Matrix Computation and its applications
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Lecture - 40
Orthonormal bases of a vector space

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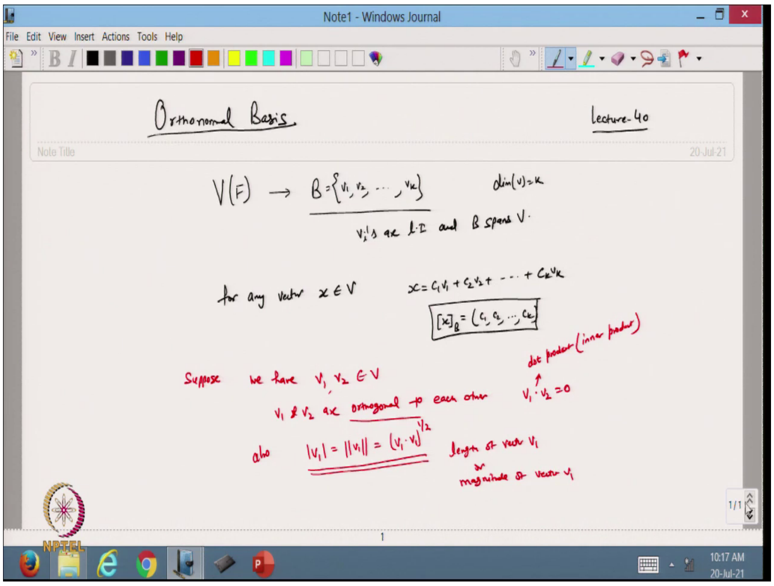

Hello viewers, welcome back to the course on Matrix Computation and its application. So, in the previous lecture we discussed vector spaces and their basis. So, today we are going to start with another thing that will happen when we are going to have Orthonormal bases. So, that is what we are going to discuss today.

So, today we are going to discuss the orthonormal basis. So, what is that? So, till now we have a vector space. We have discussed the vector space V that is defined on the given field and from there we know that we have a basis B that is made up of the vectors $\{v_1, v_2, \ldots, v_n\}$. So, this is the basis for vector space V, it means that the dim(V)=K. Now, we know that this is the basis then that vi’s are linearly independent and B spans V.

So, this is what we have seen till now. Now, you have seen that whenever we want to show that for any vector say I call it $x \in V$. If I need to write that x as a linear combination of this vector i.e $x = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$ then we need to find this coordinates and I know that I can write the coordinate of this x with respect to basis. So, this is a $[x]_B = (c_1, c_2, \ldots, c_k)$

Now, finding these coordinates is a very cumbersome job because whenever we have a large number of systems like we have a v with the dimension may be 10. So, in that case we will solve this x and to find this coordinate $c_1, c_2, \ldots, c_k$ we need to solve the system of 10 by 10 matrix and then with that one will be able to find this value.

So in fact, in real life we may have a very large number of very large systems. So, in that case finding these coordinates is really a very tough job. So, to get rid of this one we take an alternative and that alternative we are going to discuss is based on an orthonormal basis. So, let us see what we are going to have from here. Now, I will start with just a simple theory about the orthonormal basis.

So, suppose we have two vectors v_1 and v_2 , so, that belongs to the vector space V. Now, I will say that v_1 and v_2 are orthogonal to each other. It means that if I take the dot product $v_1 . v_2$. So, this is we are talking about dot product or I can also call it inner product. So, if I take the dot product of two vectors and if this is going to be 0 i.e $v_1 . v_2 = 0$ then we say that the vectors v_1 and v_2 are orthogonal to each other.

And also we call it now based on this one. I know that we can find the magnitude of the vector sometime we also write like this one. So, this is basically a symbol of norm. So, this is we can represent by v_1 taking the dot product of v_1 itself and then taking the under root. So, that is the basically magnitude of the vector v_1 or we also call it length of vector v_1 or magnitude of vector v_1 i.e $|v_1| = \|v_1\| = (v_1 . v_1)^{1/2}$

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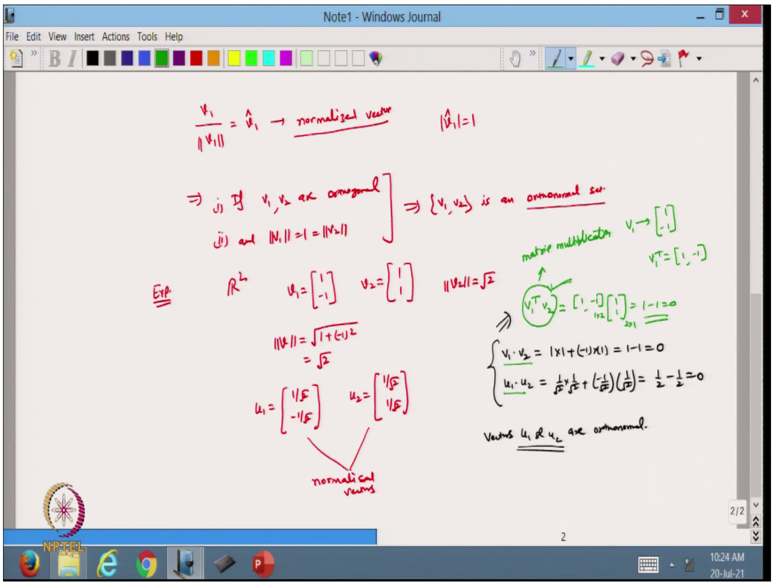

So, what do we do? Now, I take any vector v_1 and I will divide by its magnitude. So, now, whatever the vector we are getting. So, this can be written as a hat. So, this is called a normalized vector. So, it is a normalized vector and if you see now then its length is 1. So, what have we done? We have taken a vector divided by its magnitude. So, that process is called normalization.

So, we have normalized the vector and then it is a magnitude of 1 and this vector v_1 , v_2 already was orthogonal. So, now, from here v_1 and v_2 , so, now, what I do is that now if vectors v_1 and v_2 are orthogonal 1st one. 2nd $\|v_1\| = \|v_2\|$ or the length is one then from these two we call it that set v_1 and v_2 is an ortho; ortho coming from the orthogonal and normal means it is normalized. So, it is called an orthonormal set ok.

It means the set the v_1 and v_2 orthogonal and their magnitude is 1. So, then it is called an orthonormal set. Now, the question comes to how we can talk about the basis of that. So, for example, suppose I take a vector for example, I take a vector we talk about R^2 then let I take $v_1 = (1,-1)$ and $v_2 = (1,1)$

Now, if you take the magnitude of this vector, $||v_1|| = \sqrt{(1+(-1)^2)} = \sqrt{2}$, $||v_2|| = \sqrt{2}$ So, its magnitude is not 1. Similarly, if I take the magnitude of v_2, that is also under root 2. Now, I divide it by this one. So, maybe I can call it u_1 now.

So, $u_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ and similarly I can construct my $u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. So, these two vectors are normalized now. These are normalized vectors. Also now I just check what will happen.

So, they know the difference between orthogonality. Now, what I am going to do is that I will check from here $v_1 .v_2$. So, if you see from $v_1 .v_2$ we are taking the dot product. So, dot product is component y. So, I will call it $v_1 .v_2 = 1*1+(-1)(1)=1-1 =0$. I can write from here

$$u_1 .u_2 = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{2} = 0$$

So, in this case we can call it these two vectors. So, vectors u_1 and u_2 are orthonormal; it means they are orthogonal and their magnitude is 1.

So, now from here this has become the orthonormal set. So, this dot product if you see from here then, this dot product I can write in this form also. So, this is in the vector form. We know how to take the dot product, but what I can do is that I can write this dot product in terms of matrices because we know the matrices. So, v_1 is the vector.

So, I call it because a vector V we always take in the column form like this one we have taken and v_1^T transpose if I take that will be a row vector. So, this dot product I can write as $v_1^T .v_2$ So, this is basically matrix multiplication. So, I can write from here that what is v_1^T? v_1^T will be a row vector. So, this is [1,-1] and v_2 is [1 ,1] is a column vector.

So, this is the matrix of 1 cross 2 and this is a matrix 2 cross 1. Now, if I multiply those matrices, this will multiply by this one. So, it will be 1 - 1 and that will be 0 because we know that this is orthogonal. So, this dot product we can also write in this form. So, that is the multi matrix multiplication and this is a vector dot product, but the things are the same.

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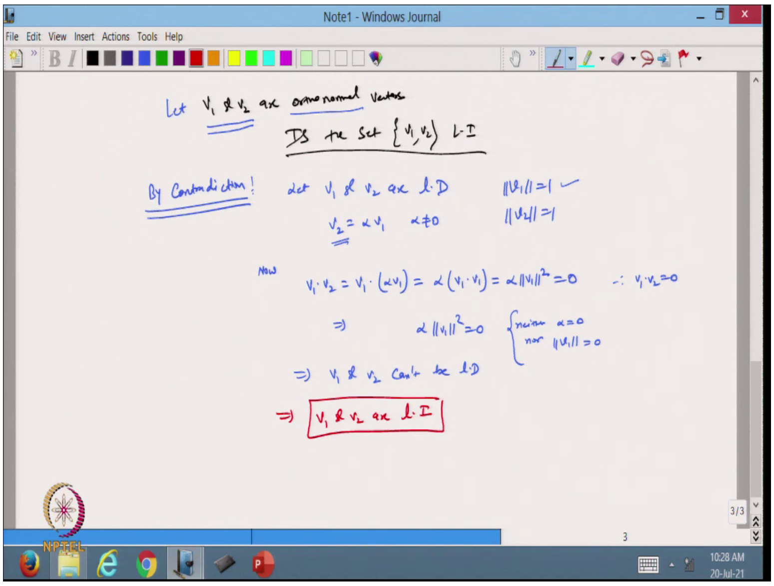

So, after doing this one now the question is that now, from here I can say that this v_1 and v_2 are orthonormal vectors then what about linear independence? The question is that is the set v_1 ,v_2 linearly independent because if we discuss the orthonormal basis then we know that it should be linearly independent.

So, let us check whether this v_1 and v_2 are linearly independent or not ok. So, this one I can just check. So, by contradiction we can say by contradiction. So, let v_1 and v_2 are linearly dependent, L D. It means if one vector. So, I can write vector $v_2 = \propto v_1$ where $\propto \neq 0$ ok. So, I am taking v_1 and v_2 any orthonormal vectors.

So, it means that $\| v_1\| =1$ and $\| v_2\|= 1$. So, I can say that let v_1 and v_2 are orthogonal vectors. Now, this is what I have written. Now, I take $v_1 .v_2$. So, I can write here

$$v_1 . v_2 = v_1 (\propto v_1)= \propto(v_1.v_1) = \propto||v_1||^2 =0$$

because v_1 , v_2 are orthonormal. So, from here which implies that $\propto||v_1||^2 =0$,Now, we know that alpha is not 0

So, if the magnitude is equal to 0 then we know that its magnitude is 1. So, I can say from here that neither alpha is 0 nor v is 0. So, from here I can say that which implies that it cannot be linearly dependent. So, from here we can say that the vectors v_1 and v_2 are linearly independent. It means if I take the two vectors which are orthonormal then definitely they will be linearly independent to each other.

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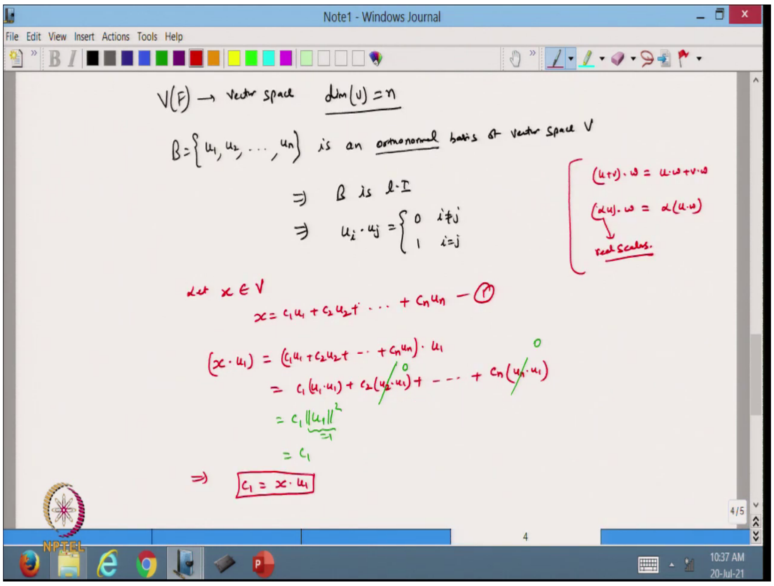

So, this is a very important result of linear independence. Now, the question comes that suppose we have a vector space V(F) and so, this is a vector space and now from here I call it finite dimensional. So, suppose dim(U)= n. So, it is a finite dimensional space over the field F.

Now, what I do is that I take a basis B= $\{u_1, u_2, \ldots, u_n\}$. ok. So, let this is an orthonormal basis of vector space V ok because here we are talking about n dimension vector space and I am taking the basis of this one. So, generally we are talking about the square type of matrices.

But it can be suppose if I take the dim(U)= n and if we take the basis which contain the vectors less than n then you know that this basis will create a subspace of V, so, but now we are taking that it is an orthogonal basis of vector V now from here. So, which implies that B is linearly independent also and which also implies that $u_i \cdot u_j$ that is equal to 0, when $i \neq j$ and if it has the same elements then it will be 1 because it is an orthonormal basis. So, that is there.

Now, I want to see that because we know that this is linearly independent. Now, what is going to happen now? Let us check how it is useful for finding the coordinates. So, let we take any vector $x \in V$. So, then I can write as

$$x = c_1u_1 + c_2u_2 + \ldots + c_nu_n$$

$$(x.u_1) = (c_1u_1 + c_2u_2 + \ldots + c_nu_n).u_1$$

$$= c_1(u_1.u_1) + c_2(u_2..u_1) + \ldots + c_n(u_n.u_1)$$

$$= c_1||u_1||^2 = c_1$$

$$\Rightarrow c_1 = x.u_1$$

So, we need to find the value of c_1 c_2 this one. So, I can call it 1. Now, it is a vector. So, I want to find c_1 c_2 . So, for this one I know I will do that. So, I will define the x this vector taking the dot product with u_1, I define it like this one. So, it is a dot product of $x.u_2$. So, x is this one. So, I can write it $x.u_2 = (c_1u_1 + c_2u_2 + \ldots + c_nu_n).u_2$. So, this is my vector taking the dot product with u 1. Now, we know that in the dot product suppose I have a vector (u+ v). w.

So, it can be written as u .w + v .w and I also know that alpha we are talking about the real scalars. So, alpha u taking the dot product with w that can be written as u w. So, this is real scalars we are talking about. So, we already know this property. So, from here I can write like this one.

So, this will be equal to c_1 $(u_1.u_1)$ taking the c_1 $(u_1.u_2)$ and I can write from here c_n this will be u_n taking the dot product with u_1. Now, I know that this is the orthonormal basis. So, from here I know that this is 0, this is 0. So, all other terms will be 0 except.

So, this one can be written as c_1 and this is u_1 taking the dot product of u_1. So, I can write from here that this is equal to u_1 square. Now, I am taking the orthonormal basis. So, this quantity is 1. So, from here I will get c_1. So, from this I can say from this that my c_1 is taking the x $.u_1$. So, that is my c_1.

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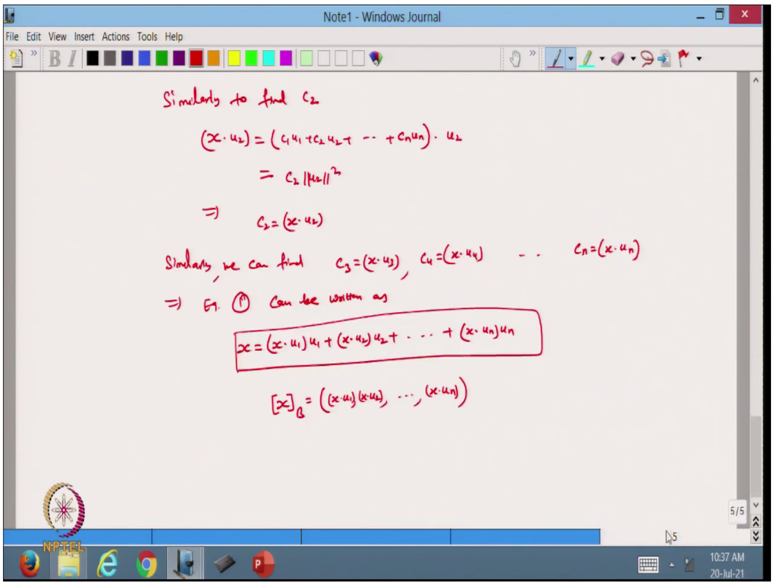

Similarly, to find c_2, what do I do? I take the dot product of $x.u_2$. So, it is the same way. It is $c_1u_1 + c_2 u_2 + \ldots + c_nu_n$ taking the dot product with u_2. So, ultimately if you see from here the same way will go then will get c_2 u square only left with this one.

All other terms will be cancelled out the same way because they are orthonormal basis. So, this means the orthonormal basis means that if you choose any two vectors from a different vector their dot product will be 0 and if you take the same 1. So, it will be one. So, that is the meaning of this orthonormal basis. So, from here I will care that my $c_2 = (x.\, u_2)$. So, from this way I can find. So, similarly I can find.

So, similarly we can have or we can find $c_3 = x\ u_3$, $c_4 = x\ u_4, \ldots\ldots,$ $c_n = x\ u_n$ So, from here, now this equation 1. So, equation 1 can be written as I can write x.

$$x= (x\ .u_1).u_1 + (x\ .u_2).u_2 + \ldots\ldots + (x\ .u_n).u_n$$

So, by this way we are able to find the coordinates of a vector very easily because here just we are taking a dot product and that is it we are able to do that one. So, now, from here you can see that my coordinates of x with respect to the basis are

$$[x]_B = (x\ .u_1, x\ .u_2, \ldots.., x\ .u_n)$$

So, this is the advantage of dealing with the orthonormal basis. So, let us take one example.

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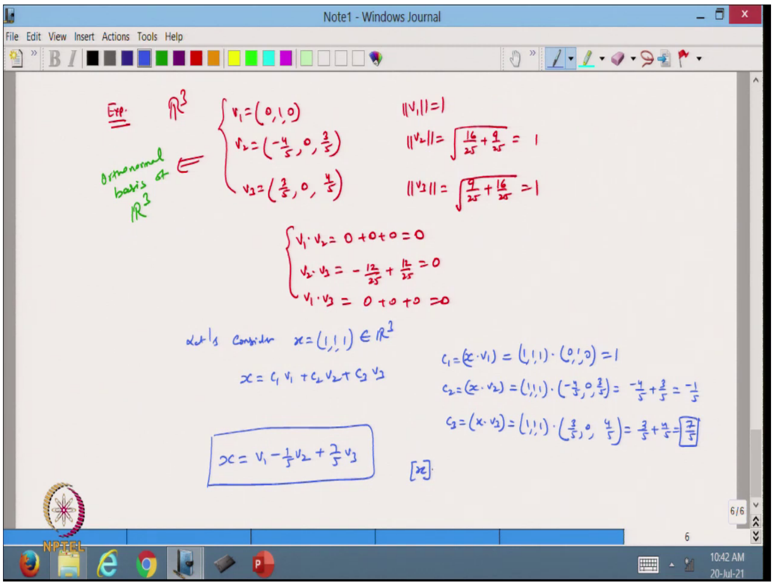

So, in this example I will just take some orthonormal basis. So, let us take a vector space R^3 and I choose the basis. So, I just take v_1 =(0, 1, 0); v_2 =(-4/5, 0, 3/5) and v_3=(3/5, 0, 4/5) ok. So, this is the vector I am taking from here. Now, you can see from here that $\| v_1\|=1$, $\| v_2\|=1$, $\| v_3\|=1$.

It means these are normalized vectors and also you can see from here that $v_1.v_2 = 0+0+0=0$ the first two vectors component wise dot product. Now, $v_2.v_3 =0$; $v_1.v_3 = 0$. It means that these vectors are orthogonal to each other. So, they are mutually orthogonal and their magnitude is 1. So, I can say that these are the orthonormal basis.

So, this is what I can call the orthonormal basis of R^3 . Now, suppose I choose. So, let us take one. So, let us say consider a vector x from R^3. So, I call it suppose x=(1, 1, 1) I take that belongs to R^3 . Now, I want to find

$x= c_1v_1 +c_2 v_2 + c_3v_3$.

$c_1 = (x. v_1)=(1,1,1).(0,1,0) =1$

$c_2 = (x. v_2) =(1,1,1).(-4/5,0,3/5)= -4/5 +3/5 =-1/5$

$c_3 = (x. v_3)=(1,1,1).(3/5 ,0,4/5)=3/5+4/5 =7/5$

$x= v_1 -1/5 v_2 + 7/5v_3$

So, this is the linear combination we can write and these are the coordinates we have taken.

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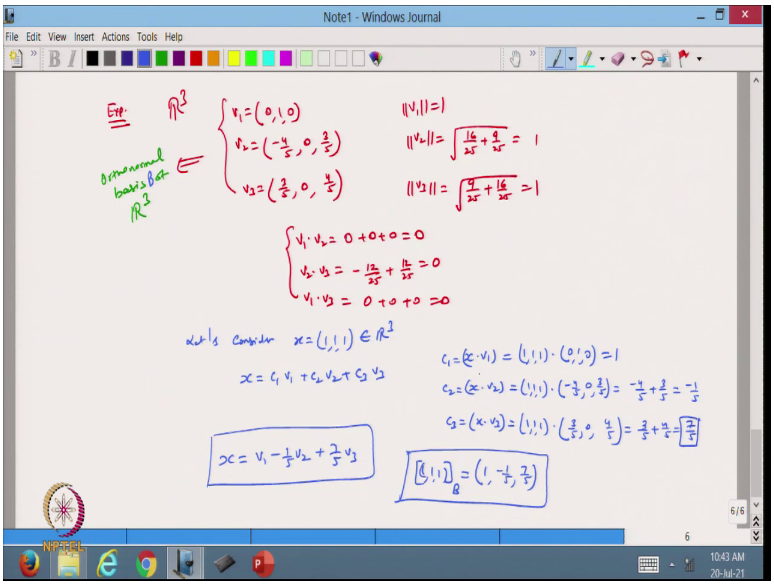

So, from here the coordinate of x or maybe I can just call it that coordinate of vector $[1, 1, 1]_B$ with respect to the. So, I can call it an orthogonal basis. So, I just call it B. With respect to B it is just (1,-1/5, 7/5). So, that is the answer to this question. So, it is very convenient for us to find out the coordinates or the vector x with respect to the orthonormal basis. So, that is one of the advantages of taking the orthonormal basis.

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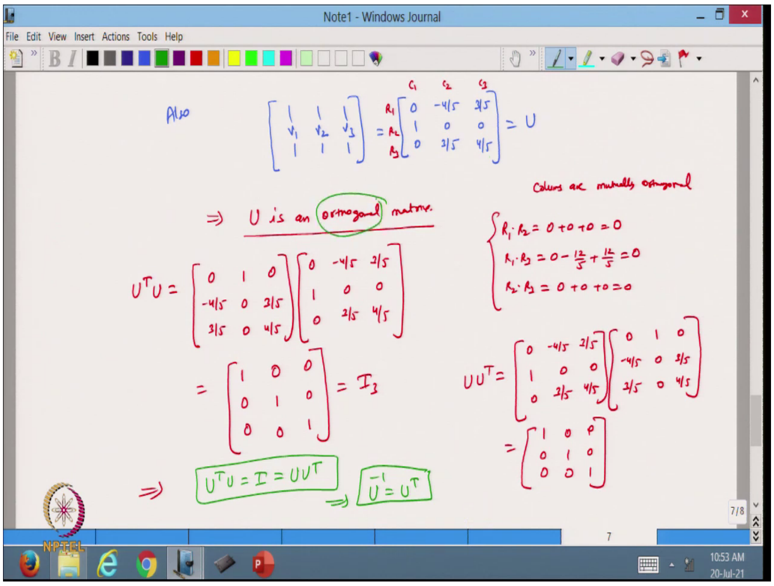

Also, if you see from here, these are orthogonal vectors if I am taking, so, what I am going to do is that I will make a matrix made up of this basis. So, I just take here v_1 first column here, then v_2 and v_3 as second and third column. So, I call this matrix.

$$\begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix} = U$$

Now, if you see from here we have already seen that these columns are orthonormal. So, it means that if I take the dot product of these columns then this is going to be 0. So, this vector I can from here I can say that this is my column 1, column 2, column 3. So, if I take the dot product of these columns, so, these columns are mutually orthogonal. Columns are mutually orthogonal and their magnitude is 1, that we already know.

Now, we call it R1, R2, R3, these row vectors. So, why are we now calling the row vector? So, what about this one? So, if I take the dot product of R1 .R 2, so, if you see from here it is 0 +0 + 0=0. So, that is 0. I call it R1.R3 . So, R1. R3 = 0-12/25 +12/25 =0.

And R2 . R3 =0 + 0 + 0=0 that means, that in this case my columns are mutually orthogonal to each other as well as the rows are also mutually orthogonal to each other. So, such a type of matrix if you get then from here you will see that U is an orthogonal matrix. So, what is the orthogonal matrix?

So, an orthogonal matrix is that in which the columns are orthogonal to each other and rows are also orthogonal to each other and the magnitude of each column is or the rows is 1. So, that is the definition of an orthogonal matrix. So, this is the orthogonal matrix and now from here starting with the orthogonal matrix what I am going to do is that, the orthogonal matrix I am taking.

Now, if you see from here U transpose U. So, this is what we are going to find out. Now, U transpose will be this one. So, it will be

$$U^T.U = \begin{bmatrix} 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \end{bmatrix} . \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Now, the same thing, I can verify what will happen if I take U U^T. So, in this case if you see then this is my same matrix.

$$U\,U^T = \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

So, from these two you can see that which implies now that $U\,U^T = I = U^T.U$. So, this is possible. So, from here I can say that $U^{-1} = U^T$. So, that is another advantage of dealing with the orthonormal basis that whenever we have a matrix made up of orthonormal basis then in that case if you want to take the inverse that is just the transformation transpose of the given matrix.

So, it is also very easy to find out the transpose of a matrix as compared to finding the inverse of the matrix. And from here also I just want to find the determinant of this.

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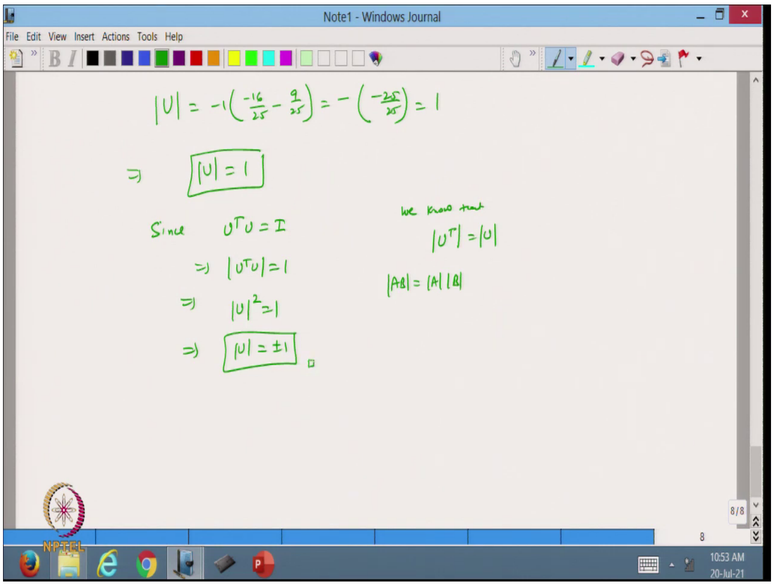

So, if you see the determinant I want to find. So, I can just say it will be -1. I just take this one going from the first row to the first column. So, I am taking this one. So, this and this; so, it will be |U| = -1(-16/25 -16/25) =1.

So, from here you see that this is equal to 1 and these things we can check from here also. Since $U^TU = I$ and we know that matrix U transpose taking the determinant is same as the determinant of the matrix. So, from here if I take the determinant of I is 1 always.

Now, if I take two matrix and determinant that is equal to determinant of A then determinant B. So, that should be equal to $|U|^2=1 \Rightarrow |U| = \pm 1$. So, it means that the orthogonal matrix always has the determinant either plus 1 or equal to - 1. So, this is the some properties of the orthogonal matrix that is made up of the orthonormal basis. So, that is why this matrix is called the orthogonal matrix.

So, I will stop here. So, today we have discussed the orthonormal vectors and orthonormal basis and then we talk about the orthogonal matrix. So, we will continue with this one in the next lecture also. So, thanks for watching.

Thanks very much.