Matrix Computation and its applications
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Lecture - 36
Matrix representation of a linear transformation relative to ordered bases

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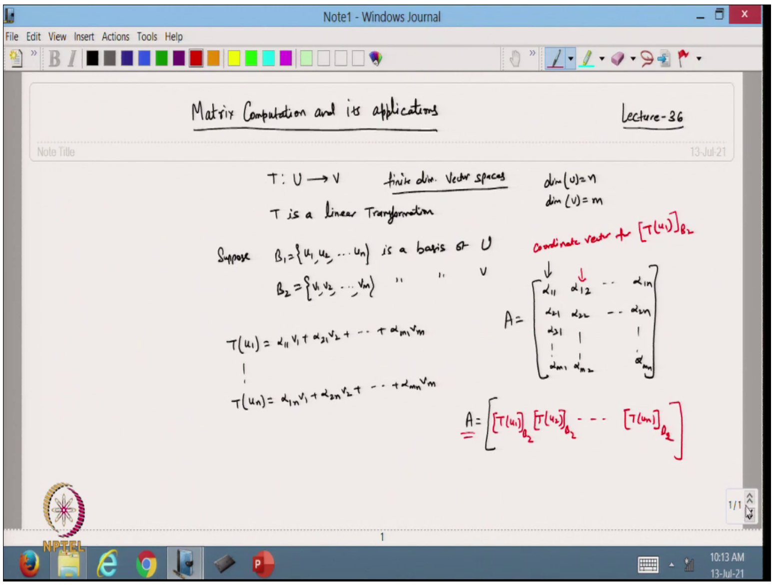

Hello viewers, welcome back to the course on Matrix Computation and its application. So in the previous lecture we have just started how we can write a matrix corresponding to the linear transformation, now we will continue with that one.

So, in the previous lecture we have discussed that, suppose we have a linear transformation $T : U \rightarrow V$. Here we are talking about finite dimensional vector spaces. Then, based on this linear transformation where T is a linear transformation then suppose I have some basis. So, suppose

$B1 = \{u_1, u_2, \ldots, u_n\}$ is a basis of U,

$B2 = \{v_1, v_2, \ldots, v_m\}$ is a basis of V.

I am taking that the dim(U) = n and the dim(V) = m. Now, based on this basis I know that my linear transformation T can be obtained by taking the image at u_1 and this can be written as

$$T(u_1) = \alpha_{11} v_1 + \alpha_{21} v_2 + \alpha_{31} v_3 + \ldots + \alpha_{m1} v_m$$

$$\vdots$$

$$T(u_n) = \alpha_{1n} v_1 + \alpha_{2n} v_2 + \ldots + \alpha_{mn} v_m$$

Now, if you see from here then I can write my matrix A as

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$$

$$A = [[T(u_1)]_{B2}, [T(u_2)]_{B2}, \ldots, [T(u_{n1})]_{B2}]$$

So this way we can write. Now, from here we will see how we can write the matrix A corresponding to the linear transformation. So, before that one we want to discuss a bit more about coordinate systems.

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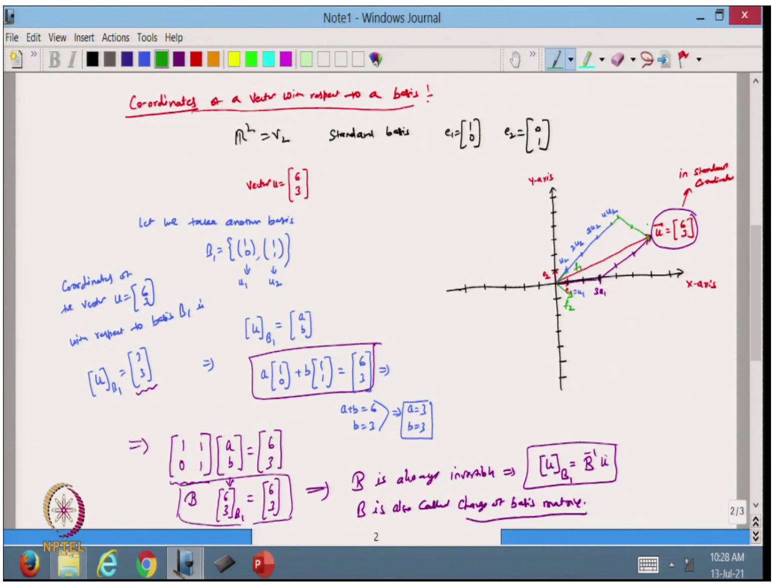

So, next what do you mean by coordinates of a vector with respect to a basis. So, this one we want to discuss; what is the meaning of this?

Now suppose, I just take 2 dimensional it is I just take $R^2 = V_2$ and I know that it is the standard basis this is $e_1 = (1\ 0)$; $e_2 = (0\ 1)$. Now, if I take the coordinate system because this is my R^2 and suppose it is 1, 2, 3, 4 ,5 ,6 ,7 ; -1, -2 ,-3 ,-4, - 5 like this one we can go and 1, 2, 3, 4, 5, 6 like this one.

Now, if you see from here I take the base e_1. So, e_1 is this base which is moving in the direction of X axis, because this is my X axis and this is my Y axis and e_2 is a unit vector in this direction. So, this is my e_1 and this is my e_2. Now suppose, I have a vector, so I choose a vector, I take a vector, maybe I will take. I just take maybe 6 and 3, so I take this vector.

Now, from here this vector I can take now I can represent this vector here. So, it is 1 2 3 4 5 6 and 3, so it is 1 2 3. So this is my vector. I can draw from here and this is my vector, I can call it u and it is given by 6 ,3; where we are going 6 in e_1 direction and 3 in the e_2 direction, so 1 2 3 4 5 6 and 1 2 3. So this is my vector, I am able to represent it in my standard basis.

So these are called the standard basis or the physical basis. Because, if I ask you the vector (6,3) then you will definitely know that it is going 6 times in the X direction and 3 times in the Y direction then we are able to find this vector 6, 3. So, this is a vector we are writing in standard coordinates.

Now after doing this one, now suppose I change the basis. So, let us take another basis. So, this I just take another basis B may be B1. So, I take it like (1, 0) the same basis I am taking, but here I am taking (1 ,1). So, I am taking the basis and these bases are linearly independent. They span the whole space. So it is the basis of R^2. Now, if you see from here, my B1 is again the same as e_1, so this is my I can call it basis u_1 and this is u_2.

So, here you can see that e_1 is equal to u_1 no problem, but if you take the u_2, so u_2 is going one direction in the X direction and one in the Y direction. So this is my u_2. So now, it is my u_2. Now suppose, I take the same vector (6, 3) so I want to write so I want to write the u that is my (6, 3) with respect to the basis B1.

So, this one how can we write this one? So, this one I want to write it as

B1 = {(1,0),(1,1)} and $[u]_{B1} = \begin{bmatrix} a \\ b \end{bmatrix}$

- ⇨ $a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$
- ⇨ a+b = 6 ; b=3
- ⇨ a=3 , b=3

Now, from here I can write that the coordinates so I can write from here the coordinates of the vector u = $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ with respect to basis B1 . So, from here I can write that

$[u]_{B1} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

- ⇨ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

So, I have the same vector u. Now, you can see from here what I am going to do. So, this is my u_2 then I will go a little bit here it is my $2u_2$, then I go it is my $3u_2$, then I go it is 4 u_2 like this one.

Now, what I am going I am taking a = 3 and b = 3, it means now I am going maybe I will use this one. So, I am going in one direction (2, 3). So 3 I have taken a =3. So, I am going now 3 times in my u_1 direction and then 3 times in my u_2 direction. So, u_2 direction is this one, so I will go like this one.

So this is 1 u_2, 2 u_2 and 3 u_2. So, I am going till here 3 times u 1 and then taking the direction of u_2 and this is also 3 times in u_2 direction and I will reach the same point. So, from here you can see from this diagram that the coordinate vector of this point is the same, whatever the point we have taken (6,3) that was a point in the physical domain and it was related to the basis that is the standard basis.

Now, I am changing the representation of this vector with respect to the basis. So, what I am going to do is that I will write this vector as a linear combination of the given basis and based on that when I will find out the coordinates and that will be the coordinates of this vector related to the new basis B1. So, that is why we also called these as coordinates, because it is the same as the coordinate or the vector u, but we have changed the basis.

So now from here, you can see that this expression I can write like this one, so I take the matrix made up of vectors this first basis (1 1), this is the second basis and then I write the coordinates of the vector with respect to this matrix and this is right hand side vector which we wanted to write in terms of the basis. So, this is the matrix you can call it the basis matrix B and this is the coordinates of the vector (6 3) that we are going to write with respect to the basis B2 and this is my vector whose coordinates are given to us in the standard form. So, this way we are able to write this one.

Now from here, now you know that this matrix B is always invertible, because it is made up of these basis vectors. So, I can write from here, so from here I can write my u vector, coordinates of this u vector with respect to the basis B1 not B2, it is B1. So, with respect to B1, so that will be equal to I can write directly by taking the B inverse and then the vector u. So, this way we can directly write the coordinates of 1 vector in terms of the basis B1 when the coordinate of that vector is given to us in another basis form.

So, this is the way we are able to write and B is also called change of basis matrix and it is always made up of basis. So, that is why it is invertible here and we are able to write this. So, from here we can see that we are taking the same vector, but changing in this one.

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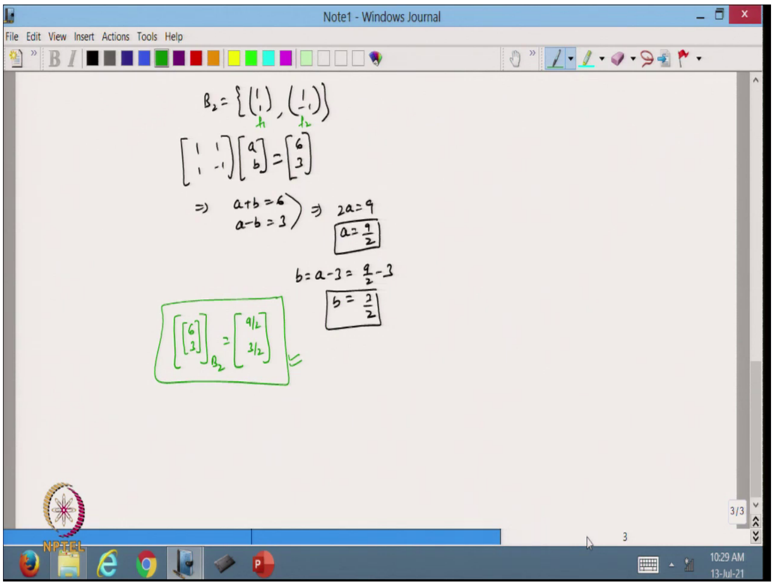

Now, maybe I can change another vector. Somebody can take the another basis

B2 = {(1,1),(1,-1)}. So these are the basics I can take. Now, from here based on this one, I will write my matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

- ➪ a+b = 6 ; a-b = 3
- ➪ a=9/2 ; b= 3/2

So (1 ,1) is again this one, so I can take this as my next base. So maybe I will represent this one as f1 and this is my base f2. So, I am going this way. So, it is my f1 and f2 is 1 minus 1. So I am going 1 minus 1. So, that will be this one. So it is my f2. Now, if I want to reach this place. So, what we are going to get here. So now from here, I can write the coordinate of the vector (6, 3). So, this is my vector with respect to the basis is a= 9/2 and b = 3/2.

So I can go 4.5 times. So, I will go here, you just check I will go 4.5. So, 1 2 3 4 and maybe this 4.5 and then I will go in this direction half, 1 and half time. So, I will go 1 time this way and then another half is this 1 like this one, so will reach here on this point. So, this is just what we are writing here. So, now from here you can see that the same point (6, 3) now can be represented with the new coordinates that are 9/2 and 3/2, it is yeah. So, it is 9/2 and 3/2 of the same coordinate with respect to the basis B2.

So this way we can have an infinite number of basis, we know that and then we can represent the same point with the different basis and having, so we can have the different coordinates of the same point. So, this way we are able to have the relation between the coordinate of a vector with respect to a different basis. So, I will use this one thing when we write the linear transformation in some other basis. So, let us do one example.

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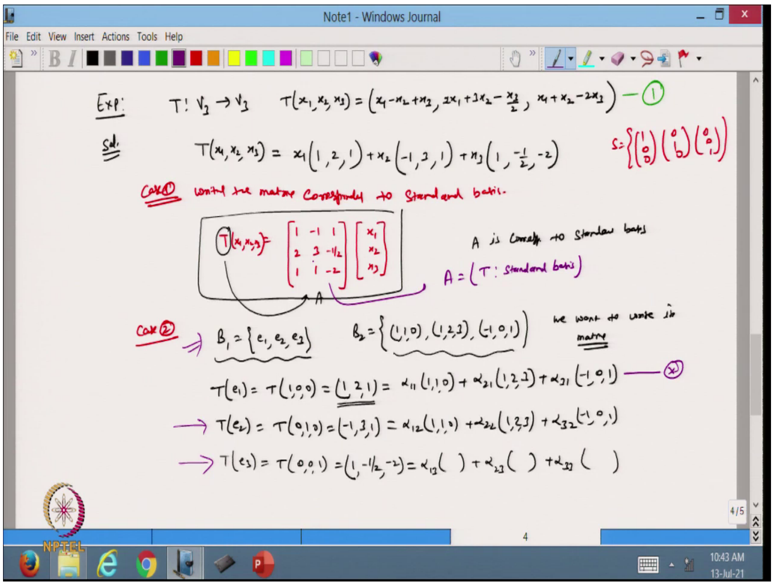

So, I take one example; I have a linear transformation $T : V_3 \rightarrow V_3$ is defined as

$$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 + 3x_2 - x_3/2, x_1 + x_2 - 2x_3)$$

So, this linear transformation is given to us. Now, I want to write the corresponding matrix.

So, we want to write this. So now what will I do? I will write this

$$T(x_1, x_2, x_3) = x_1 (1,2,1) + x_2 (-1,3,1) + x_3(1,-1/2,-2)$$

Now, from this one I can write directly from here. Now we are writing, so from this one first,

case 1: writing the matrix corresponding to standard basis. So, standard basis I know

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$T(x_1, x_2, x_3) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -1/2 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This matrix A is corresponding to the standard basis. So, this way we can write the matrix. we are able to write this matrix with respect to the standard basis. Now if you do the calculation you will see that you will get the same terminology here.

Now, I will take case 2. Now, we change the basis. So let us take another basis. So, I will choose the basis B1, we are writing it as the standard basis $\{e_1, e_2, e_3\}$and I take the basis B2 =$\{(1\ 1\ 0), (1\ 2\ 3), (-1\ 0\ 1)\}$. So, this is the basis I am taking. Now, you have changed the basis. So, the B1 basis I am taking for V_3 because this linear transformation is from V_3 to V_3. So, basically I am taking this B1 for V_3 and this B2 is the image set that is also V3. So, we are taking this basis.

Now we want to write, so the question is that we want to write its transformation or its matrix corresponding with the linear transformation T. So, now, we know that the transformation is uniquely determined by the value or its basis. So what we are going to do now is, I will take the image of e_1 that is T(1 0 0).

$T(e_1) = T(1\ 0\ 0) = (1,2,1) = \alpha_{11}(1,1,0) + \alpha_{21}(1,2,3) + \alpha_{31}(-1,0,1)$

$T(e_2) = T(0\ 1\ 0) = (-1,3,1) = \alpha_{12}(1,1,0) + \alpha_{22}(1,2,3) + \alpha_{32}(-1,0,1)$

$T(e_3) = T(0\ 0\ 1) = (1,-1/2,-2) = \alpha_{13}(1,1,0) + \alpha_{23}(1,2,3) + \alpha_{33}(-1,0,1)$

So now I want my matrix. I call this matrix as we represent by A which is a matrix with respect to the standard basis, so I just write this matrix as a C matrix.

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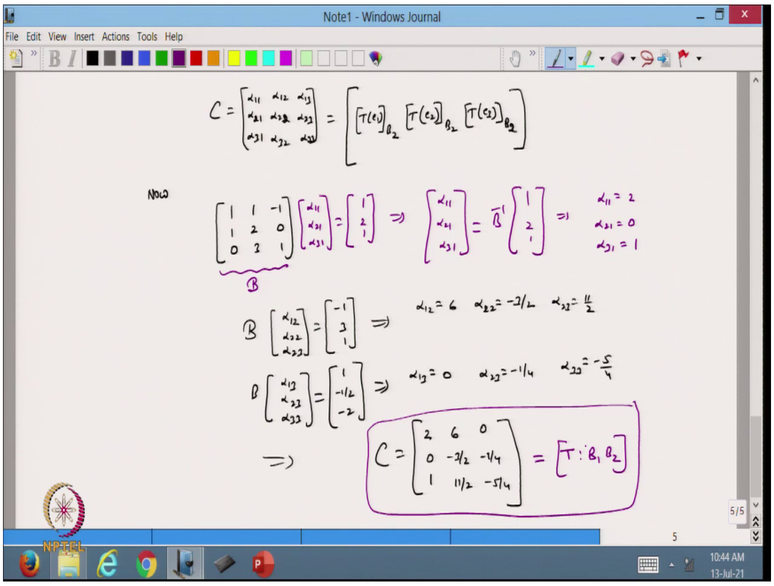

So C = $\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} = \left[[T(e_1)]_{B2} \quad [T(e_2)]_{B2} \quad [T(e_3)]_{B2} \right]$

Now

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{31} \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

⇨ $\alpha_{11} = 2,\ \alpha_{21} = 0,\ \alpha_{31} = 1$

So, this is the first vector. Now, we can solve the other system, this one by the same way only thing is that we have to change this right hand side vector. So, the next one will be

B $\begin{bmatrix} \alpha_{12} \\ \alpha_{22} \\ \alpha_{23} \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \Rightarrow \alpha_{12} = 6, \alpha_{22} = -3/2, \alpha_{23} = 11/2$

$$B\begin{bmatrix}\alpha_{13}\\ \alpha_{23}\\ \alpha_{33}\end{bmatrix}=\begin{bmatrix}1\\ -1/2\\ -2\end{bmatrix}\Rightarrow \alpha_{13}=0,\alpha_{23}=-1/4,\alpha_{33}=-5/4$$

$$\Rightarrow \quad C=\begin{bmatrix}2 & 6 & 0\\ 0 & -3/2 & -1/4\\ 1 & 11/2 & -5/4\end{bmatrix}=[T:B_1B_2]$$

So, you can write from here that this matrix is corresponding to the linear transformation T with respect to the basis B 1 and B 2.

The same way I can write this matrix whatever the matrix is there. So, this is the matrix with a standard basis and this is the matrix with respect to the new basis that we have taken. So, this is the linear matrix corresponding to the linear transformation with respect to the new basis.

So, here you can see that the matrix here is completely different from the matrix here, because this is the coordinate system of the T [u_1] with respect to the new basis. So, that is the relation between these two matrices. So, we will discuss in the future how one matrix is there then how we can find the another matrix. So, we will stop here today.

So in today's lecture we discussed how the coordinates of one vector can be written with respect to the basis and then we have shown how we can write a linear transformation, the matrix for a linear transformation corresponding to a new basis. So, in the next lecture we will continue with that one. So, thanks for watching.

Thanks very much.