**Matrix Computation and its applications**
**Dr. Vivek Aggarwal**
**Prof. Mani Mehra**
**Department of Mathematics**
**Indian Institute of Technology, Delhi**

**Lecture - 43**
**Inner product spaces**

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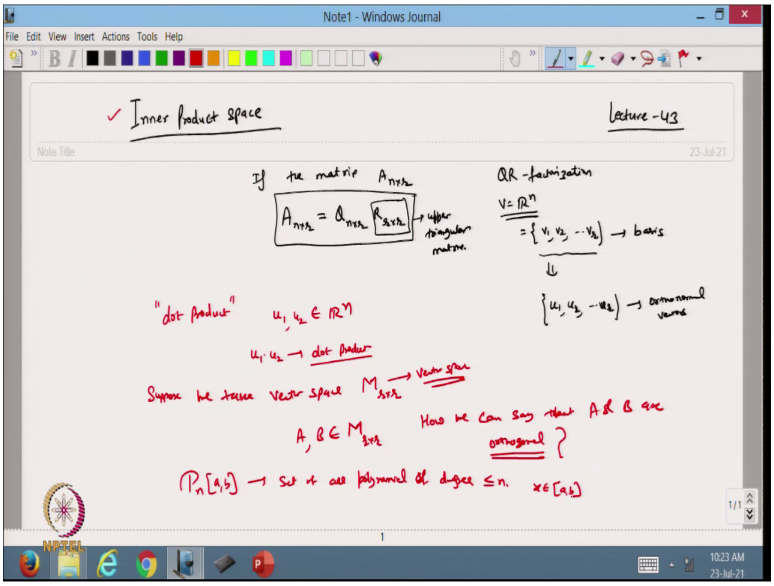


Hello viewers. So, welcome back to the course on Matrix Computation and its Application. So, in the previous lecture we have discussed the Gram Schmidt orthogonal process and the QR factorization. Now today we are going to generalize the concept of dot product and we are going to introduce the Inner product space. So, let us do that one.

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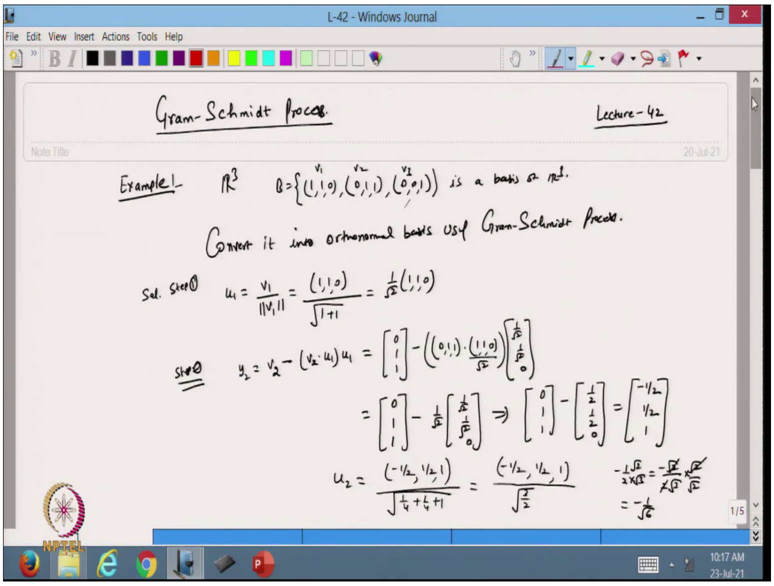


So, as in the previous lecture we have discussed that if we have a set of linearly independent vectors then with the help of the Gram Schmidt process we can make them orthogonal and then normalized.

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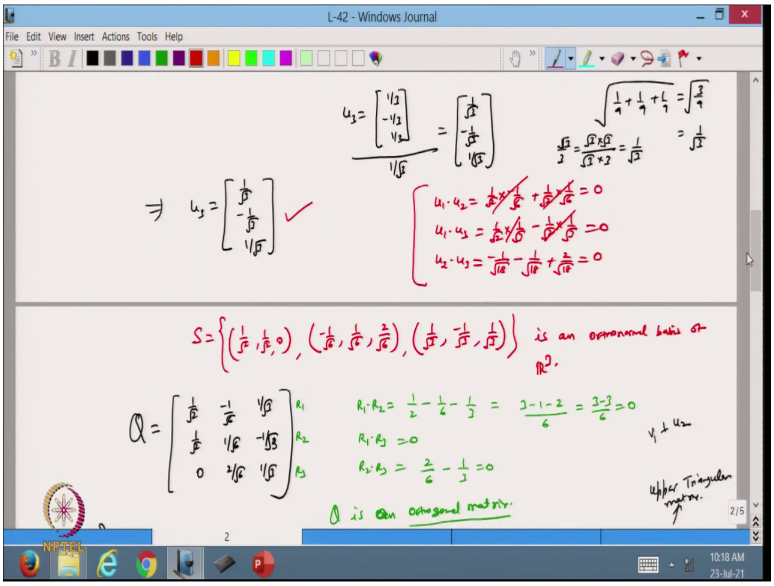


So, we can make them an orthonormal set of vectors.

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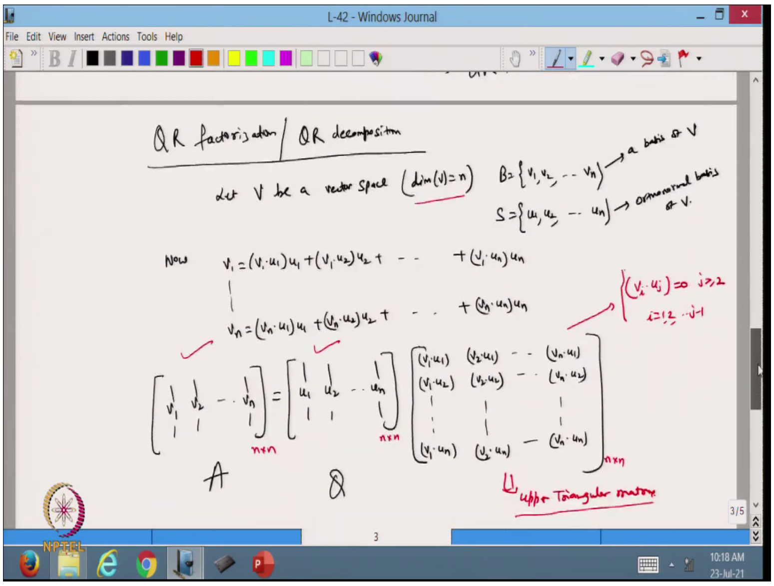


And using those vectors we are able to factorize the given matrix A into the QR form.

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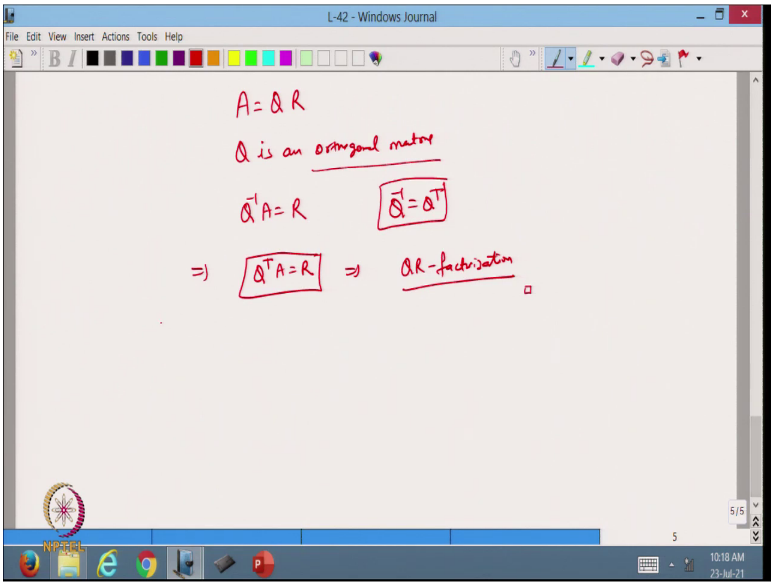


So, the Q is the corresponding orthogonal matrix and this is the upper triangular matrix. So, that we have discussed in the previous lecture and in this case we just want to say that it may happen sometime that this QR factorization.

So, whatever we are doing here I can write here that if the matrix A that is n cross r in the Q R factorization, then also we can have a A that is n*r. So, this can be written as A Q that is also n*r because A will contain the vector r vectors each of dimension this. So, in this case we have a V. We suppose we are dealing with $R^n$ and then I take the set containing $v_1$, $v_2$,…, $v_r$.

So, it is basically a basis of some subspace of $R^n$. So, based on this one we will convert this one into the orthogonal set of vectors we represent by u. So, here we are represented by u. So, this is my basis and then I will convert this one into orthonormal vectors with the help of Gram Schmidt process then it will be a will be of n*r, where a is the matrix corresponding to the putting this vector as a column vector this will be n*r and then r will be r*r.

So, this matrix we are writing here is always an upper triangular matrix because their dimension is a rectangular matrix. This is a rectangular matrix, but this will be a square matrix and this will be an upper triangular matrix. So, QR factorization is also possible when we are dealing with the basis which is a basis of a subspace instead of the complete vector space.

Now, after doing this one, today we are going to discuss the inner product space. So, we have discussed that whenever we take the Gram Schmidt process we deal with the dot product. So, that we know that this is what we have used for the vectors, suppose we have a vector may be $u_1$ and $u_2$ belongs to some vector space R n then we have defined $u_1.u_2$. So, that is the dot product of dot product because using this dot product only we are able to tell whether these vectors are going to be orthogonal or not.

Now, suppose we take vector space of all the matrix of order r*r. So, it is also a vector space, it is a vector space, then suppose I take the matrix A and B belonging to this vector space, then how can we say that A and B are orthogonal? How can we say that? Or maybe I will take a set of polynomials if I suppose I take the $P_n$ defined over an interval [a b]. So, it is a set of all polynomials of degree less than equal to n and x belongs to the interval suppose [a, b].

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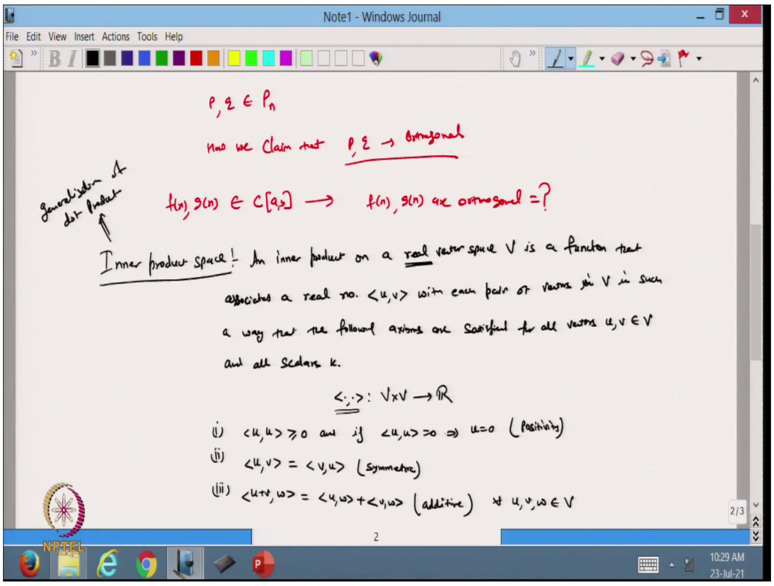


So, in this case also suppose I have a polynomial p and q ∈ $P_n$ then how can we say how we can claim that p and q is orthogonal or suppose I take the function f (x) and maybe f(x) and g(x) that belongs to the vector space of all the continuous function defined from [a , b].

So, in this case also I want to check that f(x) and g(x) are orthogonal. So, how will I check? Because we do not know how to take the dot product in this case. So, to define this type of thing we do the generalization of the dot product and then we define the generalization and we call it inner product space.

So, you can say that this is a generalization of dot products. So, what is this one? So, let us write the formal definition because in this case. So, we call it an inner product on a real vector space. So, we are talking about real vector space V as a function.

So, it's a function that associates a real number. So, we represent this one. Suppose I take u and v from the vector space v, then we represent this inner product in this way. So, by the real number this one with each pair of vectors each pair of vectors in V in such a way that the following axioms are satisfied for all vectors u and v belongs to the vector space V and all scalars k.

So, it means that we are defining a function that is going from V cross V into a set of real numbers. So, this is called the inner product if it satisfies the following axioms. So, first what is the first axiom? So, first axiom is that if I take u and taking its inner product with itself then this is always greater than equal to 0 i.e <u,u> $\geq 0$

So, it should be always positive and if u with u itself is 0 that implies that u = 0. So, this is called the positivity axiom. The second one is that if I have an inner product with <u , v> = <v u> , then its value should be the same and this is called symmetric property.

Third one is we have to suppose <u + v ,w> =<u,w> + <v,w> and this is true for all u, v and w. So, this is called the additive property.

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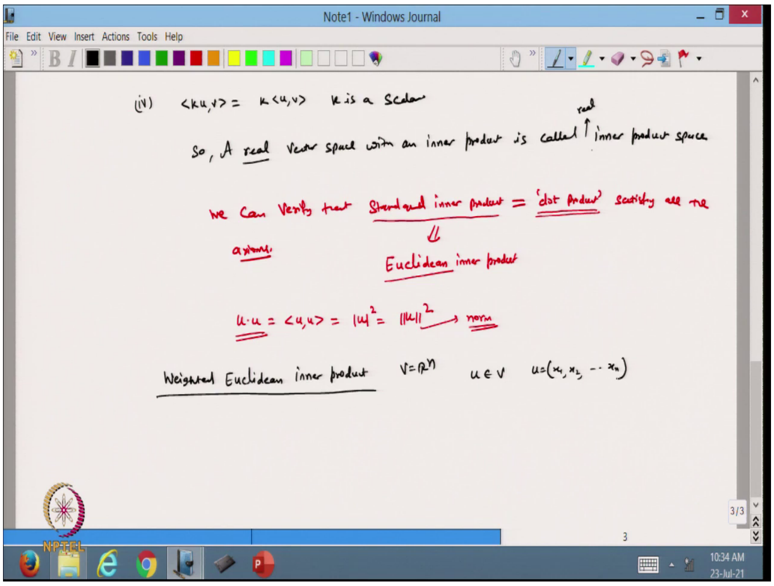


And the fourth one is, suppose I have a scalar <ku , v> = <u,v>. So, if this is true. So, I can write here that this is true for all u, v, and w belonging to the vector space V. This is also true for all vector spaces where k is a scalar. So, in this case if all these four properties are satisfied, then we say that whatever we are defining here is an inner product and so, from here I can write that.

So, a real vector space with inner product. So, I can say that an inner product is called inner product space. So, here we are talking about the real. So, it is called the inner product space or maybe I can write here because we are talking about real, real inner product space.

So, these things we just change the name and then we are able to do that. So, all these properties are to be satisfied now you can verify this. So, we can verify that the standard dot product or I can say that the standard inner product that is equal to the dot product satisfies all the axioms.

So, if we call it the standard inner product, it means we are talking about the dot product we have discussed about the vectors in $R^n$. So, that is called the basically standard inner product or we also have a some other type of name.

So, this is also called the Euclidean inner product. So, Euclidean means we are talking about Euclidean geometry. So, in that case we take the dot product inner product as a dot product. So, that is also called the Euclidean inner product, the standard inner product or the dot product or the Euclidean inner product all are the same.

Now from here if you see whatever we have defined u dot product u I can write as u, u. So, we are defining this inner product as a dot product and this things we already know that $u.u = \langle u,u \rangle = |u|^2 = ||u||^2$

So, whenever we put this line then we call it norm. So, the norm of the vector here in this case is also associated with the inner product. So, this is satisfied that is always positive or if it is 0 then this value will be 0 that is also symmetric. This is additive or maybe I can call it distributive and forth property. So, all the properties we can verify that this is satisfied by the standard inner product or the dot product.

So, now we will discuss the inner products in other vector spaces. Now we can discuss after discussing this one, we can discuss weighted Euclidean inner product. So, the Euclidean inner product is the same as a dot product, but here we are defining the weighted inner product. So, suppose we have.

So, we suppose this is $R^n$. So, it is a Euclidean space then in this case I take suppose I take a x $\in V$. So, my u is basically I can write it as $x = (x_1, x_2,.. ,x_n)$ or maybe I will take it as a x then I define the another vector $y = (y_1, y_2,\ldots, y_n) \in V$.

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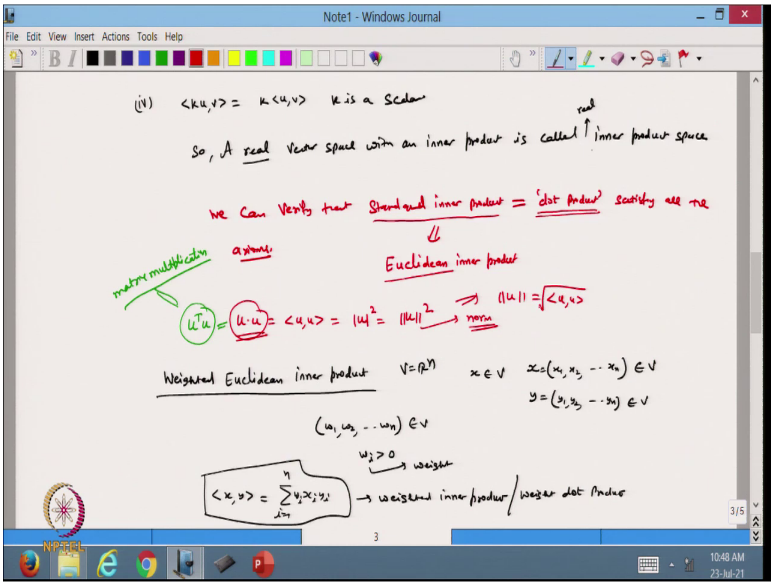


So, if it belongs to v it also belongs to the given vector space v, then we define a vector that is $(w_1, w_2,\ldots, w_n) \in V$ such that each $v_i$ is positive. So, this $w_i$ is what we represent as a weight, then we can define the weighted Euclidean inner product. So, we are represented by this one.

So, we are writing inner product $$< x, y >= \sum_{i=1}^{n} w_i x_i y_i$$

So, this one is basically what we are representing. If we take the normal dot product then it is equal to $x_i$ into y I, then we multiply by $w_i$ and i is from 1 to n. So, in this case we have taken the dot product multiplied by $w_i$. So, this is if we take this one this is called the weighted inner product or weighted dot product.

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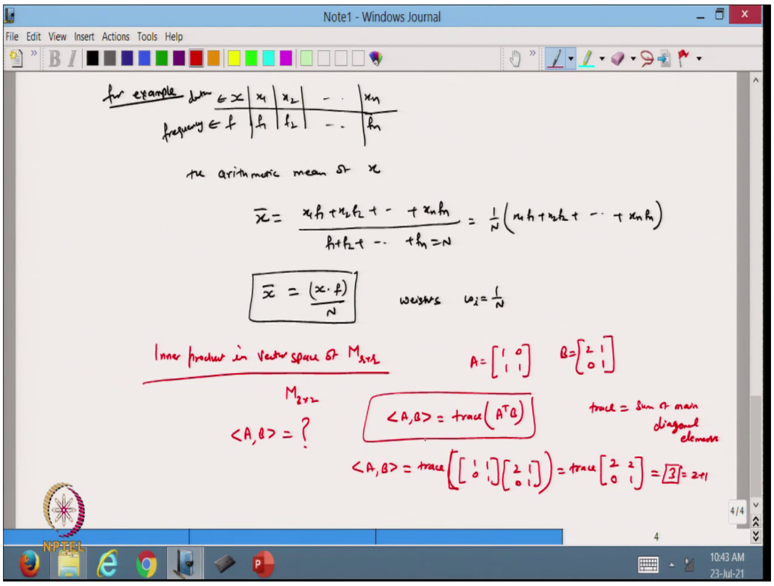


For example, suppose we have some random variable like suppose I have a x that is a random variable and it has the value $x_1, x_2, \ldots, x_n$ some data is given to me and suppose I have the frequency of each x. So, it is $f_1, f_2, \ldots, f_n$ then we know that the arithmetic mean if I am I want to take then the arithmetic mean of x.

$$\bar{x} = \frac{x_1 f_1 + x_2 f_2 + \ldots\ldots + x_n f_n}{f_1 + f_2 + \ldots\ldots + f_n = N}$$

$$= \frac{1}{N}\left(x_1 f_1 + x_2 f_2 + \ldots\ldots + x_n f_n\right)$$

$$\bar{x} = \frac{(x.f)}{N}$$

So, in this case I can say that the weight function weights. So, I can represent here the weights $w_i$ in this case is 1/N ok.

So, from here we can find out the arithmetic mean with the help of taking the weights that is 1/N in this case. So, this is a weighted dot product. We take in the case of a data whenever we have a data and this is the frequencies and this is my some data numerical data, then I want to find out the mean then we apply this formula to find out the arithmetic mean.

So, this is one of the ways to find this one. Now, after doing this one I will want to find the inner product in the space of matrices. So, let us define the inner product in the vector space of matrices M of order r cross r. Suppose I take for example, I take M 2 by 2.

So, I just take 2 by 2 suppose I take the matrix A

$$< A,B >= trace\left(\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}\right) = trace\begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix} = 0$$

$$A^T A = \begin{pmatrix} a_{11}^2 + a_{21}^2 & a_{11}a_{12} + a_{21}a_{22} \\ a_{11}a_{12} + a_{21}a_{22} & a_{12}^2 + a_{22}^2 \end{pmatrix}$$

$$trace(A^T A) = a_{11}^2 + a_{21}^2 + a_{12}^2 + a_{22}^2 \geq 0$$

$$U = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}, V = \begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}$$

$$= \frac{U}{\|U\|} = \frac{\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}}{\sqrt{3}}$$

$$\sqrt{1+1+1} = \sqrt{3}$$

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

Now I want to say that the matrices A and B are orthogonal to each other or I want to take the inner product of A and B.

So, how can we define it? So, I want to define this one A and B inner product. So, this one I want to define ok. So, if we define an inner product then it should satisfy all these four conditions, then only we say that the inner product is well defined. So, in this case how we define it is let us take the definition.

So, what I do is that, I define A, B the inner product I take it as because if you know then the dot product here whatever we have defined this one, this dot product can also be written as u transpose u because whenever we take u as a in the matrix form, then we know that u is a column. So, u transpose becomes the row vector and multiply by u. So, it is a matrix multiplication matrix multiplication.

So, now I know that these matrices are also made up of vectors. So, I take the idea from here and then I define the inner product here as I will take the trace of A transpose B. So, this one I will define and we know that the trace of a matrix is basically the sum of diagonal elements and the sum of main diagonal elements.

So, I am defining the inner product in this way. So, for example, let us take this matrix and I want to define the inner product of A and B. So, in this case I will take the inner product of A and B. So, I will take the trace of trace of a matrix.

So, this is the matrix we are going to define.

$$< A, B >= trace\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\right) = trace\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} = 3$$

Now, the question is whether it satisfies all the four conditions or not, so that we need to find out.

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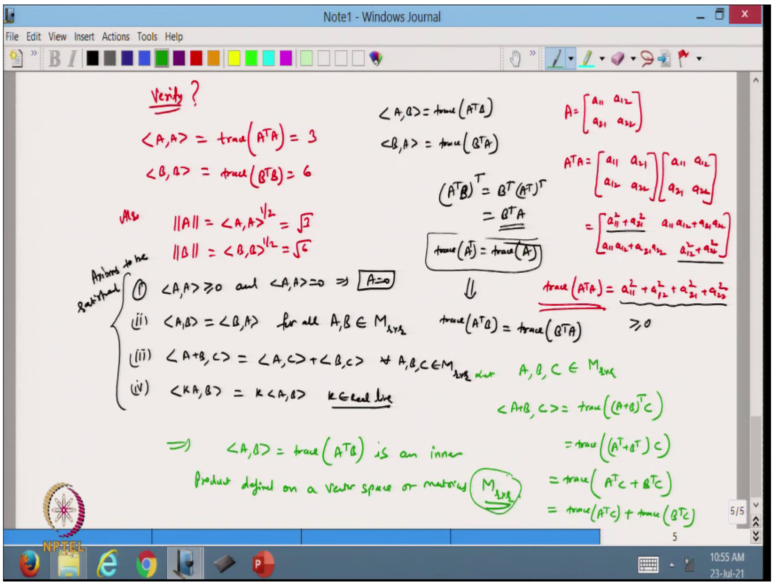


So, verify that. So, this verification we can do, one thing I want to see from here that let us see what will happen to when I am taking the inner product of A with itself then I can take from here this is I will defining trace of A transpose A and if you take a matrix any matrix suppose I take in the matrix A

A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$

$$A^T A = \begin{pmatrix} a_{11}^2 + a_{21}^2 & a_{11}a_{12} + a_{21}a_{22} \\ a_{11}a_{12} + a_{21}a_{22} & a_{12}^2 + a_{22}^2 \end{pmatrix}$$

$$trace(A^T A) = a_{11}^2 + a_{21}^2 + a_{12}^2 + a_{22}^2 \geq 0$$

So, if you see from here then this is equal to taking the square of each and every element of the matrix A. So, that is going to be the trace of this one. So, I am using this one here. So, from here then it should be equal to and always you can see from here that it is taking the square.

So, it is always greater than equal to 0 unless until all the elements are 0 it is always greater than 0. So, it is always positive. So, from here I can say that the trace of A will be 1+ 1 +1. So, it will be 3 in this case; because I am taking this square plus this square plus this square so it is 1 so, no problem.

Or maybe I can take a trace or the inner product of B with itself that is trace of B transpose B. So, it is equal to 6 and this is positive that we know also we know that if I take the norm of A here. So, this will be taking the inner product of itself by the square root.

So, from here this will be equal to under root 3 similarly I can define the norm or we used to call in this case the length. So, this is what we used to define. So, from here you know that the length is always taking the square root. So, this is always taking B. So, this is under root 6.

So, the first property is satisfied that this is always positive and whenever it is 0. So, whenever it is 0 if this is equal to 0 it means all the values of the matrix will be 0. So, that property is satisfied. So, I can say from here properties or axioms to be satisfied. So, the first one is that A with this is always positive and if this inner product is 0 which implies that A= 0.

So, it is true that all the elements should be 0. The second one is that, the second one we want the symmetry. So, we know that we are defined by A, B. So, here we have defined $\langle A, B\rangle = \text{trace}(A^TB)$. Now, I want to see what will happen if I take it $\langle B, A\rangle$. So, $\langle B, A\rangle = \text{trace}(B^TA)$.

$$(A^TB)^T=B^T(A^T)^T= B^TA$$

So, if I take the transpose of this one then I am getting this value and we also know that the trace is just the matter of the elements of the diagonal and if we take the transpose then the diagonal element is not going to change. So, we also know that the trace of some matrix A transpose is always equal to the trace of A that we always know.

So, from here we can say that trace of $\text{trace}(A^TB)=\text{trace}(B^TA)$. So, from here I can say that this is equal to $\langle B, A\rangle$ for all matrices A and B belonging to the space. So, this is I am taking any matrix of order r*r. So, the third one is also. So, now, I from here I know that 1 more thing that suppose we have two matrices let we take matrices A, B, C ∈ r*r then I want to take

$$\langle A+B,C\rangle = \text{trace}((A+B)^TC).$$

$$= \text{trace}((A^T+B^T).C)$$

$$=\text{trace}(A^T.C+B^T.C)$$

$$= \text{trace}(A^T.C) + \text{trace}(B^T.C)$$

$$= \langle A,C\rangle + \langle B,C\rangle$$

So, this is true for all A, B, C belongs to the vector space of the matrix r*r and the fourth one is I take any scalar k. So, it is very easy to say that we just take the k common then I can write this as the inner product of this one for k belongs to the real line.

So, here it is a scalar. So, I am taking the real vector space. So, it is a real number. So, all these four properties are satisfied and from there we can say that this inner product is defined on.

So, the inner product is equal to trace($A^TB$) is an inner product defined on a vector space of matrices M of order r cross r. So, this is the inner product I can define on these matrices. Now this you know once we define this inner product on the matrices then we can talk about orthogonality.

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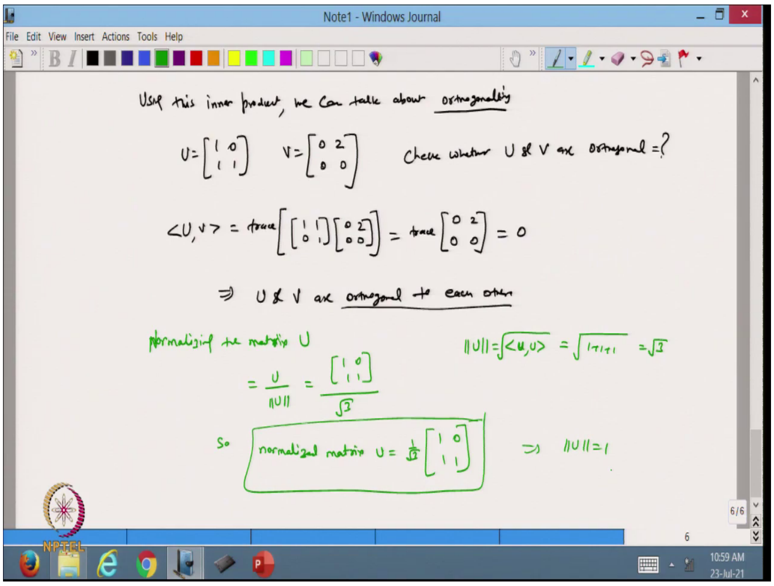


So, using this inner product we can talk about orthogonality. So, how can we say? Let I take the matrix some matrix I just take

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Suppose I take this matrix and I want to check whether these two matrices are orthogonal or not. So, check whether U and V are orthogonal. So, this one we want to check. So, what do we do? I will take the inner product. So, in taking the inner product I will take the trace of the matrix.

<U,V> = trace($U^TV$) = $trace\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right) = trace\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 0$

So, from here I can say that the inner product in this case is coming 0 which implies that U and V the matrices are orthogonal to each other. Now suppose I want to normalize this matrix. So, these are orthogonal to each other now let us normalize this one. So, how to normalize? So, by normalizing the matrix U; so, how can we do the normalization? First of all I want to find normalization here.

So, that will be equal to U divided by its norm or its length. So, this one I can write now $= \frac{U}{\|U\|} = \frac{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}{\sqrt{3}}$ and this one I know that this is equal to taking the dot inner product and the square root. So, this is we already found that this is equal to under root and this is the square of all the elements of the given matrix. So, it is it was $\sqrt{1+1+1} = \sqrt{3}$ . So, from here I will get this matrix. So, normalized matrix U is $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

So, this is the way we are able to do this normalization. So, now, if you can from here you can see that if you take its norm then it will be equal to 1. So, this way we are able to find how we can define two matrices orthogonal to each other and how we can normalize this one.

So, we stop here. So, in the today's lecture we have introduced the generalization of the dot product that is the inner product and then we have defined that how we can define the inner product on the vector space of matrices and we have shown that how we can say that two matrices are orthogonal to each other. So, the next lecture will also continue with this one. So, thanks for watching.

Thanks very much.