Matrix Computation and its applications

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Lecture - 34
Application of rank-nullity theorem and inverse of a linear transformation

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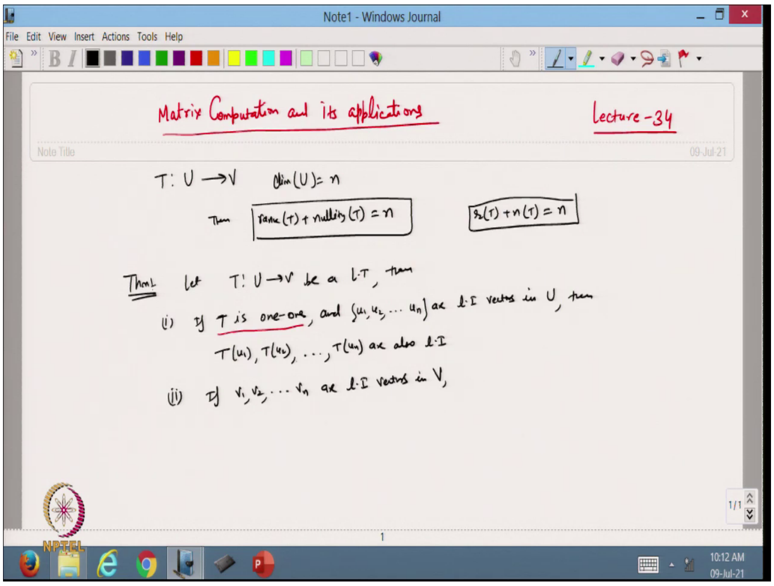

Hello viewers, welcome back to the course on Matrix computation and application. So, in the previous lecture we discussed the rank and nullity theorem. So, today we are going to discuss some problems based on that. So, in the previous lecture we have discussed that if I have a linear transformation $T: U \rightarrow V$ and the $\dim(U) = n$, then by the rank nullity theorem the $\text{rank}(T) + \text{nullity}(T) = n$.

So, this is the theorem that we have done. So, based on this one we will discuss some other results.

Theorem: let $T : U \rightarrow V$ be a linear transformation then

(1) if T is one-one and $\{u_1, u_2, \ldots, u_n\}$ are linearly independent vectors that belong to U, then $T(u_1), T(u_2), \ldots, T(u_n)$ are also LI.

So, this is the condition that if I am taking. So, this T is one- one and if I take a set of vectors that are linearly independent in the given vector space then their image is also linearly independent. So, this one is one of the result.

(2) If I take v_1, v_2 ,…v_n are linearly independent vectors in R(T) and u_1 , u_2 , …,u_n are vectors such that $T(u_1)=v_1$, $T(u_2)=v_2$, …, $T(u_n)=v_n$ then u_1 , u_2 , …,u_n are linearly independent.

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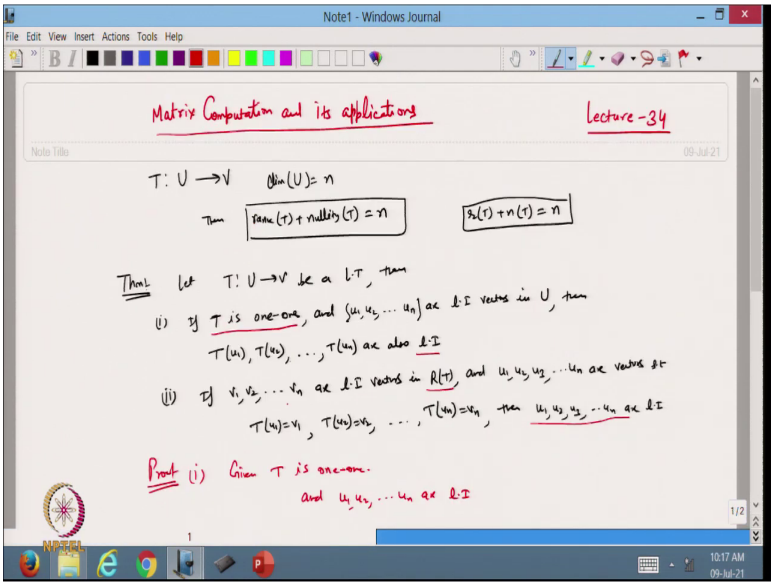

So, in this case it is just the opposite one. Here we have T is one-one and then if the set of vectors in u are LI then their image is also LI, but here it is opposite that if I have the vectors v_1, v_2 ,…v_n that are linearly independent vectors in the range space T.

And suppose I choose n number of vectors in u such that satisfying this condition then this set of vectors is also LI. So, this is the statement of the theorem. So, now, we want to prove this one. So, let us prove the 1st part. Now, given that T is one-one and u_1 , u_2 , …,u_n are linearly independent. So, now, I need to show that these are also independent.

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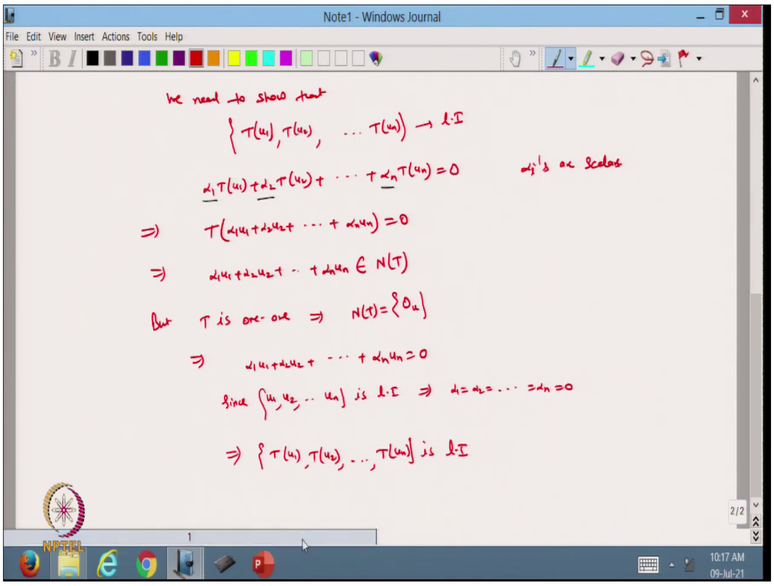

Now, we need to show that the set of vectors $\{T(u_1), T(u_2), \ldots, T(u_n)\}$ is LI. So, for this LI will take the linear combination. So, let I take some linear combination

$\alpha_1 T(u_1) + \alpha_2 T(u_2) + \ldots.. + \alpha_n T(u_n) = 0$, α_i's are scalars

$T(\alpha_1 u_1 + \alpha_2 u_2 + \ldots.. + \alpha_n u_n) = 0$

$\alpha_1 u_1 + \alpha_2 u_2 + \ldots.. + \alpha_n u_n \in N(T)$

But, $T\ T$ is one one $\Rightarrow N(t) = \{0_\alpha\} \Rightarrow N(t) = \{0_\alpha\}$

$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$

Since, $\{u_1, u_2, \ldots, u_n\}$ $l \cdot I \Rightarrow \alpha = \alpha_2 = \cdots = \alpha_n = 0$ $l \cdot I \Rightarrow \alpha = \alpha_2 = \cdots = \alpha_n = 0$

=> $\{T(u_1), T(u_2), \ldots, T(u_n)\}$ is $l \cdot I$ $l \cdot I$

So, from here I can say that the set $\{T(u_1), T(u_2), \ldots, T(u_n)\}$ is a linearly independent set and the vectors belonging to these are linearly independent. So, this is the proof of this first part. In the 2nd part we are going the opposite way that these are linearly independent in the range space and we need to show that these are linearly independent.

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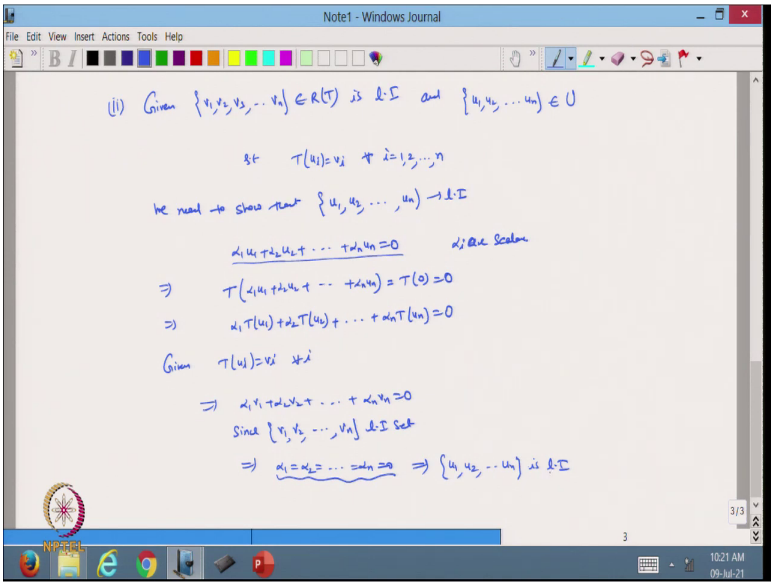

Now, the 2nd part we are going to prove is that the set $\{v_1, v_2, \ldots v_n\} \in R(T)$ is a linearly independent set. So, that is given to us and this belongs to the range space of T and I take a set of $u_1, u_2, \ldots, u_n \in U$ such that $T(u_i) = v_i$, $i = 1, 2, \ldots, n$.

So, I want to show that this $u_1, u_2, \ldots, u_n$ are linearly independent. So, for this one I will take the linear combination. So, let

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots.. + \alpha_n u_n = 0, \quad \alpha_i\text{'s are scalars}$$

Now, in this case I take the transformation the linear transformation T. So, that will be

$$T(\alpha_1 u_1 + \alpha_2 u_2 + \ldots.. + \alpha_n u_n) = T(0) = 0$$

$$\alpha_1 T(u_1) + \alpha_2 T(u_2) + \ldots.. + \alpha_n T(u_n) = 0$$

Now, given that $T(ui) = vi$, for all i's.

$$\Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 + \ldots.. + \alpha_n v_n = 0$$

and since the set $\{v_1, v_2, \ldots v_n\}$ is a linearly independent set.

$$\Rightarrow \quad \alpha_1 = \alpha_2 = \ldots. = \alpha_n = 0$$

So, which implies that the set $\{ u_1 , u_2 , \ldots, u_n \}$ is LI because if all the scalars are coming 0 then from this linear combination I am able to find that this $u_1 , u_2 , \ldots, u_n$ is a linearly independent set and the this is what we want to show that this set is also linearly independent. So, this way we can do the proof of this theorem.

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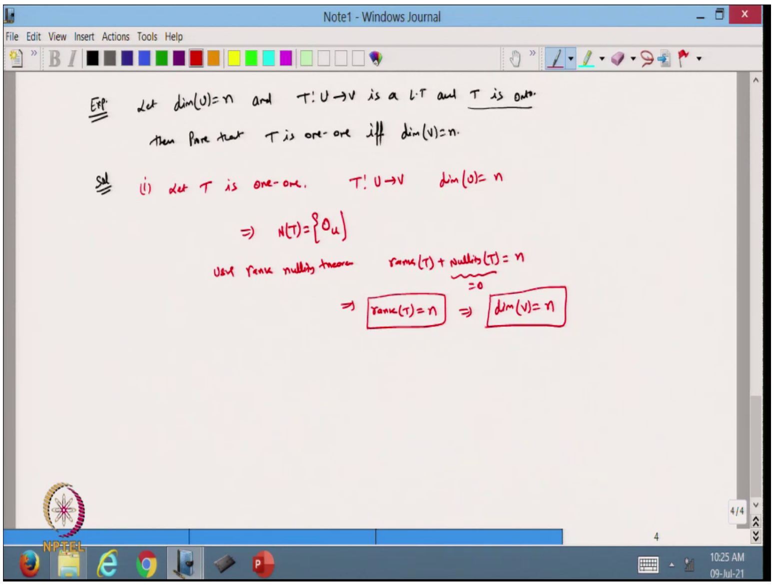

Now, based on this one. So, let us do one example of how we can use these previous theorems for finding some solution. Now let $\dim(U) = n$ and $T : U \rightarrow V$ is a linear transformation and T is onto then prove that T is one-one if and only if T if and only if $\dim(V) = n$.

So, this one is the question that we have a $\dim(U) = n$ and I take a transformation that is a linear transformation and it is onto, then it says that T is one-one if and only if $\dim(V) = n$. So, this one we need to prove now from here. So, let us start with the 1st part, let T be one-one.

Now, I have the transformation $T : U \rightarrow V$ and $\dim(U) = n$. Now if T is one -one.

$$\Rightarrow \; N(T) = \{0_u\}$$

So, we know about this one now using rank nullity theorem rank(T) + nullity of (T) = n. But now in this case this nullity is 0 because it contains only one element. So, from here we can say that the rank(T)= n and if rank(T)= n then from here we can say that dim(V) = n.

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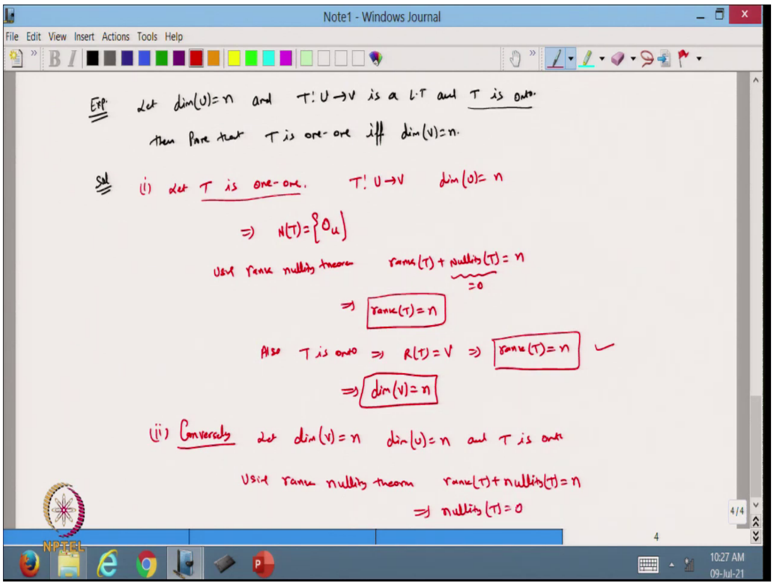

And after doing this one so, we should use this concept also that not directly we can say then from here I can say that the rank(T) = n also T is onto which implies that

$$R(T) = V \Rightarrow rank(T) = n \Rightarrow dim(V) = n.$$

So, this is what we wanted to show that if T is one-one and the map T is onto, then definitely the dim(V) =n.

Conversely, let dim(V) =n , dim(U) = n and T is onto. So, that is given to me now using 2 is onto. we need to show that T is one-one. Now, using rank nullity theorem, I can say that the rank(T) + nullity(T) = n because the dim(U) = n and from here I can say that nullity(T) = 0.

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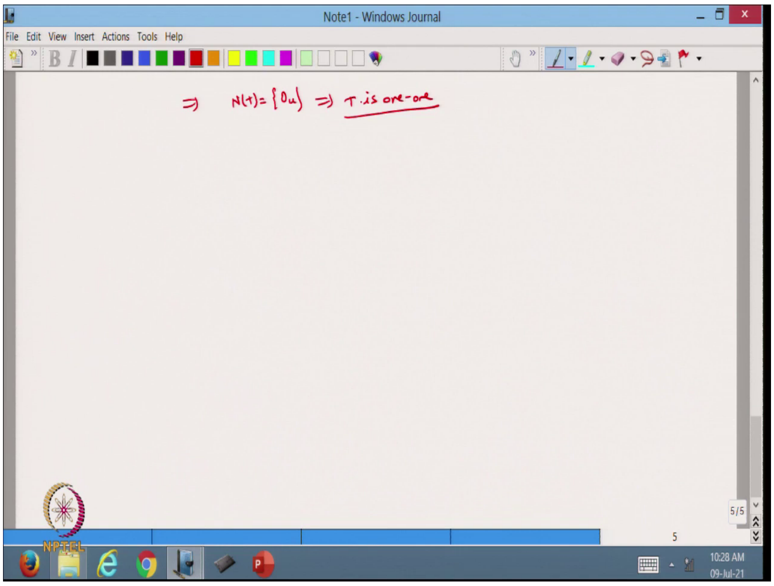

and if the nullity(T)= 0 so, which implies that N(T) = $\{0_u\}$ and which says that T is one-one. So, this way we are able to show that T will be one-one or maybe not directly we can right from here then maybe I can extend this condition.

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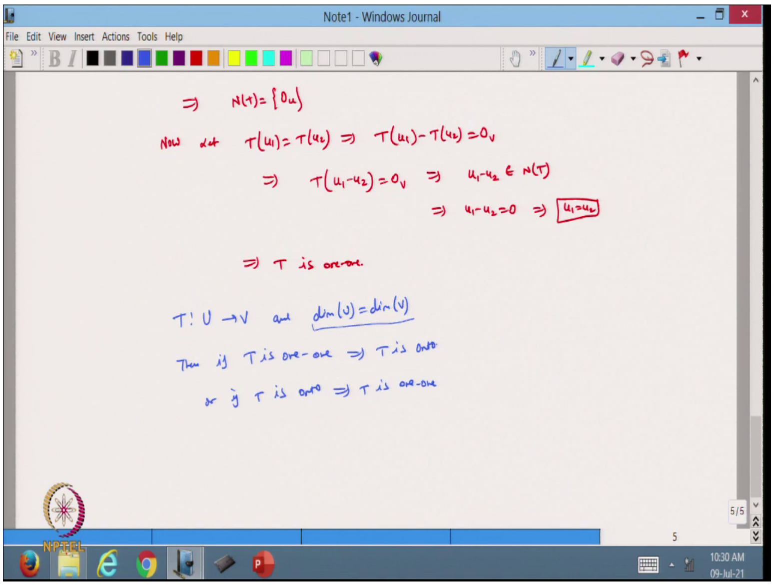

Now, we know that we check the condition T is one-one we can do like this one that, let => $T(u_1) = T(u_2)$

⇨ $T(u_1) - T(u_2) = 0_v$

⇨ $T(u_1 - u_2) = 0_v$

⇨ $u_1 - u_2 \in N(T)$

⇨ $u_1 - u_2 = 0 \Rightarrow u_1 = u_2$

⇨ T is one-one.

So, which implies that T is one-one. So, we are able to show this result so in fact, from here we can see that it says that that if I take the transformation

$T : U \rightarrow V$ and $\dim(U) = \dim(V)$ then if T is one-one then implies T is onto or if T is onto that implies that T is one-one. So, the only condition is that it should be from the same dimension of vector space U to the same vector space V are the same dimensional then these things are true.

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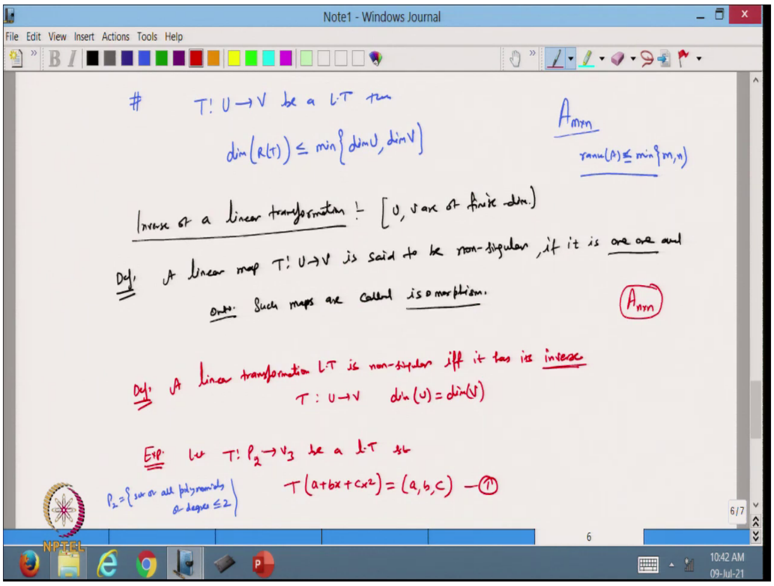

Now, from here I can say that, let us take another example or maybe one more result. I want to show that if I have a transformation $T : U \rightarrow V$ be a linear transformation then $\dim(R(T)) \leq \min\{\dim U, \dim V\}$.

So, it will be the minimum of this one. So, we already know that because we have seen this condition in the case of a matrix of order I am crossing and in that case we have seen that the rank of this matrix A is always less than equal to the minimum of {m, n}. So, the same thing we are writing here.

So, now after this one, once we know that T is one-one onto then we can talk about the other thing is how we can take the inverse of a linear transformation that how we can define. So, before that one we will define a definition that a linear map $T : U \rightarrow V$ is said to be non singular; it is said to be nonsingular if it is one-one and onto.

So, if it is one-one and onto then we say that this linear map from U to V is non singular and here we are talking about that finite dimensional map. So, U and V are of finite dimensionality and now if I have this one such maps are called isomorphism.

Now, we will define how we can take the inverse of this one. So, from here I can write the definition that a linear transformation LT is non singular if and only if it has its inverse. So, that way we can define the inverse of the linear transformation. If it is non singular like the matrix A and we know that we can find the inverse of this matrix, the only thing is that I should talk about that it is of the same dimension and then we can discuss how we can define the inverse of this one.

So, for finding the inverse, now we will define that U and V are finite dimensional and the dimension is the same, so that we can use. So, here because whenever we define the inverse of a given matrix we find that it should be non singular and the other thing is that it should be of the same dimension or it should be a square matrix.

So, in this case also we are saying that the linear transformation T is from U to V and dim(U) = dim(V).So, let us take one example of how we can find the inverse because there is no specific formula to find the inverse depending upon the problem.

So, let we have a transformation $T : P_2 \rightarrow V_3$ be a linear transformation such that

$$T(a+bx+cx^2) = (a,b,c)$$

where P_2 is a set of all the polynomials of degree less than or equal to 2. So, this way I defined this linear transformation. So, here I am talking about second order polynomials. So, the coefficient of this goes to the vector space V_3. So, now I want to check whether the inverse exists or not.

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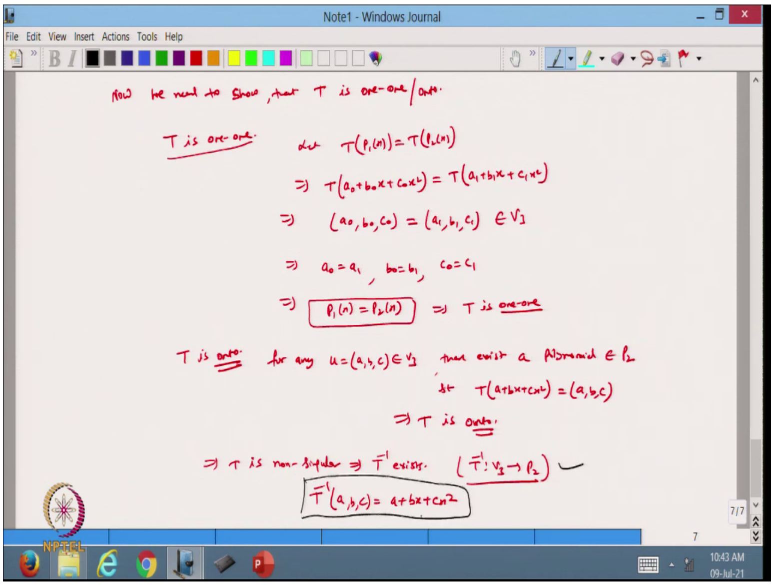

Now, so, first I need to check that this linear transformation is one-one onto or not. One thing is here that the dimension of P_2 is 3 and $\dim V_3$ is 3 . Now, we need to show that T is one-one onto. So, this one we need to check. The first one is that it is well defined because from any polynomial I can have my coefficient .

So, from here I can say that T is one-one. T is one-one because if I can check that? So, let $T(P_1(x)) = T(P_2(x))$

$$\Rightarrow T(a_0+b_0x+c_0 x^2) = T(a_1+b_1 x+c_1 x^2)$$

$$\Rightarrow (a_0 ,b_0 ,c_0) = (a_1 ,b_1 ,c_1) \in V_3$$

$$\Rightarrow a_0 = a_1 , b_0 = b_1 , c_0 = c_1$$

$$\Rightarrow P_1(x) = P_2(x)$$

$\Rightarrow$ Tis one -one

Now, I also claimed that T is onto. So, it is true that that for any u suppose I take some any u = (a,b,c) ∈ V3 there exist a polynomial ∈ P_2 such that $T(a+bx+cx^2) = (a,b,c)$

So, this is true for all elements from the V_3, then from here I can say that the transformation T is onto.

Now, you can see from here that it is moving from $P_3 \rightarrow V_3$ both have the same dimension 3 and this is one-one and onto also. So, now, from here I can say that T is non-singular implies that T^{-1} exists and the $T^{-1}(a,b,c) = a+bx+cx^2$ and this T inverse will be from V_3 to P_2. So, after doing this one we can say that this T inverse will exist because it is one-one onto and T inverse is the transformation we have defined like this one.

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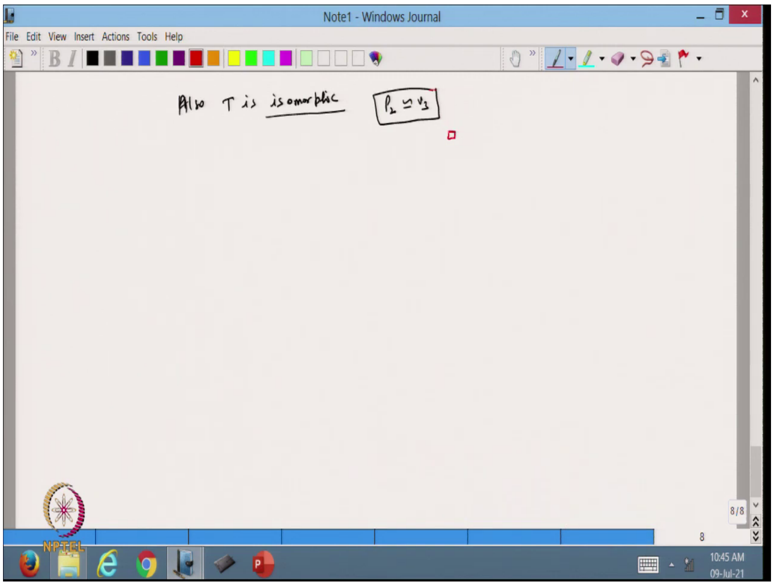

And from here also T is isomorphic means, when T is one-one onto and moving from one vector space to the another vector space of the same dimension, then we can say that the T is isomorphic and from here I can say that P_2 is isomorphic to V_3 , this is a sign that we can define for isomorphism.

They are isomorphic to each other although this seems that they are different because one contains the polynomial of degree less than equal to 2 and another contains the vector in the 3 dimension that we are talking about the real vectors.

But now we are able to show that this is one-one onto, it is a non singular and inverse edges. So, from here we can say that this is isomorphic to each other. So, this way we can show that T is isomorphic and T inverse exist. So, I will stop here. So, in the today's lecture we have discussed some application of rank nullity theorem and then we have also defined the inverse of the linear transformation T from a vector space u to vector space v. So, in the next lecture we will continue this one.

Thanks for watching. Thanks very much.