is also notation; but this is basically a set of complex numbers. Let I take some vector I take; so, let us take

$$z_1 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}, z_2 = \begin{bmatrix} 1-i \\ i \end{bmatrix} \in C^2$$

Now, I want to take so suppose I choose now for any vector z belongs to C^2, suppose I take $z = \begin{bmatrix} i \\ 1 \end{bmatrix}$ Now, I want to find what will be the coordinates of $\begin{bmatrix} i \\ 1 \end{bmatrix}$ with respect to this z_1, z_2; it means, I want to find what are the coordinates of this vector with respect to these two vectors;

z_1, z_2. So, what I am going to do is to define a linear combination. So, in this case, I am now defining the scalars. If you see from here the scalar is also a complex number.

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = (a_1 + ia_2)\begin{bmatrix} 1 \\ 1+i \end{bmatrix} + (b_1 + ib_2)\begin{bmatrix} 1-i \\ i \end{bmatrix}$$

So, I have defined the linear combination putting this vector equal to 0. So, you have to keep in mind here that the scalars here, this is my scalar, this is my scalar that will belong to the C, this also belongs to C. So, we have to take it as a complex number.

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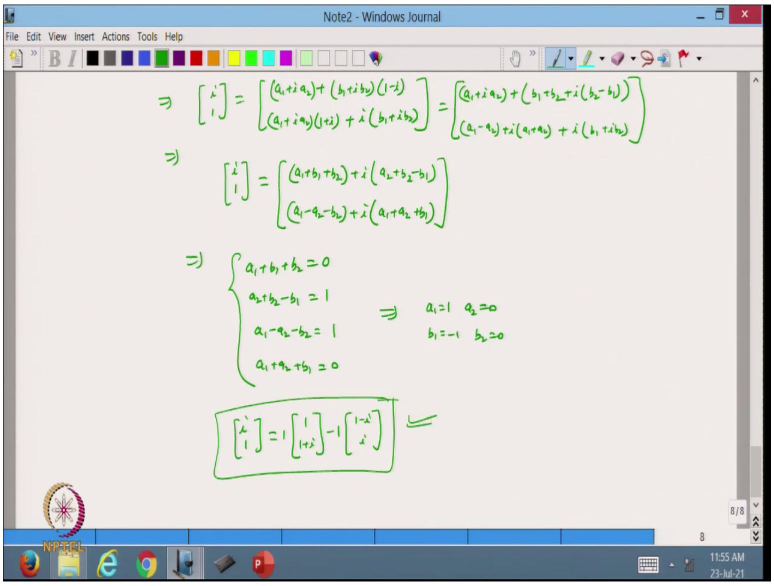

Now, from here I can define this as. So, this one I can write component wise. So, from here, I can write this

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} (a_1 + ia_2) + (b_1 + ib_2)(1-i) \\ (a_1 + ia_2)(1+i) + (b_1 + ib_2)i \end{bmatrix}$$

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} (a_1 + b_1 + b_2) + i(a_2 - b_1 + b_2) \\ (a_1 - a_2 - b_2) + i(a_1 + a_2 + b_1) \end{bmatrix}$$

Now, from here, I will compare this one. From here, now we need to find out the four coefficients; a_1, a_2, b_1, b_2. So, we need the four equations.

$$a_1 + b_1 + b_2 = 0$$
$$a_2 - b_1 + b_2 = 1$$
$$a_1 - a_2 - b_2 = 1$$
$$a_1 + a_2 + b_1 = 0$$

So, from here, I can say that $a_1=1$, $a_2=0$, $b_1=-1$, $b_2=0$. So, here we can write the linear combination.

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = 1.\begin{bmatrix} 1 \\ 1+i \end{bmatrix} - 1\begin{bmatrix} 1-i \\ i \end{bmatrix}$$

So, this is just the simple one we have discussed. So, this way, we have to find out the linear combination of the given vector.

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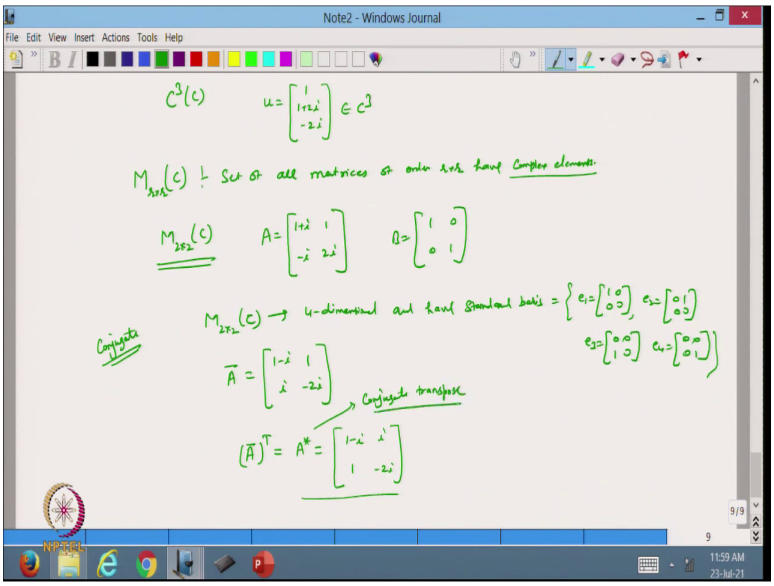

Now, so, if we are able to define the vectors in the complex plane, similarly I can define if I take $C^3(C)$, then maybe I can define a vector u as $u = \begin{bmatrix} 1 \\ 1+2i \\ -2i \end{bmatrix} \in C^3$ So, the same way we have to deal. Now, once we define the vectors, I want to define the matrices.

So, the matrices we want to define are of order r cross r over complex. It means that the elements of the matrix can be complex numbers. For example, so set of all matrices of order r cross r having complex elements; complex elements means complex numbers.

For example, maybe I will define a 2 by 2 matrix over the complex number. So, suppose I take my matrix A = $\begin{bmatrix} 1+i & 1 \\ -i & 2i \end{bmatrix}$, another matrix I am defining $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So, that is also a complex number. So, because I know that each real number can be written as a complex number.

So, this also belongs to this one. Now, suppose, I take these matrices. So, I know that this is dimension four. So, in this case, I know that M 2*2 over the complex number is four-dimension and its standard basis and having standard basis as

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So, this is my standard basis and this is four-dimension. Now, in this case, for the complex number I know that we can define the conjugate. So, conjugate of A will be

$\bar{A} = \begin{bmatrix} 1-i & 1 \\ i & -2i \end{bmatrix}$ So, this is a conjugate. So, I have to take the conjugate of each of the elements in the matrix Q, Qn matrix.

Now, for the complex matrices, we define the terms A conjugate transpose, we generally write this as A star. So, in this case, what do we take? We take the conjugate first and then take the transpose. So, I can write from here that this is equal to

$$\left(\bar{A}\right)^T = A^* = \begin{bmatrix} 1-i & i \\ 1 & -2i \end{bmatrix}$$

I have taken the conjugate and then, I have taken the transpose. So, this is called the conjugate transpose and that is represented by a star.

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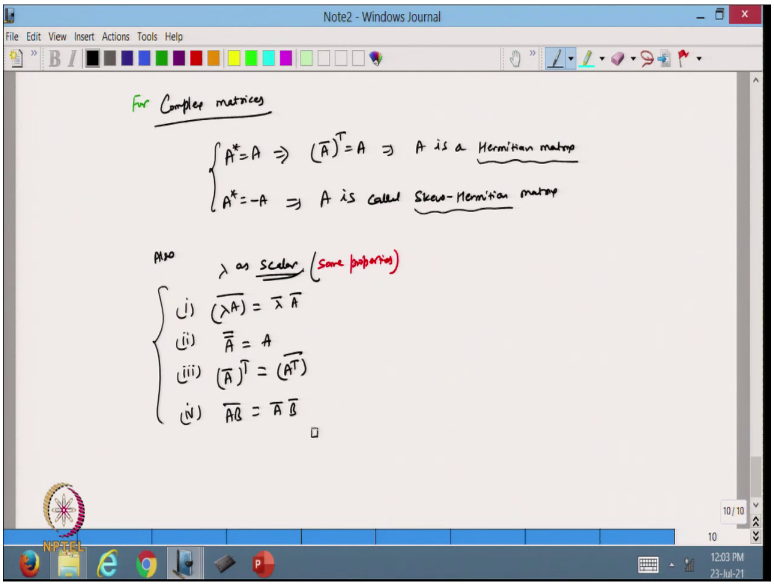

Now, for maybe I can for complex matrices or matrices having the complex numbers, if I take

A^* = A => $\left(\bar{A}\right)^T$ =A So, this is called A is an or A is a Hermitian matrix. When I say A^* = A, then it is called Skew-Hermitian.

So, it is analogous to the symmetric matrix in real numbers and this is analogous to the skew symmetric matrix in the real number. So, one thing extra we need to do, whenever we are dealing with the set of complex numbers; then also, if I take any λ as scalar.

So, in this case, scalar is coming from the complex number, then if I take A any matrix taking the conjugate, then it is equal to $\overline{(\lambda A)} = \bar{\lambda}\,\bar{A}$, some observations you can write. Also, if I take the matrix A and taking two time conjugate $\bar{\bar{A}} = A$. We can write down some properties. Then, if I write $(\bar{A})^T = (\bar{A^T})$, no problem and the fourth one is if I take $(\overline{AB}) = \bar{A}.\bar{B}$.

So, this way, we can define the terms in the set of matrices in the complex form. So, once we are able to do this one, then I want to define the inner product in the complex vector space. So, that we are going to discuss in the next lecture. So, we will stop here.

So, in today's lecture, we discussed the inner product in the space of continuous functions defined over an interval [a, b] and then, we want to define the inner product in the case of complex vector species. So, we have discussed some properties or how we can deal with the complex vector spaces having the vectors or having the matrices.

So, now, in the next lecture, we will discuss how we can define the inner product in the complex vector spaces. So, thanks for watching this one.

Thanks very much.