Matrix Computation and its applications
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Lecture - 41
Gram-Schmidt orthogonalization process

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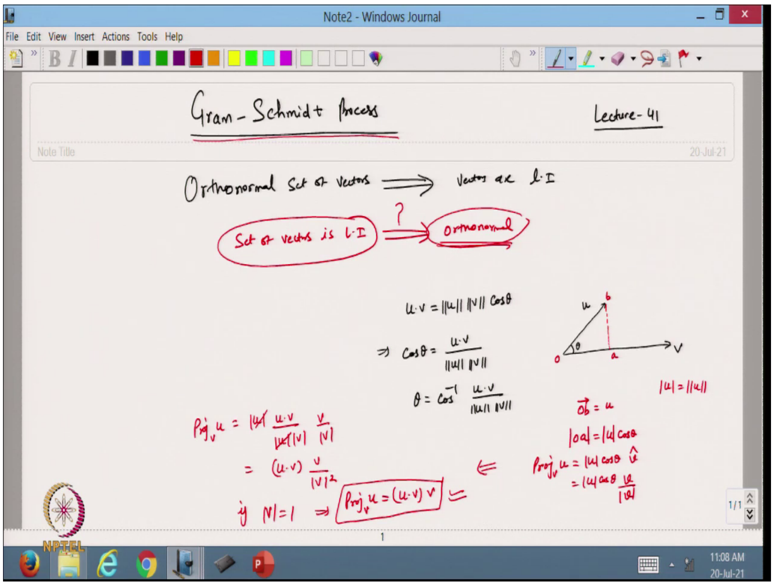

Hello viewers. So, welcome back to the course on Matrix computation and its application. So, today we are going to discuss a very important concept: how a linearly independent vector can be made orthonormal and this process is given by the great mathematician Gram-Schmidt process. So, this process is called the Gram-Schmidt process. So, let us do that.

Now, in the previous lecture we have seen that if we have an orthonormal basis, ortho we have a orthonormal set of vectors, then automatically these vectors are linearly independent that we have seen. Now, the question comes that what about if the set of vectors is linearly independent can we say that they are orthogonal or orthonormal? This process that if we have a set of linearly independent vectors can we make them orthonormal to each other? So, that is what we want to do.

So, this process is how we can make the linearly independent vector to the orthonormal basis that is given by the Gram-Schmidt process. So, let us see what is going to be in the Gram-Schmidt process. So, before that we will discuss a few concepts about the projection.

So, now, I know that suppose I have a vector, this is my vector I call it maybe v and suppose I take the another vector that is my vector u. Now, this is the angle between these two.

So, I know that if I take the dot product u dot v, then this can be written as

$$\begin{aligned} u \cdot v &= \| u \| \| \mathrm{v} \| \cos \theta \\ \Rightarrow \cos \theta &= \frac{u \cdot v}{\| u \| \| \mathrm{v} \|} \\ \theta &= \cos^{-1} \frac{u \cdot v}{\| u \| \| v \|} \end{aligned}$$

Now, we know that if u and v of unit magnitude then this will be just equal to cos theta will be equal to u dot v. So, this things we already know now what I want to do, I take the projection of this vector on the vector v. So, this is suppose o and this is a this is my b. So, the position vector I am taking is the position vector ob we represent by this as a vector. So, this is basically the position vector.

Now, I want to take the projection of this oa. So, now, I can write that oa. So, this oa I am writing in the direction of v, so this o a is basically u. So, this is u I am taking cosθ. So, that is the projection I am taking. So, this is my o a or means I am talking about the magnitude. So, this is equal to the magnitude of this one.

So, if suppose I want to find out the projection of u in the direction of v. So, this is equal to

$$\begin{aligned} \overrightarrow{ob} &= u \\ |oa| &= |u| \cos \theta \\ \mathrm{proj}_{\mathrm{v}}\, u &= |u| \cos\theta \hat{v} \\ &= |u| \cos\theta\, v/|v| \end{aligned}$$

Now, from here I can write from here, this one. $\mathrm{Proj}_v u =$

$$= \frac{|u|\, u.v}{|u||v|} \cdot \frac{v}{|v|}$$

$$= (u.v) \frac{v}{|v|^2}$$

$$if\ |v| = 1 \Rightarrow \mathrm{Pr}oj_v u = (u.v)v$$

where v is a unit vector.

So, this is how we take the projection of a vector over another vector.

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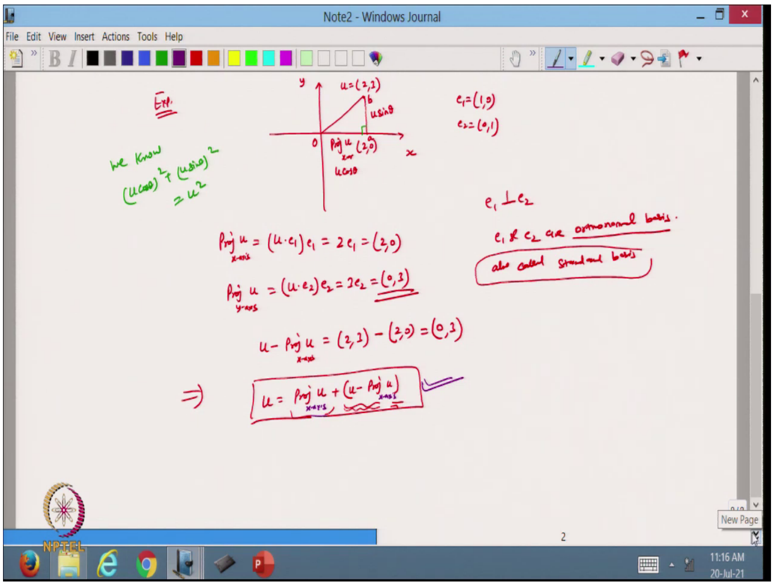

For example, now let us take one example. Suppose I take it as r square. So, it is x axis it is y axis and suppose I just take one vector. So, I call it maybe (2, 3). So, this is my vector. So, I call this vector as my vector u.

Now, I just take the projection of this u on the x axis. So, I know that x axis. So, the vector I am taking in this direction is a standard vector. So, we call it e_1 .So, I know that e_1 is the unit vector (1, 0) and e_2 = (0, 1). So, these are the standard basis in the R^2 that we already know so I just take the projection of this one.

So, I am taking the projection of this along the x axis. It means if I take u and I take a vector in the x axis.

$$\text{Proj}_{\text{x-axis}}\ u = (u.e_1)e_1 = 2e_1 = (2,0)$$

$$\text{Proj}_y\ u = (u \cdot e_2)e_2 = 3e_2 = (0,3)$$

$$u - \text{Proj}_{\text{x-axis}} = (2,3) - (2,0) = (0,3)$$

⇨ u= $\text{Proj}_{\text{v}}\, u + (u - Proj_v u)$ $\text{Proj}_{\text{v}}\, u + (u - Proj_v u)$

I know that the x axis and y axis are orthogonal to each other. We already know that e1 is orthogonal to e2 and this is. So, I can say that e1 and e2 are orthonormal bases and they are also standard bases that we already know are also called standard bases.

So, that is why we already know that working with the standard base is much easier as compared to the other basis. So, instead of the standard basis we are going to introduce other types of basis which have the same property as the standard basis. So, these are orthogonal to each other.

Now, I see what is happening u minus projection of u in x axis because if you see that this is my u. So, I can write this as a u cos theta component and u sin theta component that we already know because it is a right angle triangle. So, using the Pythagoras theorem we know that u cos theta square u sin theta square that is equal to u square.

So, this is a Pythagoras and this is happening because they are perpendicular to each other. So, this basis is here now. I am taking this projection on this.

So, from here you can say that now from here I can say that u can be written as if I am taking the projection of u in along v plus I can write u minus projection of v it means so, this is the projection in the perpendicular vector to u. So, that is what we can say. So, this projection of u we are taking in the direction perpendicular to the vector v ok.

So, that is here we are doing. So, we are taking the projection of u along v and this is the projection of u in the direction perpendicular to v. So this is my projection in the direction perpendicular to v that is perpendicular to this. So, I am taking this as a y axis. So, that is my u.

It means I can split my u into two parts; one in the direction of v another in the direction perpendicular of v. So, here I am taking v maybe I can just instead of v I just write x axis because here we are talking about x axis. So, I can take it here x axis; x axis. So, this is the projection in the y direction. So, which is a perpendicular to the x axis. So, this is the way we are doing it here.

Now, I know that if v is a unit vector then the projection would be in this form and this is also why we have discussed that it is a component in the direction of x axis and in the component in the direction of perpendicular to x axis that is in the y direction. So, we can split the vector like this one.

Now, from here let us see what is going to happen in the case of three vectors. So, these are the two vectors we have taken.

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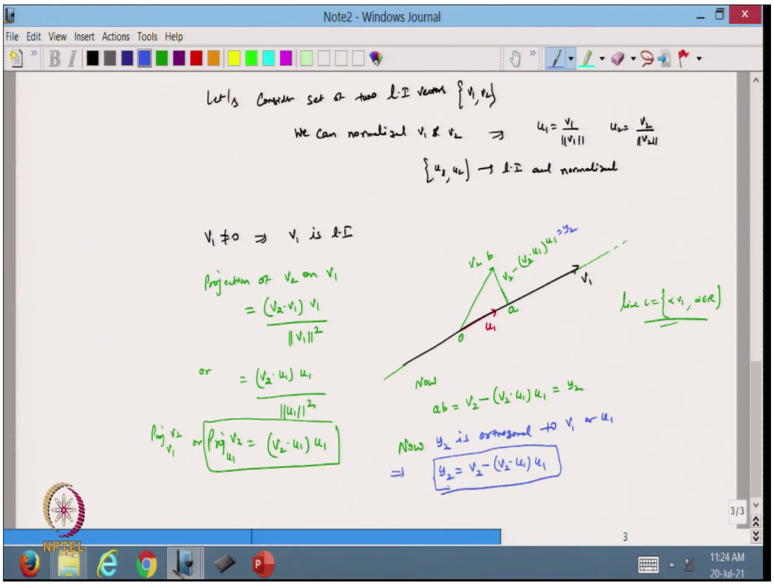

Let we have two vectors. So, let us consider a set of two linearly independent vectors. So suppose I take the set as $\{v_1, v_2\}$ this is linearly independent now . If it is linear independent then I can normalize this one.

So, we can normalize. So, what how we can normalize v_1 , v_2 and from here I can write my

$$u_1 = \frac{v_1}{\|v_1\|}, u_2 = \frac{v_2}{\|v_2\|}$$

So, from here I know that u_1 and u_2 are LI and normalized. So, nothing has changed, only the length has become 1 that is it.

Now, I take the set v_1. So, $v_1 \neq 0$ is linearly independent that I know. So, v_1 I have taken is not zero I know that. So, only one vector v_1 is linearly independent that I know. Now, this is what we are going here, this is my v_1 vector and I am taking the line passing through this vector v_1.

So, suppose this is my v_1 this is my origin. I am taking a line passing through this v_1. So, this is the line I am taking.

Now, suppose this is the unit vector I am representing by u_1. Now, one thing is true that if v_2 is linearly independent to v_1 and v_2 cannot be on this line because if v_2 lies on this line then it will be linearly independent. So, maybe it cannot be on this line. So, suppose it is lying somewhere here like this one. So, this is supposed to be my v_2 because it cannot be parallel to v_1 or lying on this v_1. So, it is tilted. So, whenever it is tilted it will definitely cut the line here. So, suppose this is this one and I call it this is the o where it is cutting.

Now, from here I can take the projection of this on this line passing through the vector v one. So, suppose this is basically my line is there and I am just taking the v_1 vector because I can write the line l as some scalar. Maybe I can take some alpha into v_1 where alpha belongs to the real line. So, this is the line.

Now, I have taken this one. So, I am taking the projection of v_2 on the v_1. So, now, from here I know that the projection of v_2 on v_1 that will be equal to I am taking the dot product with the vector v_1, then I am writing v_1 divided by the magnitude square.

So, this is what we have defined from here. If this is of this form then we can write like this or maybe I can write. So, this is what we have written or we can write it as $v_2 . v_1$. So, I am taking the v_1 and u_1 is the unit vector in the direction. So, maybe I can take it as u_1 here no problem into u_1 and I know that the u_1 magnitude square, but this is equal to 1. So, it is $(v_2 . u_1) u_1$.

So, this is my projection of v_2 in the direction of u_1 or I can call it the projection of v_2 in the direction of v_1. So, this is the way we have taken projection of v_2 on v_1 and I will take the vector u in the direction. Now, from here on, this is my projection..

Now, I want to write. So, this is suppose o a and b now I want to write what is my a b vector. So, that is the perpendicular vector we are going to take. So, what I am going to do is that I am taking my v_2 and subtracting the part of projection of this along the direction v_1. So, this is I have $ab= v_2-(v_2 . u_1). u_1 =y_2$

So, this y_2 is orthogonal to v_1 or u_1 ok. So, from here, how I can write y_2 directly. So,

$y_2 = v_2 - (v_2 . u_1). u_1$ .So, this is where I have taken my y_2.

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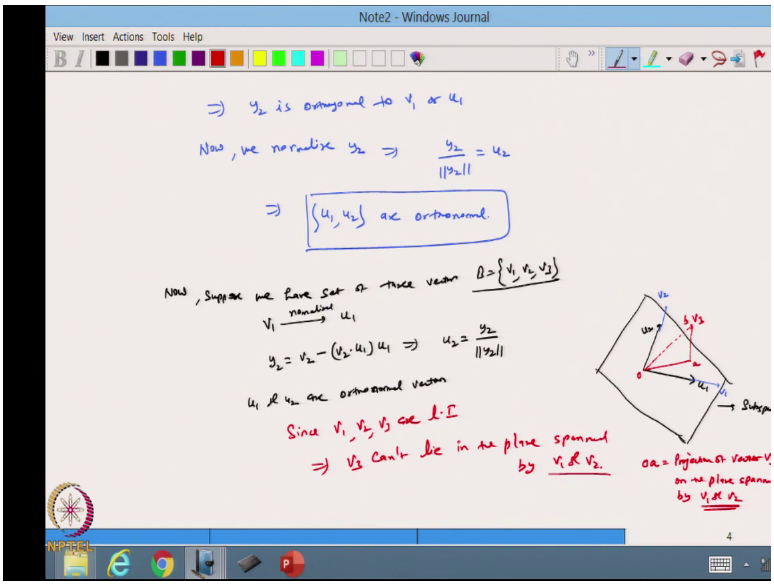

From here y_2 is orthogonal to v_1 or u_1 because v_1 is just in the vector. Now, what do we do that now we normalize y_2 and from there I call it y_2 divided by its magnitude. So, I call it u_2. So, from here I can say that the vector u_1 and u_2. So, these vectors are orthonormal to each other or the orthonormal vectors are orthonormal. So, I just stop here and write it orthonormal, but orthonormal means that these vectors are.

So, we started with the linearly independent vectors v_1 and v_2 and then from there we constructed the vectors which are orthogonal to each other and have magnitude 1. So, basically what we have now we have started with this one. So, this becomes the orthonormal basis. So, if I have a vector with two bases having the two vectors then by this process we are able to make it orthonormal.

Now, I consider the third one. So, let us take the third part. So, this is we have started with 2 vectors. Now, suppose we have a set of three vectors. So, suppose I take the basis with vectors $\{ v_1, v_2, v_3\}$

So, this is the basis and suppose this is L I. Now, from here I know that I will take my vector v_1, I will normalize it and I will call it u_1, then I will take the another vector y_2 that will be orthogonal to this one. So, I will take my

$$y_2 = v_2 - (v_2 . u_1). u_1 \Rightarrow u_2 = \frac{y_2}{|| y_2 ||}$$

So, it is a normalized vector. Now, u_1 and u_2 are orthonormal vectors. Now, what about the third one? So, let us take the v three.

So, in this case what we are going to do is it means that I have a vector suppose I represent my suppose I write here this is my suppose u_1 and I take another vector which is my u_2 they are orthogonal to each other and suppose they make a plane. So, I make a plane from these two vectors. So, this is my plane.

Now, in this plane if I take the linear combination of this one that all the vectors will lie here. So, I can say from here that this is a subspace basically a subspace of R^3 I am taking because I am taking 3 vectors. So, using two vectors from the basis they will make the subspaces.

Now, I take another vector v three. So, this v_3 is a linear combination of v_1 and v_2 this sorry this v_3; $\{ v_1, v_2, v_3\}$are linearly independent to each other. So, definitely the v_3 cannot be a linear combination of v_1 and v_2, because if v_3 is lying in this plane made by v_1 and v_2 or u_1 and u_2 then they cannot be linearly independent.

So, from here I can write now we know that since $\{ v_1, v_2, v_3\}$are linearly independent which implies. So, it means that they cannot be coplanar and all the threes cannot lie in the same plane; it means that v_3 cannot lie in the plane spanned by v_1 and v_2 . So it means that I just take. So, this is my u_1 and u_2 and I can extend it to v_1 and v_2 no problem. So, suppose this is my v_1 and this is my v_2 no problem.

So, my v_3 cannot be in this line in this plane. So, I just represent this is my supposed v_3 it is not lying in the plane made up of u_1 and u_2 and suppose I take the projection of this on this plane. So, this is my projection. So, this is the projection we have taken. So, I call this as I suppose I call it o a b. So, from here I can write that from here o a this is what I am writing that is equal to projection of vector v_3 on the plane spanned by v_1 and v_2 .

So, it is just a plane spanned by v_1 and v_2 and this is just I am taking the projection of this one. So, that is o a and a b are the projections in the direction perpendicular to this subspace.

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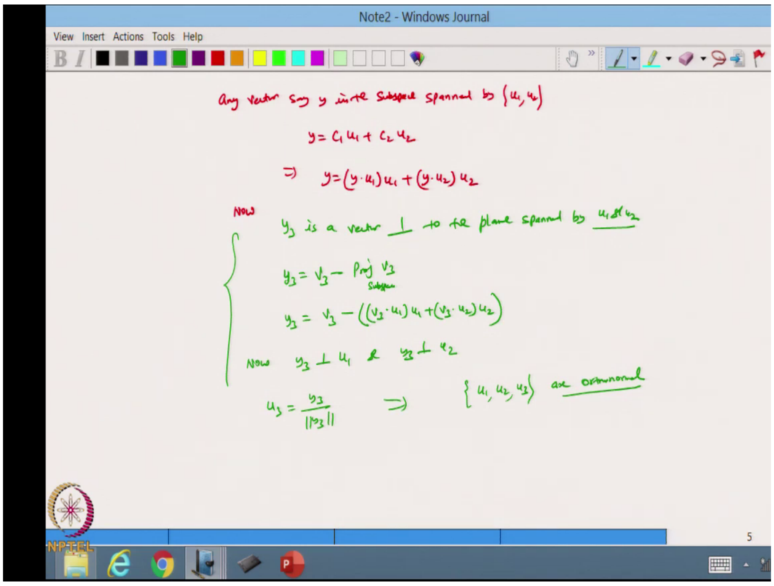

Now, from here now any vector says y in the subspace spanned by v_1 and v_2 . So, I can write this as c_1 v_1 + c_2 v_2 and these are linearly independent to each other.

So, from here I can call this y as y taking the dot product with u_1 that is the normalized vector in $y=(y.u_1).\ u_1 + (y.u_2).\ u_2$ ok so spanned by v_1 and v_2 or maybe I just instead of this one I take u_1 u_2. So, I can write from here as a linear combination c_1 v_1 + c_2 v_2 and u_1 and u_2 I know this is the normalized vector we have taken and from there I also know that the coordinates c_1 and c_2 can be found using this formula.

So, from here now I can write. So, I need a vector which should be a perpendicular to this one. So, this one I need to find out which is a perpendicular to the plane passing from u_1 and u_2. So, this vector. So, I call this vector as may be y_3 so from here I can find. Now, y_3 is a vector perpendicular so, this is the sign of perpendicular to the plane spanned by u_1 and u_2.

So, how can I find it? So, the same way we will find out. So, I can write $y_3 = v_3$ - projv_3 on the subspace. So, that which subspace; the subspace spanned by this one.

So, and subspace is spanned by from here. So, now, from here I can write that I can write my v_3. So, this is my y_3 and how I can find this one so v_3 I am taking the span. So, this one I can write as $y_3 = v_3 - ((v_3.\ u_1).\ u_1 + (v_3.\ u_2).\ u_2)$ this is what we have written because I am taking this projections here. So, if I want to take the projection of this one. So, that projection is basically a linear combination of this.

Now, I want the projection instead of y. I will just write v_3 because I need the projection of v_3 on this plane. So, I will take v_3 over this plane. So, this is$((v_3.\ u_1).\ u_1 + (v_3.\ u_2).\ u_2)$. So, this is the projection on the plane and that is v_3.

So, now from here on, y_3 is perpendicular to u_1 and y_3 is perpendicular to u_2. Now, what I do is that I will take $u_3 = \frac{y_3}{\| y_3 \|}$. So, from here and now I can say that my vectors $\{ u_1, u_2, u_3\}$are orthonormal because they are perpendicular to each other and their magnitude is also 1. So, this is basically how we can proceed from making the given linearly independent set to the orthonormal set. So, let us do this one again.

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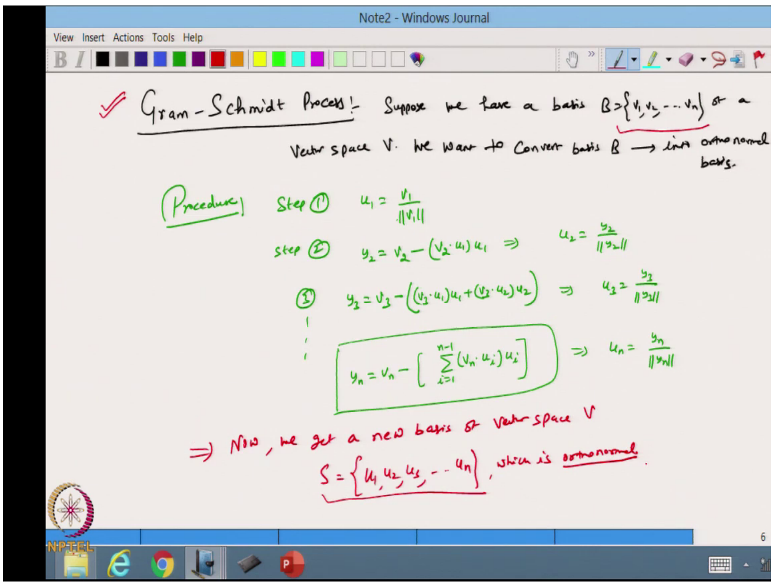

So, I will write here the process that is called the Gram process. So, what we are going to do in this case.

Suppose we have a basis I call it{ v_1, v_2, ……,v_n}. So, this is the basis of a vector space V. So, it is a n dimensional vector space. Now, what we need to do. So, we want to convert basis B into orthonormal basis. So, this is done by the Gram-Schmidt process. So, what how we are going to do is that. So, this is the procedure.

Step 1 because I will choose the vector v_1. So, I will take the vector $u_1 = \frac{v_1}{\| v_1 \|}$. So, I will make u_1 is a vector that is a unit vector. So, u_1 is a normalized vector. Step 2 now in the step 2 I take the vector

$$y_2 = v_2 - (v_2.u_1)u_1 => u_2 = \frac{y_2}{\| y_2 \|}$$

Then step 3 I will take

$$y_3 = v_3 - ((v_3.u_1)u_1 + (v_3.u_2)u_2) => u_3 = \frac{y_3}{\| y_3 \|}$$

And we keep going like this one so in the end I will get y_n. So, what is y_n? It will be

$$y_n = v_n - \left[\sum_{i=1}^{n-1} (v_n.u_i)u_i \right] => u_n = \frac{y_n}{\| y_n \|}$$

So, from here after doing all this calculation now we get a new basis of the vector space of vector space V. So, I call this new basis as we call it. I will call right { u_1, u_2, ……,u_n}. a new basis of V.

So, let us say I call it maybe I should call B 1 or I should represent it by S close to the standard basis. So, I am represented by S. So, now, we take a new basis of vector space V that is S which is S which is orthonormal. So, these are orthonormal.

So, we have started with a linearly independent basis and we convert that one into the orthonormal basis with the help of Gram-Schmidt process. So, this is the benefit of the

Gram-Schmidt process to convert or to transform a given set of linearly independent vectors in a given vector space to an orthonormal set of vectors.

So, I will stop here. So, in the today's lecture we have discussed that if we have a basis for a given vector space, then how we can convert that basis into the orthonormal basis of the same vector space and that process we have discussed is the Gram-Schmidt process that is going to convert the given set of basis into the orthonormal basis. And in the next lecture we will continue with that one. So, thanks for watching.

Thanks very much.