Matrix Computation and its applications
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Lecture - 42
QR factorisation

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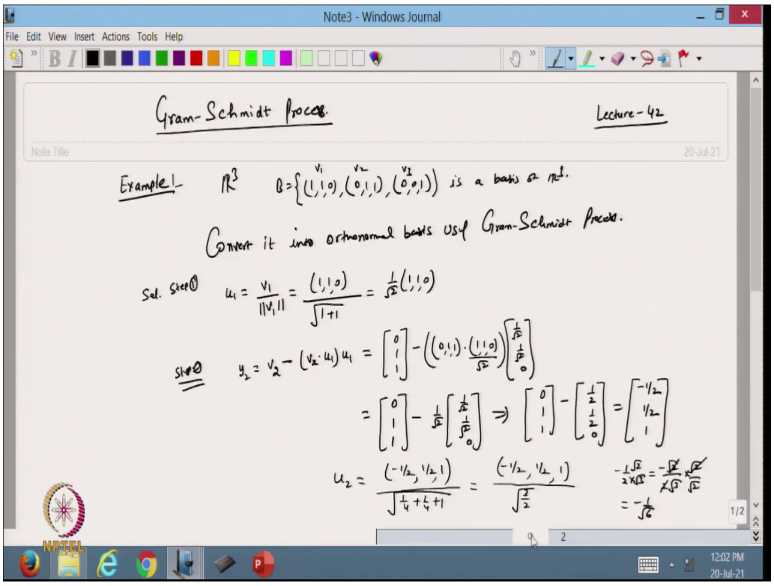

Hello viewers, so welcome back to the course on Matrix Computation and its application. So, in the previous lecture, we have discussed the process of the Gram-Schmidt that how we are able to convert the given set of linearly independent vectors into the orthonormal vectors. So, today we will discuss the application of that one.

So, let us take the example of how we can proceed with the Gram-Schmidt process. Suppose, I have a vector space R^3 and I have the basis, I am taking the basis as {(1 1 1), (0 1 1) , (0 0 1)}. So, this is the basis I am taking; the basis of R^3. Now from there, the process is that, the question is to convert it into an orthonormal basis using the Gram-Schmidt process. So now, so I call it maybe I will call it v_1, v_2, v_3.

So, this is a three vector I am taking. So, let us take u 1. So, it is $u_1 = \frac{v_1}{\| v_1 \|}$ = $\frac{(1,1,0)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \frac{1}{\sqrt{2}}(1,1,0)$. So, it is step 1.

Now, step 2, I want to make another vector. So, I will write my

$$y_2 = v_2 - (v_2.u_1)u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left((0,1,1) \cdot \frac{(1,1,0)}{\sqrt{2}}\right) . \frac{(1,1,0)}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left((0,1,1) \cdot \frac{(1,1,0)}{\sqrt{2}}\right) . \frac{(1,1,0)}{\sqrt{2}}$$

So, how I am going to solve this one is that this vector minus; so I am taking the dot product of this. So,

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

Now, from here, I will write my u_2. So, I will take this vector as

$$u_2 = \frac{y_2}{\| y_2 \|} = \frac{(-1/2, 1/2, 1)}{\sqrt{\frac{1}{4} + \frac{1}{4} + 1}} = \frac{2}{\sqrt{6}}(-1/2, 1/2, 1) = \frac{1}{\sqrt{6}}(-1, 1, 2)$$

So, that is my u_2. So, this is a normalized vector.

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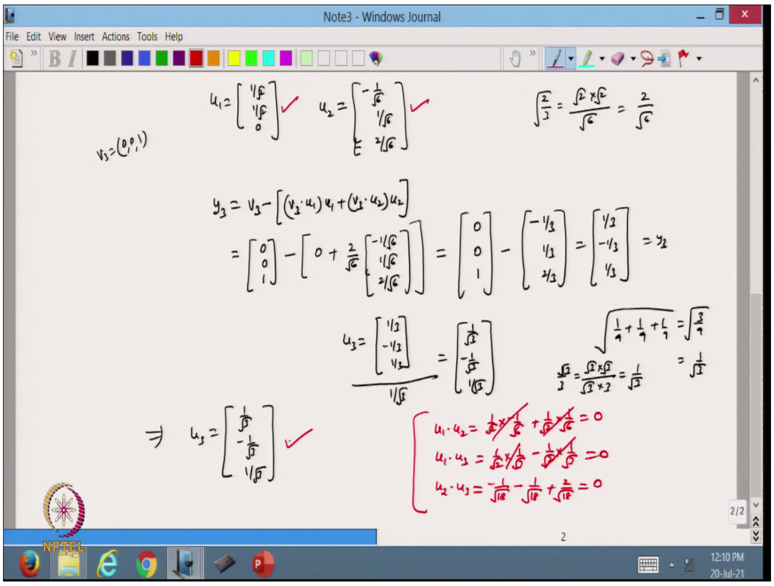

Now from here, so I am able to get my $u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$

$y_3 = v_3 - ((v_3 . u_1). u_1 + (v_3 . u_2). u_2)$ $= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left\{ 0 + \frac{2}{\sqrt{6}} \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \right\}$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -2/6 \\ 2/6 \\ 4/6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}$$

So, this is my y_3. Now, from here, I can find my u_3. So,

$$u_3 = \frac{\left(1/3, -1/3, 1/3\right)}{\sqrt{\frac{1}{9}+\frac{1}{9}+\frac{1}{9}}} = \frac{3}{\sqrt{3}}\left(\frac{1}{3}, \frac{-1}{3}, \frac{1}{3}\right) = \frac{1}{\sqrt{3}}\left(1, -1, 1\right)$$

$$U_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

So, this is my vector u_3. Now, we are able to get 3 vectors. So, this is my u_1, u_2 and u_3. Now, I want to check whether they are linearly orthogonal to each other or not.

Now, from here, we can see that if I take

$$u_1 . u_2 = \frac{-1}{\sqrt{2}} . \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}} . \frac{1}{\sqrt{6}} = 0$$

$$u_1 . u_3 = \frac{1}{\sqrt{2}} . \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} . \frac{1}{\sqrt{3}} = 0$$

$$u_2 . u_3 = \frac{-1}{\sqrt{6}} . \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} . \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{6}} \frac{1}{\sqrt{3}} = 0$$

So, they are orthonormal to each other and of course, their magnitude is 1 So, if I take this one and magnitude is 1, its magnitude is 1 and its magnitude is 1.

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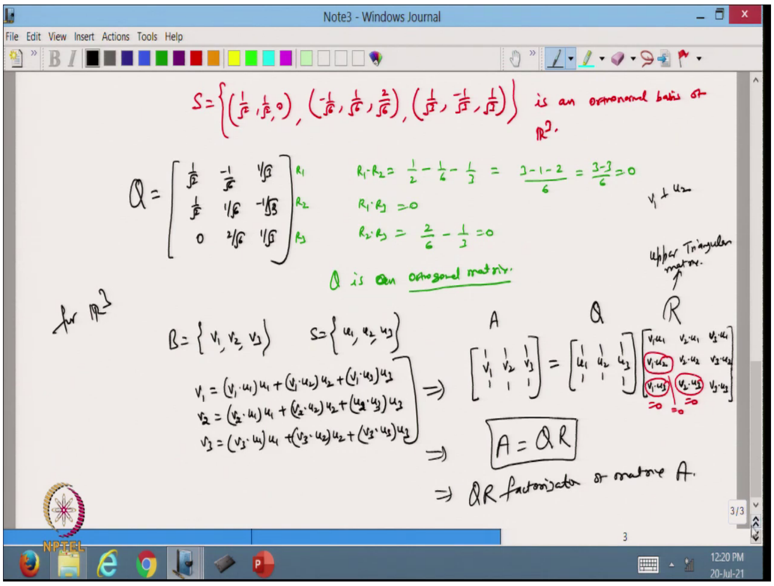

So, from here, we can say that now the set I call it is this one. So, I represent by

$$S=\left\{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right),\left(\frac{-1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right),\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\right\}$$

So, this is an orthonormal basis of R^3. So, we have changed the basis from b that is the linearly independent basis to the orthonormal basis, that is the another basis of R 3. So, you can also say that this is a change of basis.

Now, after doing this one, what we are going to do is I am going to use this basis in another form and that we are going to discuss it. Now, what I am going to write is from here, if you see then, so after doing this one, I will just write the matrix. I represent this matrix by Q.

So, in the Q, what am I writing?

$$Q=\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Now, because the columns are orthogonal to each other, even though I represent them, this is my R1, R2, R3. So, I can take my dot product with R1. R2, so it will be 1/2 - 1/6- 1/3 =0.

So, R1 is orthonormal. R1 .R3 =0 we can take. And R2 .R 3=0 that is also So, from here, I can say that my Q is an orthogonal matrix.

Now, what we are going to do is that now we have the basis. So, for the R3, we have two bases; B that is my $\{v_1, v_2, v_3\}$ and we have another basis as that is my $S=\{u_1, u_2, u_3.\}$ Now, I know that this is the orthogonal basis. So, I can write my v_1 as a linear combination of the standard orthogonal basis u_1, u_2, u_3. How can I write? I can write $v_1 = (v_1 . u_1) u_1+(v_1 . u_2) u_2+(v_1 . u_3) u_3$

$v_2 = (v_2 . u_1) u_1+(v_2 . u_2) u_2+(v_2 . u_3) u_3$

$v_3 = (v_3 . u_1) u_1+(v_3 . u_2) u_2+(v_3 . u_3) u_3$

So, I take a matrix with the first column, second and third. So, this is the matrix basically we have which is made up of the basis, the linearly independent basis, I can call it this matrix A.

So, this can be written as now I can write this as u_1, u_2, u_3. So, that is I am taking u_1, u_2, u_3 here. So, we call it Q and then, I can have a matrix here this can be written as

$$A=\begin{bmatrix} v_1u_1 & v_2u_1 & v_3u_1 \\ v_1u_2 & v_2u_2 & v_2u_2 \\ v_1u_3 & v_2u_3 & v_3u_3 \end{bmatrix}$$

Now, from here, I know that this v_1 u_1, so no problem but what about v_2 u_2? I know that v_1 is perpendicular to u_2. So, from here if you see, this is another matrix, but from here you can see that this value is equal to 0 because my u_2 is definitely perpendicular to u_1 and u_1 in the direction of v_1. So, it is perpendicular. Similarly, my v_3 and u_3, v_1 u_3 are 0. So, this is also 0. Now, in the next one $v_2 . u_1$, $v_2 . u_2$, $v_2 . u_3$ that is 0. So, this is equal to 0.

So if you see from here, all this value will be there, but this value will be 0. So, it is a 3 cross 3 matrix. So, I can call this matrix R and this I can know from here, this is the upper triangular matrix, because here it is v_2 with u_1, no problem; v_2 with u_2, no problem.

So, this can be some values, but this value definitely will be 0. So, this is a matrix that is called the upper triangular matrix. So, from here, if you see I have written my matrix A as Q, this is made up of an orthogonal basis into R. So, this is called QR factorization of matrix A.

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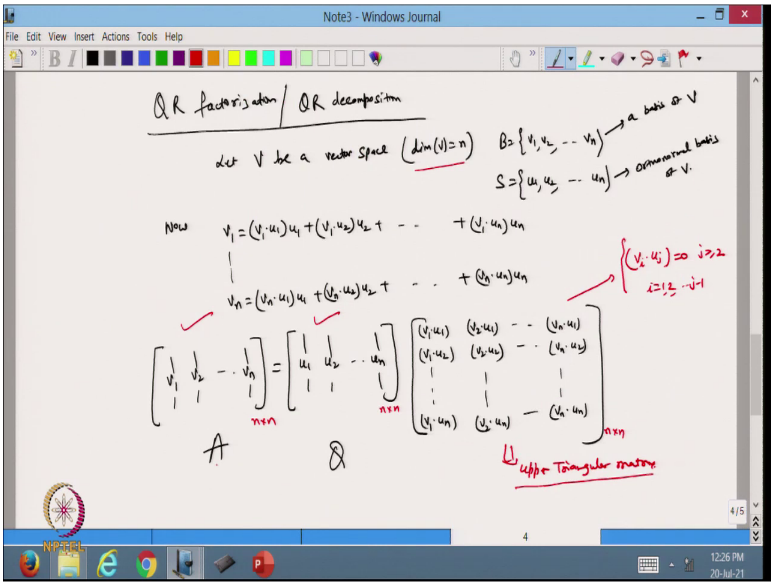

Now, based on this one, we will take the idea of another important topic that is QR factorization or we also call it QR decomposition. So, what is this? So, let we have a vector space v, v be a vector space and I call it the dimension of v is supposed to be n. So, now, I take the basis of v. So, let B is the basis that is$\{v_1, v_2, \ldots, v_n\}$, so this is the basis. Then, we take this orthogonal basis that is$=\{u_1, u_2, \ldots, u_n\}$. So, this is what we have seen.

So, this is a basis of v and this is an orthonormal basis of v. So, these are the orthonormal basis that we have taken from with the help of Gram-Schmidt. Now, based on this one, we can write again. So now, we can write v_1 as; again we can write $v_1 = (v_1 . u_1) u_1 + (v_1 . u_2) u_2 + \ldots\ldots + (v_1 . u_n) u_n$

:

$$v_n = (v_n . u_1) u_1 + (v_n . u_2) u_2 + \ldots\ldots + (v_n . u_n) u_n$$

So, from here, I can write my matrix with v_1 column vector $v_2, \ldots v_n$, it can be written as $u_1, u_2, \ldots, u_n$ and then we can have a matrix that is

$$\begin{pmatrix} v_1u_1 & \cdots & v_nu_1 \\ \vdots & \ddots & \vdots \\ v_1u_n & \cdots & v_nu_n \end{pmatrix} = A$$

Now, this matrix is my matrix corresponding to the given basis, I call it A. This is a matrix corresponding to the orthogonal basis that is my Q. And this matrix if you see, now from here in this matrix, so this matrix I can write here that from here, it is for now v_i. u_j. So, this is equal to 0 for j is greater than equal to 2 ok, and where, i is 1, 2 up to j minus 1. So, that is there.

So, if you use this one, then you will see that this matrix is an upper triangular matrix, because here we are talking about the vector space of dimension n. So, this matrix will be of order n cross n, this will be also order n cross n and this should be also of order n cross n. Because we know that when the matrix is a square matrix, only then can we talk about the upper triangular matrix.

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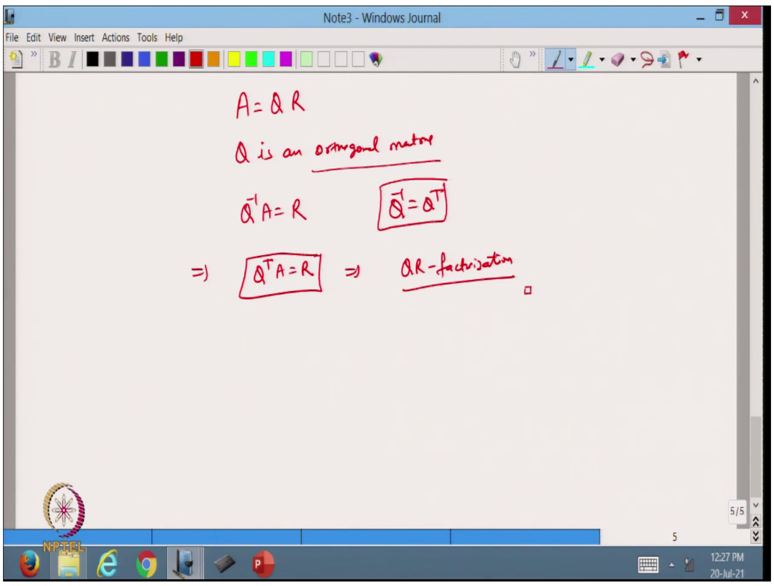

So, from here, we get my matrix A = QR. Now, my Q is an orthogonal matrix. Now, I can find the R because here, I know the value of A, I know the value of Q. So, I should be able to get the value of R directly from there.

Now, it is an orthogonal matrix, so I can take the inverse. So, Q^{-1} A= R. Now, I know that the $Q^{-1} = Q^T$ because it is an orthogonal matrix and from here, I can write that the Q^T A= R.

So, now, with the help of Q and A, we are able to find the R. So, this is my upper triangular matrix and from here, this process is called the QR factorization. So, it means I am able to factorize my given matrix A into a product of two matrices that is Q and R and Q is the orthogonal matrix corresponding to the orthonormal basis and R is my upper triangular matrix.

So, this is all we can discuss about the QR factorization. And we can also verify this one with the for the given matrix the Q and R. So, those things, we will discuss in the next lecture.

So, in this lecture, we have discussed the another important concept that is the QR factorization that in the for a given matrix, we can convert this matrix into the or we can factorize this matrix in the form of Q and R. And in the next lecture, we will discuss some MATLAB commands or Octave commands that how we can verify these things very easily in the software like MATLAB or Octave.

So, thanks for watching.

Thanks very much.