Matrix Computation and its applications
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Lecture - 35
Matrix associated with linear transformation

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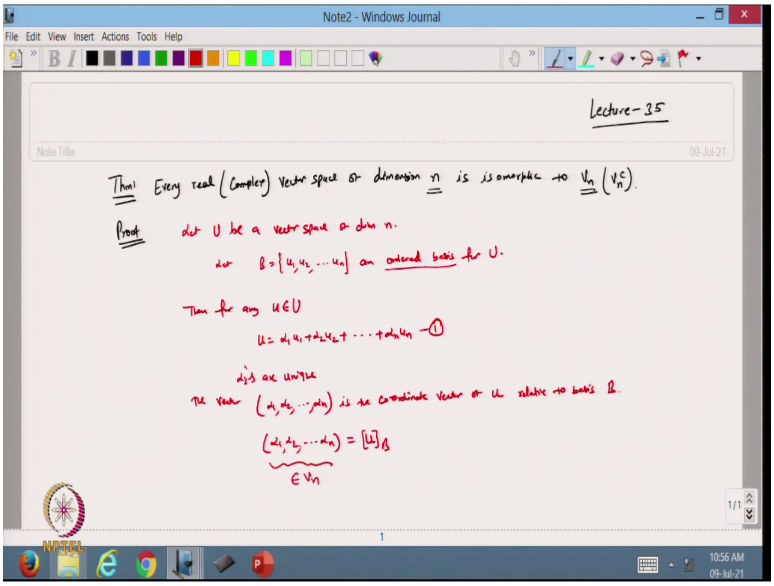

Hello viewers. So, welcome back to the course on Matrix Computation and its Application. So, in the previous lecture, we have started with the one of the definitions that is isomorphism and the inverse of a given linear transformation. So, we will continue with that one.

Now, we are going to define a very important theorem regarding isomorphism. So, it says that every real or may be complex vector space of dimension n is isomorphic. So, it is isomorphic to V_n or V^c_n if it is a complex number, but we are talking about the real, so that is the. And this V_n we know is the nth dimensional vector space. And so from here I can say that if I choose any vector space having the dimension n, then it is isomorphic to V n. So, this one we want to prove.

Now for this one; so let I choose that let U be a vector space of dimension n. Now, suppose this is the dimension n, then let we also take the set $B = \{u_1, u_2, \dots, u_n\}$ an ordered set ordered basis for U. So, we are taking the ordered basis of U and this is of dimension n. Then for any $u \in U$ that I take the vector space u, I just take any element u, we can write

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n, \ \ldots\ldots\ldots\ldots(1)$$

This one I can write because this is the basic ordered basis. So, I can define my element u as this linear combination. So, I can write that α'i's are unique because this is the basis, so it is uniquely determined. Now, from here we also know the set the vector $\{\alpha_1, \alpha_2, ., \alpha_n\}$ is the coordinate vector of u relative to basis B. And we know that this we can represent as $(\alpha_1, \alpha_2, ., \alpha_n) = [U]_B$.

Now, from here you know that this vector belongs to Vn, because it is just the elements coming from the field. So, it is a real number. So, it belongs to Vn.

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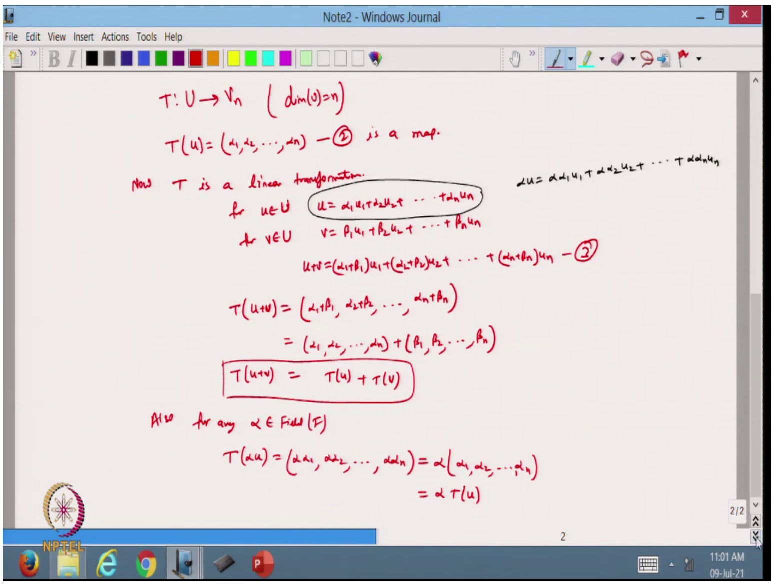

So, based on this one, we define a linear transformation $T : U \rightarrow Vn$, where $\dim(U) = n$ as. So, let's define this one $T(u) = (\alpha_1, \alpha_2, ., \alpha_n)$ …….(2) So, I define it like this one. And these are the coordinate vectors. So, these are the coordinates $\alpha_1, \alpha_2, ., \alpha_n$ are the coordinates of the vector U with respect to the basis B, so that we already know. Now, we want to show that it is a linear transformation. So, we know that

for $u \in U$, $u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$

for v ∈ U, $v = \beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_n u_n$

Now, from here I know that

$u+v = (\alpha_1 + \beta_1)u_1 + (\alpha_2 + \beta_2)u_2 + \ldots + (\alpha_n + \beta_n)u_n$ …….(3)

So, this linear combination I can write. Now, from here I want to find what will happen T(u + v). Now, T(u + v) I know that from here these are the coordinates of (u + v) related to the basis u1, u2 ,… un. So, from here I can write that

$T(u+v) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n)$

$= (\alpha_1, \alpha_2, \ldots, \alpha_n) + (\beta_1, \beta_2, \ldots \beta_n)$

$T(u+v) = T(u) + T(v)$

Then also we can define for any α ∈ field (F), thus it is a scalar I can show very easily the T(αu). So, T(αu) will be what?

$T(\alpha u) = (\alpha\alpha_1, \alpha\alpha_2, \ldots, \alpha\alpha_n) = \alpha(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha T(u)$

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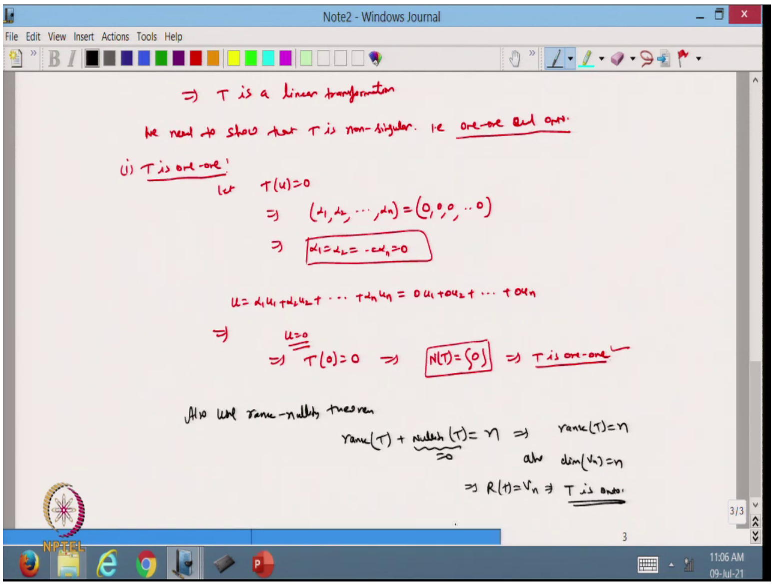

So, from these two properties, we can say that my T is a linear transformation. Now, we need to show that T is 1-1. Now, we need to show that T is nonsingular that is 1-1 and onto. Now, from here I want to check. So, let us do this one T is 1-1. So, this one we need to show. So, for this one, what I am going to do is, so I am going to show here for this one that the null space of T will contain only the 0 element.

Now, from here the T is 1-1, because I will take the element

$T(u)=0$

- $\Rightarrow$ $(\alpha_1, \alpha_2, \ldots, \alpha_n) = (0,0,\ldots,0)$
- $\Rightarrow$ $\alpha_1=0, \alpha_2=0, \ldots, \alpha_n=0$

$u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0u_1 + 0u_2 + \ldots + 0u_n$

- $\Rightarrow$ $u=0$
- $\Rightarrow$ $T(0)=0$
- $\Rightarrow$ $N(T)=\{0\}$
- $\Rightarrow$ T is one one

Also using a rank nullity theorem; so rank nullity theorem

$\text{Rank}(T) + \text{nullity } T = n \Rightarrow \text{rank}(T) = n$

Also $\dim(V_n) = n$

- $\Rightarrow$ $R(T) = V_n$

And from here I can say that the range space of T will be complete Vn, so it means the dimension. So, from here I can say that my T is onto. So, T is onto means the whole range space is equal to the complete vector space v n by the rank nullity theorem we can show this one because both have the same dimension n.

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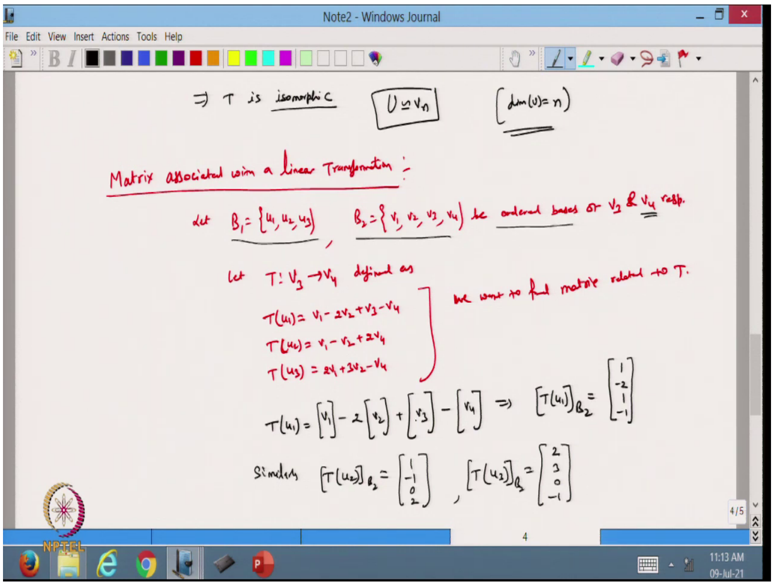

So, if it is 1-1 onto then from here we can say that T is isomorphic. And from here I can say that U is isomorphic to Vn if so that the dimension of u is n. So, this is a very powerful theorem because it shows that if I choose any vector space of dimension n, then I can always define a map which is isomorphic to Vn. So, this is the way we can use this theorem.

Now, after doing this theorem, now we want to discuss a very important topic. So, this is what we want to discuss is a matrix associated with a linear transformation. So, in the earlier case also we have discussed the matrices A, and we are sure that that works the same as a linear transformation. Now, here we want to give some rigorous idea of how a linear transformation is associated with a matrix. So, this is what we want to show.

Now let, so how can we check this one? So, let I have a basis

B1= $\{u_1, u_2, u_3\}$, B2 =$\{v_1, v_2, v_3, v_4\}$ be ordered on the basis of V_3 and V_4 respectively. Then let there is a linear transformation $T : V_3 \rightarrow V_4$ defined as

$T(u_1) = v_1 - 2\, v_2 + v_3 - v_4$

$T(u_2) = v_1 - v_2 + 2v_4$

$T(u_3) = 2v_1 + 3\, v_2 - v_4$

Now, we want to find matrix related to linear transition T, So, if you see it is a linear combination of the basis v1, v2, v3, v4; now from here I can write:

$T(u_1) = [v_1]- 2\,[v_2] + [v_3]- [v_4]$

$$\Rightarrow \quad [T(u_1)]_{B2}= \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \end{bmatrix}, \quad [T(u_2)]_{B2}= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \quad [T(u_3)]_{B2}= \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}$$

Now, these are all the coordinates we have defined corresponding to the different-different vectors.

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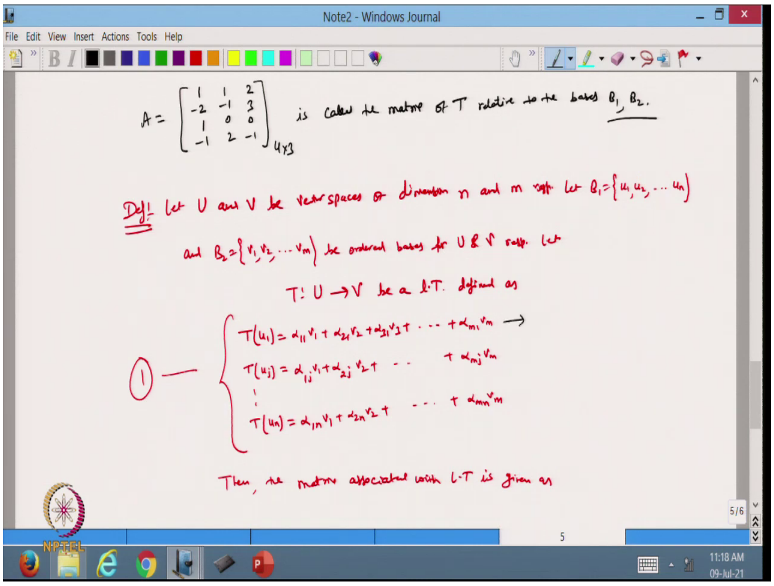

Now, what do I do? I write a matrix with these coordinates. So, I am writing the first coordinate as a first column.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & 3 \\ 1 & 0 & 0 \\ -1 & 2 & -1 \end{bmatrix}$$

So, this matrix is called the matrix of linear transformation T relative to the basis B_1, B_2, because here we are taking the basis B_1, B_2. And this is what we are taking on an ordered basis. Why are we taking an ordered basis? Because we are using the corresponding coordinate vector, that is why if we change the order the coordinate will change, so it will be a different vector in that case. So, this is the corresponding matrix we are able to define.

So, now from here we can write that, so I can define the definition now. So, what do we say in this definition let u and v be vector spaces of dimension n and m respectively. So, this is the vector spaces of dimension n and m that are respectively. So I take B1= $\{u_1, u_2, \ldots u_n\}$, B2 =$\{v_1, v_2, \ldots, v_m\}$ because it is a dimension m of ordered basis for u and v respectively.

Then let there be a linear transformation $T: U \rightarrow V$, it is given to me. So, it is given to us this is a linear transformation defined as :

$$T(u_1) = \alpha_{11} v_1 + \alpha_{21} v_2 + \alpha_{31} v_3 + \ldots + \alpha_{m1} v_m$$

$$T(u_j) = \alpha_{1j} v_1 + \alpha_{2j} v_2 + \ldots + \alpha_{mj} v_m$$

$$\vdots$$

$$T(u_n) = \alpha_{1n} v_1 + \alpha_{2n} v_2 + \ldots + \alpha_{mn} v_m$$

So, this linear transformation is defined to me. So, it is defined by (Refer Time: 25:27).

Then, the matrix associated with the linear transformation LT is given as. So, if you see from here, then this linear transformation we can define uniquely because these are the basis. And if it is the basis then we know that the corresponding system of equations will be the system will be non-singular, and then we can define the unique solution for that system. So, in this case, all these α_1, α_2, all these elements will be unique and it will be uniquely determined. And so this linear transformation will be unique.

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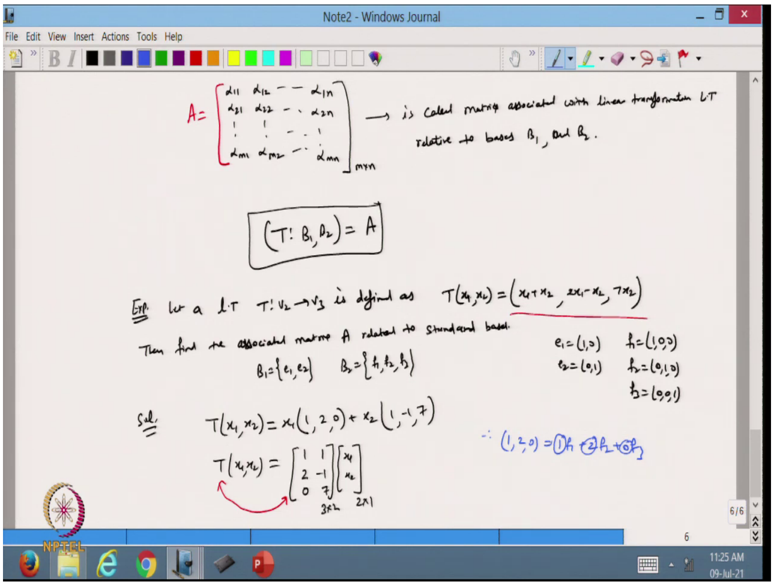

So then the corresponding matrix A, so I give the name A, so this can be given as. Now, what are we going to do?

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$$

So, this is the matrix A. So, the matrix associated with the linear term is given by this one, and is called the matrix associated with linear transformation L T related to basis B1 and B2. And we also sometime represent by T transformation corresponding to B1, B2 as

(T: B1,B2) = A

So, this basically matrix is made up of the coordinates of this one, coordinate of the vectors. So, this way we can define the matrix associated with the linear transformation.

So, let us take one example. Let a linear transformation $T : V_2 \rightarrow V_3$ is defined as

$$T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2, 7x_2)$$

So, this is a transformation given to me, it is from $V_2 \rightarrow V_3$. Then find the associated matrix A. So, we need to find the matrix A, but we need to find the basis.

So, find the associated matrix A related to standard basis. So, related to standard basis means I know that the standard basis B1 = $\{e_1, e_2\}$, and B2 =$\{f_1, f_2, f_3\}$. You know that this e_1= (1 0); e_2 =(0 1); f_1 = (1 0 0); f_2= (0 1 0); and f_3 = (0 0 1) So, these are the standard basis.

Now, from here, so how can we find this solution? So, in this case, I just define now because we deal with the standard basis. So, if you see from here, I can write here as

$T(x_1, x_2) = x_1 (1,2,0)+x_2(1,-1,7)$

$$T(x_1, x_2) = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix}$$

Now, based on this one, if you can see from here then this matrix will correspond to linear transmission T because in this case I am talking about the standard basis. So, standard basis means no change will be there in the terms of coordinates because we know that, because I can write my element (1 2 0) = 1. f_1 +2. f_2 +0. f_3 So, there is no change in the coordinates. So, I have that is why I am able to write the matrix directly from here, it is 1 to 0.

$$\Rightarrow \quad A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}$$

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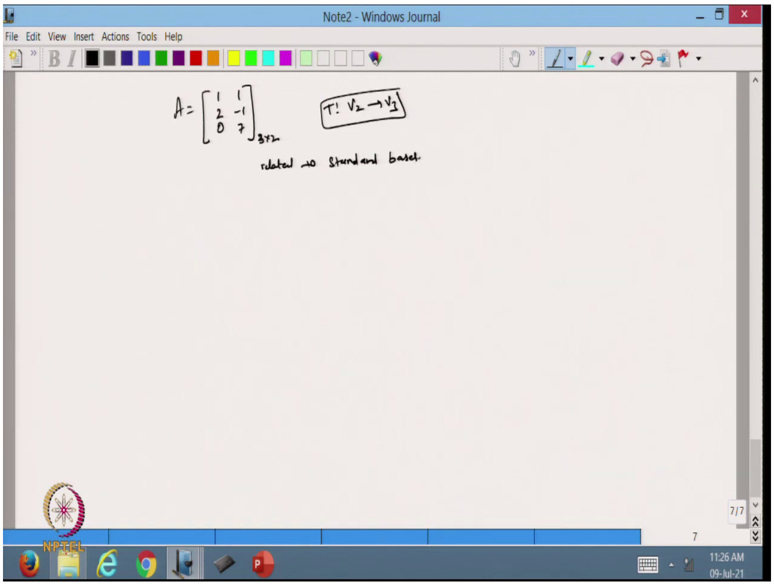

because our transformation is from V_2 to V_3. So, this is my corresponding matrix. So, this is the matrix related to the standard basis.

If we have a transformation the same transformation, but I change my basis then what will happen to the matrix. So, that matrix we can define, so this type of thing we will do in the next lecture. So, I will stop here.

So, in today's lecture, we have defined how for a given linear transformation how I can show a matrix corresponding to the linear transformation related to the given basis. And in the coming lecture, we will continue with this one. So, thanks for watching.

Thanks very much.