Matrix Computation and its applications
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Lecture - 39
Similar matrices and diagonalization of matrix

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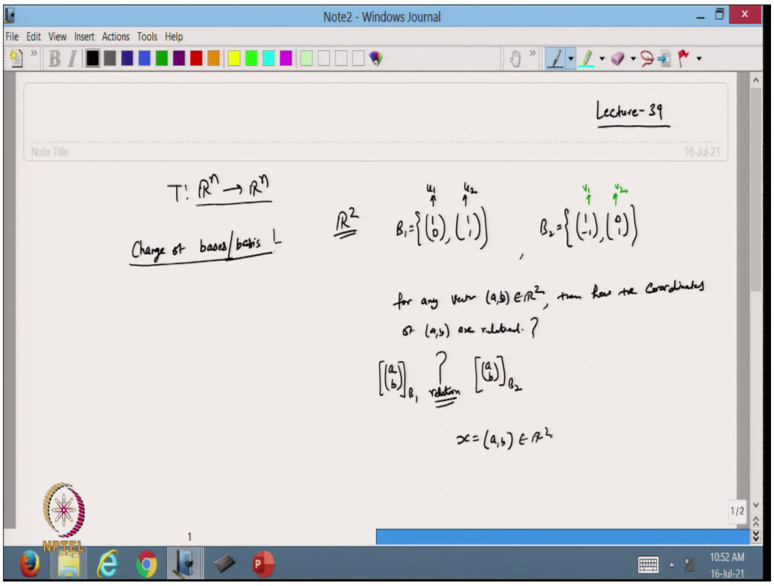

Hello, viewers. Welcome back to the course on Matrix Computation and its Application. So, today we are going to introduce the concept of similar matrices and how we can define two matrices that are similar. So, let us do that.

So, today we are going to discuss that suppose I have a transformation $T : R^n \rightarrow R^n$. So, this is what I am talking about in the same space. So, first I want to see how we can write the coordinates of a vector corresponding to a different basis.

For example: so, let us write down; first I will change the bases. It can be bases or maybe. Suppose I take R^2 in the R2. Suppose I have basis B1 that is given to me. So, I just take the very simple basis. So, let us take it as (1, 0) and (1 ,1). So, this is the basis I am taking. So, let us call it u_1 and call it u_2. Now, I take another basis B2, suppose I take (1,-1) and (0, 1) and I call this basis v1 and v_2.

Now, we want to see that for any vector for any vector may be (a, b) ∈ R^2. So, for any vector I just choose any vector (a, b) from R^2, then how the coordinates of (a, b) are related? So, this one we want to discuss because I can write the coordinates of so, let us say this vector (a, b).

So, I just take this vector as a column vector and I will write its coordinate with respect to B1. I will write the coordinates of the same vector with respect to the basis B2 and I want to find what is the relation between these two. So, this is the relationship I want to find. So, let us do this one.

Now, let us. So, from here. So, let I call this vector x. So, this vector is (a, b) ∈ R^2.

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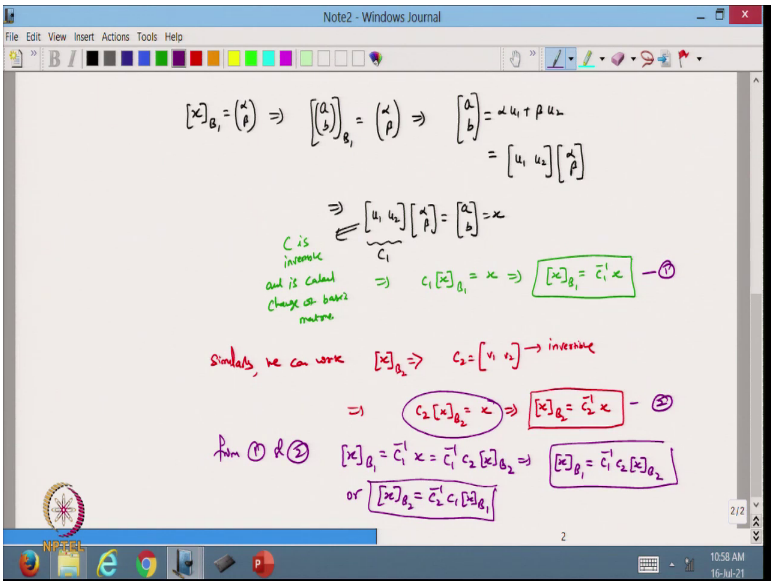

Now, from here I know the vector

$$[x]_{B1} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right]_{B1} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \alpha u_1 + \beta u_2 = [u_1, u_2]\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\Rightarrow [u_1, u_2]\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = x$$

So, I just call this matrix may be C_1. So, C_1 is the matrix made up of the basis and from here one thing I know that this matrix C is invertible and is called change of basis matrix. So, this is called the change of basis matrix C.

So, from here I can write these are the C1 and this is alpha beta. So, this is

$C_1 [x]_{B1} = x \Rightarrow [x]_{B1} = C_1^{-1}x$ ……(1)

So, this is very easy: we can write the basis coordinate of the x with respect to the basis that is equal to the inverse of the matrix made up of basis into the element we are talking about.

Similarly, we can write x of. So, here I have taken the two bases B1 and B2. So, now, I want to write with respect to basis B2 of the same element. So, this one I can write with the matrix. So, from here I can write that similarly I will make a matrix

$C_2 = [v_1 , v_2]$ as a column vector. So, I call it v_1 and v_2 this is the column I have taken.

So, this is also invertible and from here we can write that

$C_2[x]_{B2} = x$

$[x]_{B2} = C_2^{-1}x$ …..(2)

So, we are able to write like this one. Now, from here now from here I can write so, I will so, I can call it equation number 1 and equation number 2. So, from 1 and 2, what can I write? I can write as

$[x]_{B1} = C_1^{-1}x = C_1^{-1}[x]_{B2}\ C_2$

$[x]_{B2} = C_2^{-1}[x]_{B1}\ C_1$

So, this is the relation between the coordinates of the same vector with respect to a different basis.

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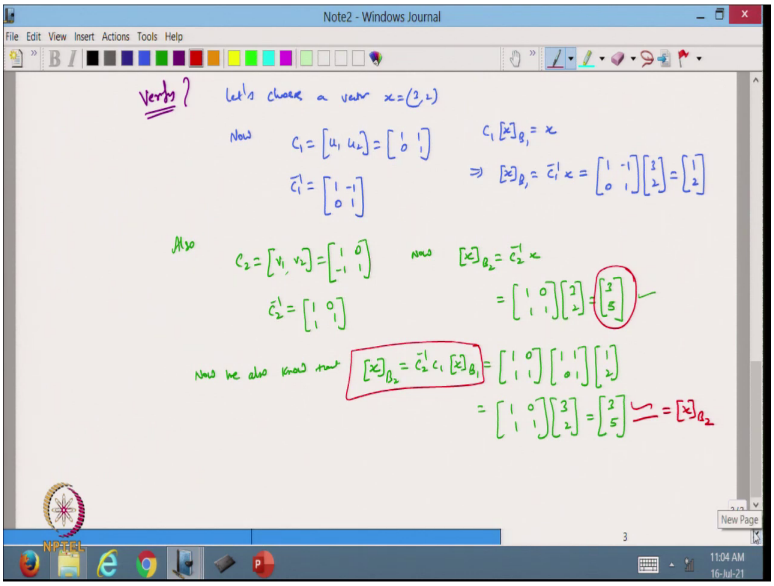

So, let us verify this one. Now, let us verify. So, I take an element. So, let us choose an element and choose a vector x. So, I just choose the vector x = (3 ,2). So, let us choose this one and I have the basis for this one. Now, first I have to make the matrix C_1. So,

$$C_1 = [u_1, u_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, C_1^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Now, I want to write so, from here what I want to do? From here I know that

$$C_1 [x]_{B1} = x$$

$$\Rightarrow [x]_{B1} = C_1^{-1} x = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So, now, this is we are able to find also

$$C_2 = [v_1, v_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$[x]_{B2} = C_2^{-1} x = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Now, we also know that

$$[x]_{B2} = C_2^{-1}[x]_{B1}\ C_1 = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}3\\ 2\end{bmatrix} = \begin{bmatrix}3\\ 5\end{bmatrix}$$

So, this way we are able to find the relation between the coordinates of the same vector with respect to different basis because once we are able to find the basis of one vector coordinate of one vector with respect to one basis then the coordinate of the same vector with another basis we can apply this formula and directly we can write that one. So, this is the relation between the change of basis matrix or the coordinate of the vector with respect to different bases.

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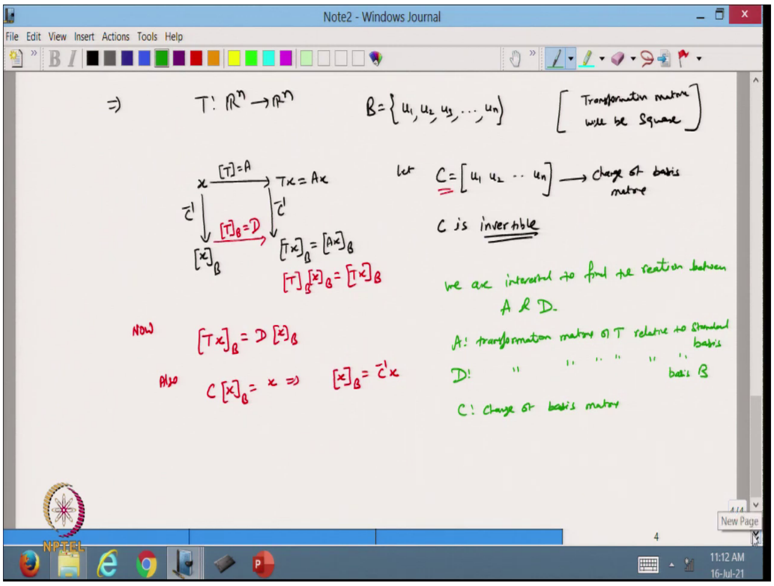

So, after this one, we are going to introduce a very important concept. Now suppose I have a transformation T : R^n to R^n and then we have the basis B1. So, this is the basis : suppose B1 $=\{u_1 , u_2 , \ldots,u_n\}$ So, now, we have the transformation from R^n to R^n then from here one thing is clear that the transformation matrix will be square. So now, we are talking about square matrices.

Now, from here I know that if I take any x apply my T the transformation. So, what I do is that, if I write the transformation and if I put the square around this it means I am talking about the matrix corresponding matrix A. Now, from here I will get Tx = A x. So, this is corresponding to the standard basis.

Now, let we take a matrix $C = [u_1 , u_2 , \ldots,u_n]$ So, I put the first column as u_1, second column as u_2. So, this is the matrix and I know that C is invertible. We know that this is linearly independent. So, it is an invertible square matrix. Now, from here I apply my C inverse to this x and I get the coordinate of this x with respect to the C. So, C is a change of basis matrix.

So, I apply this one and I will get the coordinate of the same x with respect to the basis B the same thing I apply here C inverse and I get from here the coordinate of T x with respect to B and that can be written as again Ax with respect to B. Now, from here I know that I can go directly from here by writing.

So, now, I am applying this one. So, I can apply my transformation T, the matrix with respect to basis and this is the matrix we have previously represented by D. Let me check how we have written. So, here I am writing with the C matrix. So, let us check that one, ok. So, because there also I am writing by C. So, I should, but here I am writing with this matrix C. So, let us say write it D.

So, now, from here I can write that the

$[T]_B\ [x]_B = [Tx]_B$

Now, $[Tx]_B = D[x]_B$

So, this is what we are writing. So, this is the matrix D corresponding to the change of basis that we have done before and we have represented by C. So, let us say no problem because C I am writing here. So, let us write this D. So, this is the way the transformation is taking place.

Now, also just in the form here we have written like this one that coordinates of this one can be written in this form. So, I am using this concept here.

Now, $C[x]_B = x \Rightarrow [x]_B = C^{-1}x$

Now, I am using this concept here. So, now, what I am going to do is that. So, now, let us see that what is the relation between D and A. So, we want to find we are interested to find the relation between A and D, where my A is the A is the transformation matrix.

So, this is the transformation matrix of T related to standard basis and D is my transformation matrix of T related to basis B the new basis I am writing here and what is C? C is change of basis matrix. So, it is a matrix made up of the basis what is the basis it is given. So, I know now we want to find the relation between these two.

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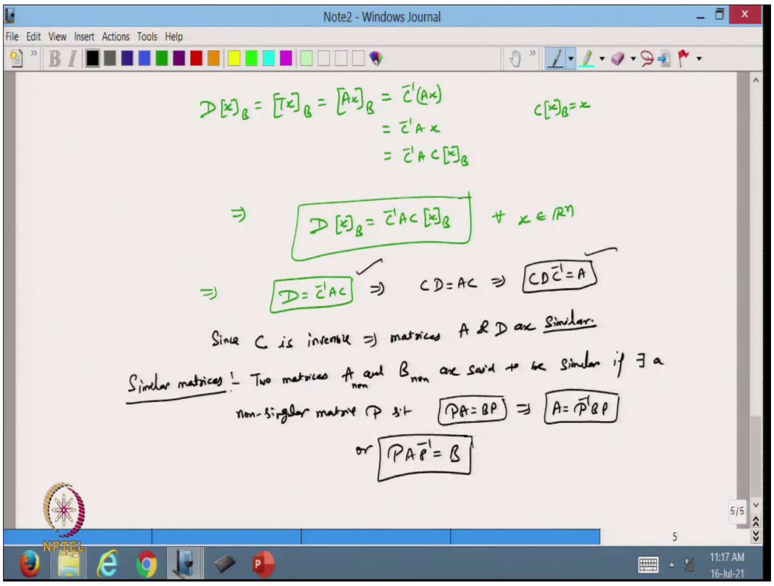

Now, so, from here I just write from here that I can write now

$D[x]_B = [Tx]_B = [Ax]_B = C^{-1}(Ax) = C^{-1}Ax = C^{-1}Ax[x]_B$

because I know that with respect to basis $C[x]_B = x$. So, from here I am able to write that

$D[x]_B = C^{-1}AC[x]_B$, for all $x \in R^n$

$D = C^{-1}AC \Rightarrow CD = AC \Rightarrow CDC^{-1} = A$

So, in both ways I can write like this one and now since C is invertible, then from here I will write down that which implies that the matrix matrices A and D are similar because we know that the definition of a similar matrix.

So, what is the definition? I just write the similar matrix similar matrices. What is the definition? Suppose I so, we are talking about that the square matrix the two matrix matrices A and B. So, this is of the same order two matrices A and B are set to be similar if there exist if there exist a there exist a non singular matrix P such that PA = BP or so, from here so, this is the definition of that PA = BP or PB = AB no problem.

So, from here I can write that because P is notable, I can write from here that A will be equal to B. So, maybe I can take the inverse. So, it will be $P^{-1}BP$ or from here I can write that PAP^{-1} = B. So, this is a definition of the similar matrices. So, if you see from here that now I can say that the matrix A and D are similar and the matrix A is the transformation matrix corresponding to the standard basis and D is the transformation matrix corresponding to the new basis. So, that is the relation between these two matrices.

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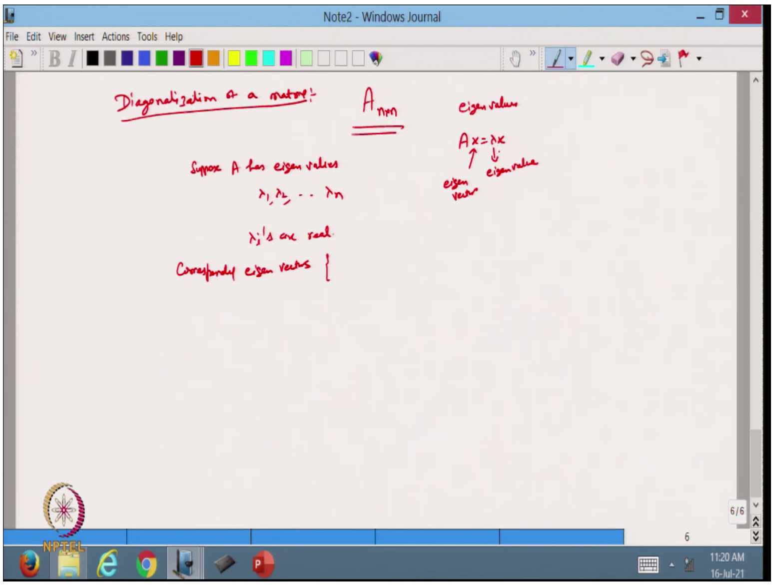

Now, I will introduce the concept we have already seen is that diagonalization of a matrix. So, with this diagonalization of the matrix we have seen from the matrix theory that suppose I have a matrix A that is n*n that is given to me I would talk about the real matrix. Now, from here A we go to the matrix. Now, from here I know the concept of eigenvalues.

So, this is already what we have introduced from the matrix theory from my previous knowledge that suppose I have a matrix A and then I have its eigenvalues. So, eigenvalue we generally write by $Ax = \lambda x$ and from there I will find out the eigenvalue and from here x this is called the eigenvector. So, that is there.

Then from here if I have a n*m matrix, then suppose A has eigenvalues $\lambda_1, \ldots\ldots, \lambda_n$. So, it can have a n number of eigenvalues and suppose I consider that these eigenvalues are real. So, that we are considering that suppose this is a real eigenvalue, we are not talking about the complex and then we take the corresponding eigenvectors.

So, the corresponding eigenvector I am writing as, so, this is a x. So, I can maybe instead of x I can represent by $Au = \lambda u$ because we are writing the vectors may be in the terms of this one.

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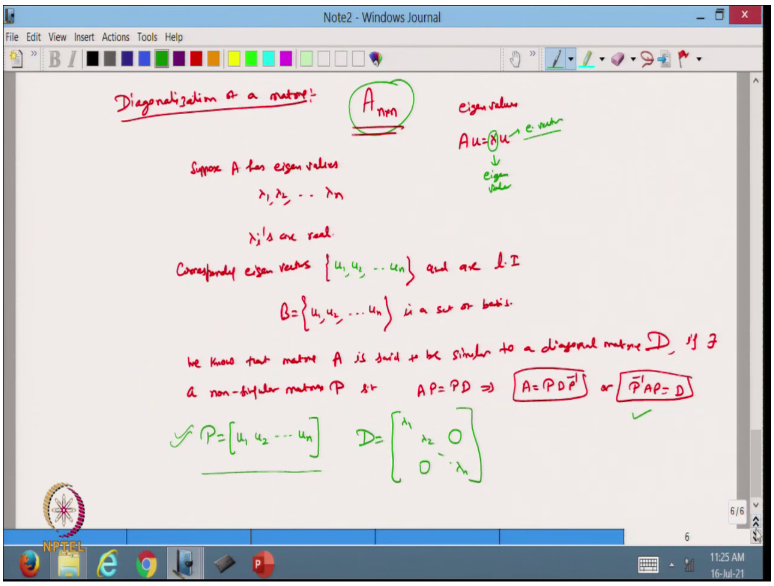

So, the same thing, eigenvalue and eigenvector. Now, I take the corresponding eigenvectors. So, suppose we are able to write the eigenvectors $\{u_1, u_2, \ldots, u_n\}$ and we are able to get this n number of eigenvectors. So, n number of eigenvectors we are able to get and corresponding eigenvectors are this one and are linearly independent.

So, I know that if these eigenvalues are distinct, then definitely these eigenvectors will be linearly independent, but since suppose that some eigenvalues repeat, then still we are able to find the vectors which are linearly independent. So, that is there. So, now, if we are able to get the so, I take the basis now B as this vector eigenvector n number of eigenvectors I will get and I take this as a set of basis because this is a linear independent any number. So, it is the basis of R^n.

Now, from here I will write that now. So, we know that the matrix A is so, in this case is said to be similar to a diagonal matrix we call it D, then whether the matrix A is said to be similar to the diagonal matrix D if there exist a non singular matrix P such that now from here you can write down because by the definition we have written this one that PA= BP.

So, from here I will write that AP = PD and from here I know that this is equal to I can be written as $A= PDP^{-1}$ or $P^{-1}AP = D$ and this matrix P. So, now from here I should write that the matrix $P = [u_1, u_2,\ldots, u_n]$. So, this is my corresponding invertible matrix we are writing and D = $\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$ is the diagonal matrix and it is made up of the diagonal elements the eigenvalues at the as a diagonal elements.

So, what we have done here is that now we have started with any matrix and then corresponding to this matrix we have calculated the eigenvalues and the corresponding eigenvectors and from here we are able to get n number of linearly independent eigenvectors.

So, I have constructed using these eigenvectors a matrix P which is invertible and then from the concept of a similar matrix; what have we done here? We have shown that this matrix A is similar to the matrix T. So, how can I find the matrix P? So, the P matrix is made up of eigenvectors. Now, from this we are able to see that this P inverse AP is equal to D.

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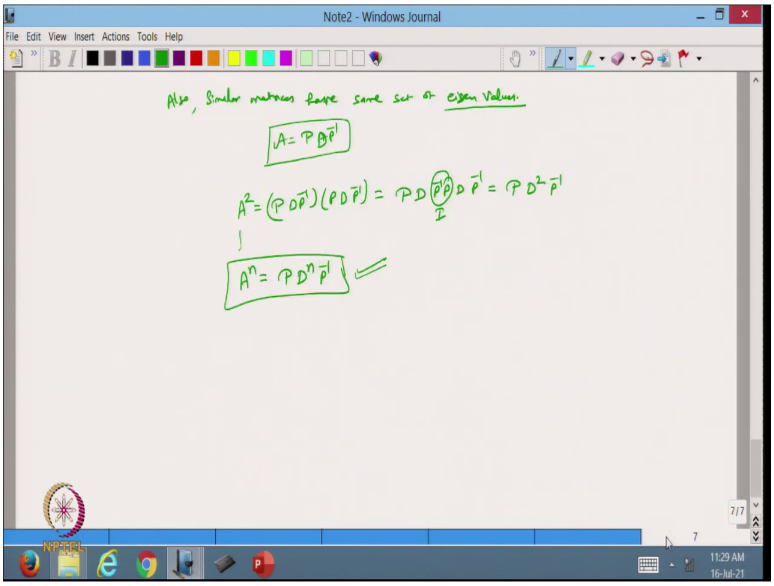

So, from here I can say that this matrix is similar and also similar matrices have similar matrices with the same set of eigenvalues. So, they have the same set of eigenvalues that is the property of the similar matrix because I am able to write AP = PD. So, A is equal to PDP^{-1} =A can be written as equal to P inverse sorry, D.

So, it is the earlier we have done this concept in my in the course of matrix theory, but there we have not seen that what is the meaning of this A and D in the terms of linear transformation, but now we are able to see that we are writing the same matrix A in the different basis and the basis are coming as the eigenvectors and then we are able to show that we are able we are finding the new matrix transformation matrix D that is made up of the eigenvectors.

So, this way we are able to convert this one and now, we can see that the matrix A suppose somebody wants to ask me what is the value of A 5, then maybe I can define it using this formula. So, I can write my P. So, suppose I want to find A square, so, it is $A^2 = (PDP^{-1})(PDP^{-1}) = PDP^{-1} P D P^{-1} = PD^2 P^{-1}$ So, I can write P and D the diagonal element D square. So, it will be D square P inverse.

So, the same way I can write that my $A_n = PD^n P^{-1}$ So, by doing this transformation we can easily handle such type of things very easily because I want to find n just I will write the D^n

and that is the diagonal elements diagonal matrix. So, it will be just the raise n to all these eigenvalues and that will be the solution of this form.

So, we are generally doing this transformation to make life simpler to deal with the corresponding transformation matrix of the linear transformation. So, this is the use of the change of basis for the given linear transformation. So, this is the way we are defining this one.

So, let me stop here today. So, in this lecture we have discussed a very important concept that is the similar matrices that how the change of basis of the corresponding transformation are used to introduce the two different type of matrices that is we call it the similar matrices and this concept is useful for using for satisfying the other properties of the given matrix or to deal with the properties of the other matrices. So, I hope that you have enjoyed this lecture. Thanks for watching.

Thanks very much.