Matrix Computation and its applications
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Lecture - 33
Rank-nullity theorem

Hello viewers. So, welcome back to the course on Matrix Computation and its application. So, in the previous lecture, we have discussed a few examples about how we can find the rank and the nullity of a linear transformation. So, today we are continuing with that one.

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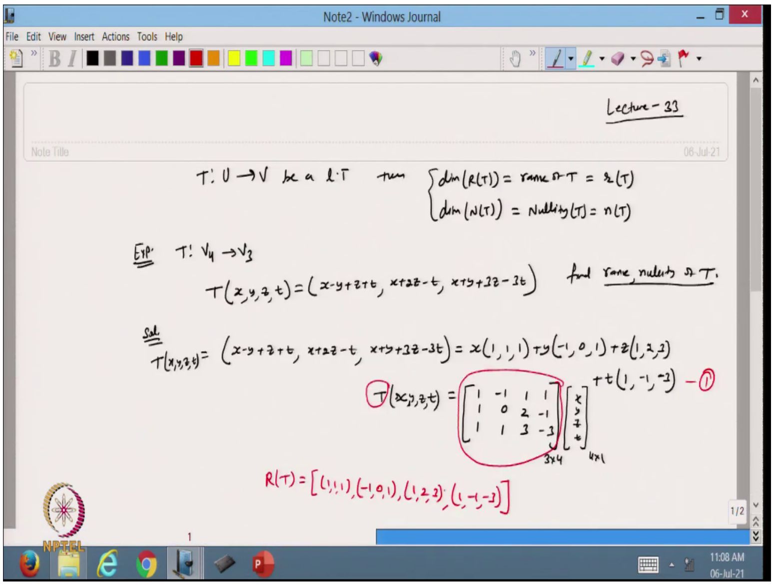

So, I know that if I have the linear transformation T: U → V. So, suppose this is a linear transformation then, the dim(R(T)) is called the rank of T. And also, sometimes we are represented by R(T). And the dim(N(T)) is called the nullity of T that sometimes you represent by a small n(T).

So, this one we have discussed. So, let us take one example. Suppose, I take a linear transformation T : V_4 → V_3. In the previous case, we have taken a transformation from V_3 to

V_4. Now, we are taking from V_4 to V_3 by the transformation T. So, we are representing it as (x, y, z, t).

Suppose, T(x, y, z, t) = (x - y + z + t, x + 2z - t, x + y + 3z - 3 t). So, this is my linear transformation. Now, we need to find the rank and nullity of T. So, this one we need to find. Now, from here, I know that the range space is made up of this type of element. So, now, I can write from here.

T(x, y, z, t) = (x - y + z + t, x + 2z - t, x + y + 3z - 3 t)=x(1,1,1)+y(-1,0,1)+z(1,2,3)+

+t(1,-1,-2)……..(1)

$$T(x, y, z, t) = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

So, now from here, you can say see that this T the transformation basically, we can represent this T with this matrix, and this matrix is of order 3 * 4, and T is a linear transformation from V_4 to V_3. So, we can represent this transformation in terms of this matrix. Now, from here, I know that. So, this is my equation number 1. Now, from here I can say that my R (T) range space is spanned by these vectors

R(T) = [(1,1,1),(-1,0,1), (1,2,3),(1,-1,-3)]

But the range space I know is a subspace of V_3. Since R (T) is a subspace of V_3. So, it has a 4 vector in its span. So, it cannot be linearly independent, and also, it belongs to V_3.

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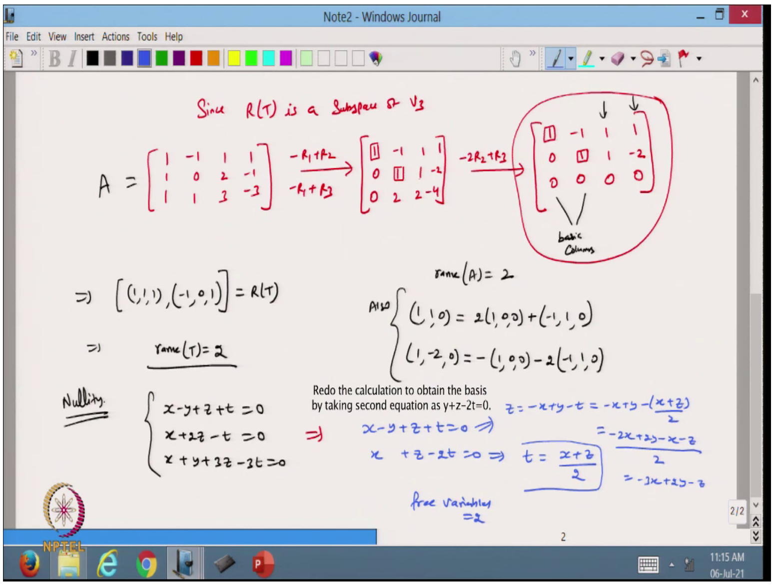

So, from here I know that. So, now, I need to check how many vectors are linearly independent in this case. So, I will take this matrix.

$$A=\begin{bmatrix}1 & -1 & 1 & 1\\ 1 & 0 & 2 & -1\\ 1 & 1 & 3 & -3\end{bmatrix} \to \begin{bmatrix}1 & -1 & 1 & 1\\ 0 & 1 & 1 & -2\\ 0 & 2 & 2 & -4\end{bmatrix} \to \begin{bmatrix}1 & -1 & 1 & 1\\ 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

So, using these operations R_1 as $-R_1 + R_2$ and R_2 as $-R_1 + R_3$, $-2\,R_2 + R3$ we get the above matrix . So, I can get this as my pivot element. So, I know that these two vectors (first two columns) are basic columns and this is a non-basic column.

So, from here I can see the rank of this matrix. So, let me call it A. So, the rank(A) = 2 that we can check from here. Also, I can write this non-basic column this and this. So, I can write this

(1 1 0)= 2(1 1 0) +(-1 1 0)

(1,-2,0)=-(1,0,0)-2(-1,1,0)

So, it is -1 and it is 2. So, it will be 1 and then it is – 2. 0. So, this non-basic column can be written as a linear combination of the basic columns. So, from here, I can find that in the set (1 1 1) the first and the next (-1 0 1). So, the whole this one they span the whole R(T) and they are linearly independent. So, from here I can say that the rank of my linear transformation is 2.

Now, I want to find the nullity. So, for nullity, this is my image. So, I will put this image equal to 0. So, now, I need to put the image. So, that image is x - y + z + t = 0 and y+z-2t = 0. So, these are the conditions. I am putting this element equal to 0 and this one.

Now, from here I have 4 variables and 2 equations and I know that the rank is 2. So, now, from here they no need to convert this one into the echelon form, because we can use this matrix here. So, I will use this matrix and from there, I will get that we can have x - y + z + t =0 and I can have y+z-2t = 0. So, with the Gauss elimination, I can convert this into this form, because the rank is 2. So, this is the way we can find out.

y= 2t-z,

x-2t+z+z+t=0 => x=t-2z

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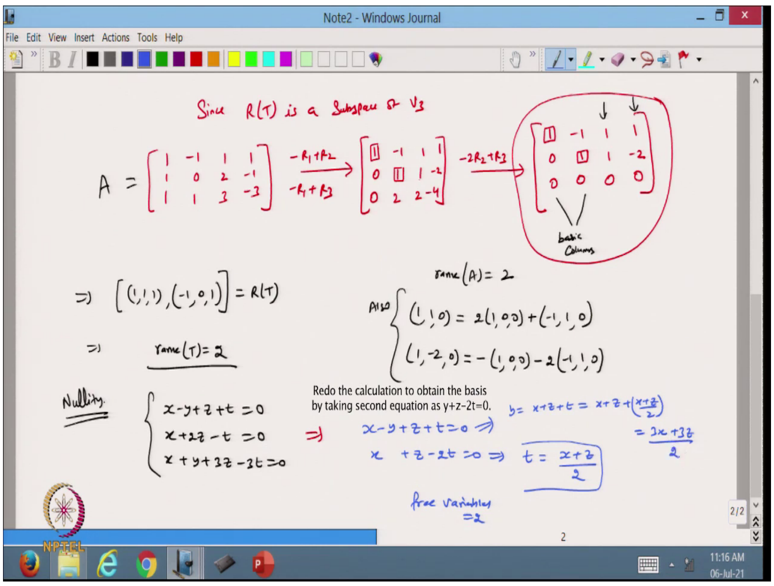

I can take my y = x + z + t. So, that is ok. So, I can write x + z + t is x + z by 2, and that gives me. So, 2 x + x 3 x and 2 z + z 3 z by 2. So, this is the way we can write our variables.

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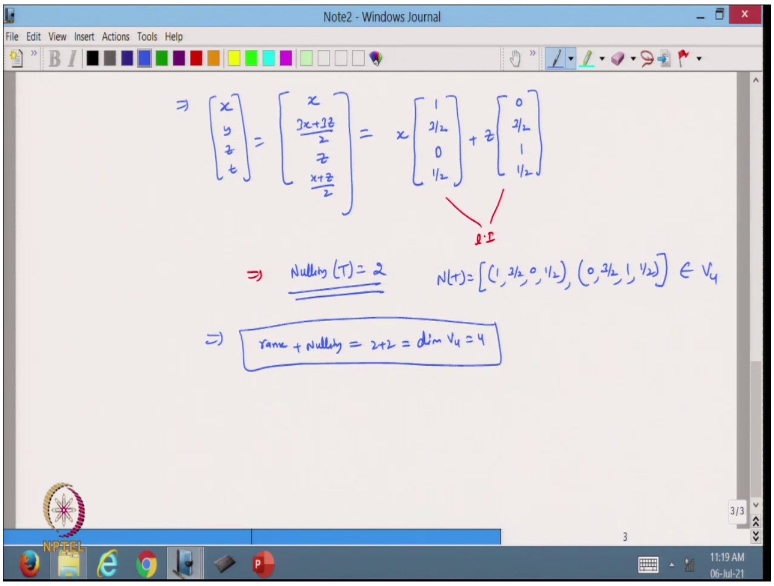

So, now from here x, y, z, and t. So, this is my vector. It can be written $\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} t-2z \\ 2t-z \\ z \\ t \end{bmatrix}$

$$t\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

So, and also, I know that from here, these 2 vectors are linearly independent, because here, it is the 0 element, and here is the 0 limit. So, I cannot make this element 0 by a scalar multiple.

So, from here, I can write that the null space is spanned by. So, from here just I can write that the nullity of T is 2 and the null space is spanned by this vector. So, I can write that N(T) = [(1,2,0,1),(-2,-1,1,0)]. And this belongs to definitely, we know that this is a subspace of V_3. So, it belongs to V_4 sorry, because we have a linear transformation from V_4 to V_3. So, from here also we can check that the rank plus nullity that is 2 + 2 is equal to the dimension of V_4 that is 4.

So, based on this one. Now, we are going to introduce a very important theorem and that is called the rank nullity theorem.

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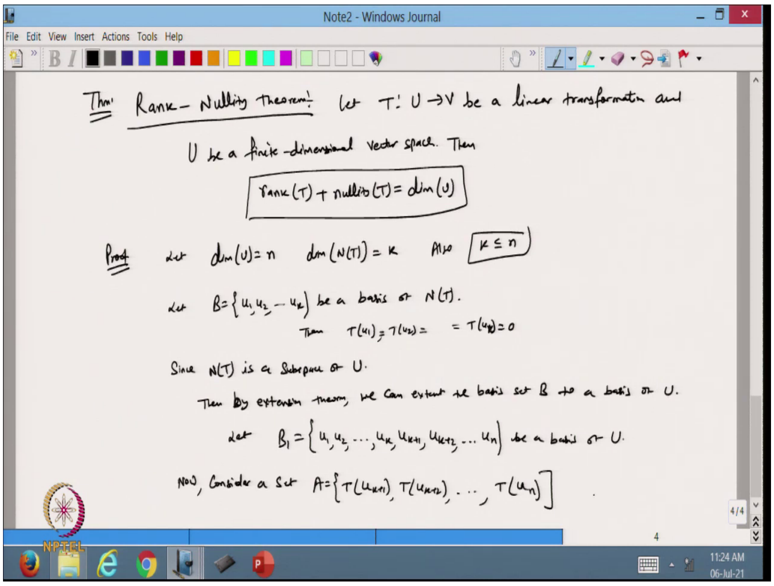

So, we are going to introduce a rank nullity theorem. So, what is this rank nullity theorem? Now, let I have a linear transformation from U to V be a linear transformation and U be a finite-dimensional vector space. Then, the rank(t)+nullity (T)= dimension(U). So, this is called the rank-nullity theorem. You can check whether although the range page is a subspace of v, the rank plus nullity satisfies this condition.

So, this is a very important theorem. We can prove this one. So, let I assume that the left dimension of U is maybe n let this one. Also, the dimension of N (T) is the nullity that it is equal to K. So, this is what we have decided. And also, we know that the K will be less than equal to n, because null space is a subspace of U. So, definitely the dimension of N T is always less than equal to the dimension of U. So, that is true. Now, it is given to me that null space is having the dimension K that is the null t is K. So, let us take a base B $\{u_1, u_2, \ldots, u_k\}$.

So, let be a basis of null space T. So, if it is the basis of this, then we know that $T(u_1)$, $T(u_2), \ldots, T(u_k) = 0$, because this belongs to the null space now. Since, N(T) is a subspace of vector space U then, by extension theorem, we can extend the basis set B. So, this is the basis of N(T) to a basis of U. So, that we can do.

So, let me just take B_1 as the basis. So, that will contain ($u_1, u_2, \ldots, u_{K+1}, u_{K+2} \ldots u_n$) because this is equal to u_n. So, let B_1 be the basis of U now. So, this we have shown. Now, consider a set. So, I consider a set A. What is this set? This set is I am taking $T(u_{K+1}), T(u_{K+2}) \ldots. T(u_n)$. So, this contains n - K elements ok. So, I consider set A.

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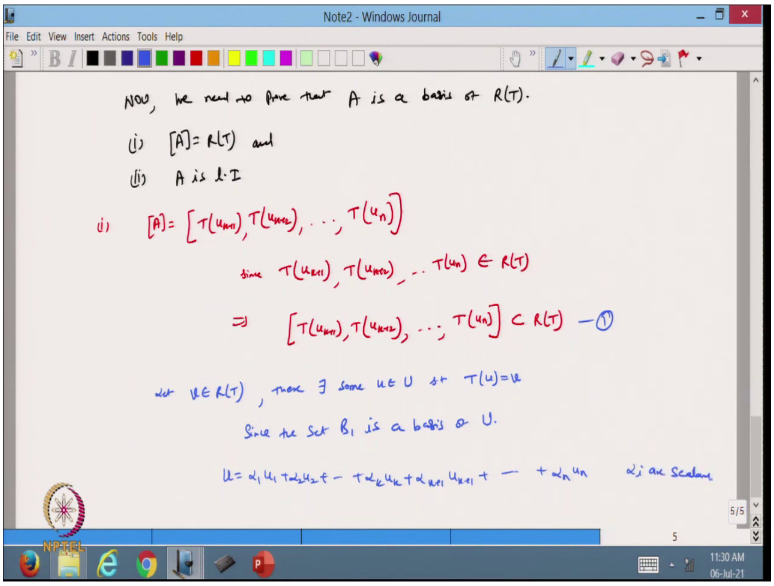

Now, we need to prove that A is a basis of range space R(T). So, to prove this one, what do I need to do? First, I need to show that A this A basically yeah. So, that is, I need to show that the span of A is equal to R(T), and the second one I need to check that it is linearly independent.

So, if this set of A I is taken is linearly independent and it spans R(T), then we can say that this is the basis of R(T). So, how can we do that one? So, let us prove two things. So, the first one. So, this I need to prove now. So, what am I going to do? So, let now A span means the span of $T(u_{k+1}), T(u_{k+2}) \ldots.., T(u_n)$ this is. Now, since $T(u_{k+1}), T(u_{k+2})$. And all these $T(u_n)$ belong to the range space T and the range space of T is a subspace of the vector space V.

Then from here, we know that the linear combination $T(u_{k+1}), T(u_{k+2}) \ldots.., T(u_n)$ So, if I take their linear combination that is contained in the range space of T because their linear combination also belongs to R(T). So, if I take this set of all the linear combinations that are spanned, this is contained in R(T) now. So, after doing this one. So, this is the one way I have done it.

Another way I can say is that let some V belong to R(T). So, this I can write as 1. So, let me take V from R (T). Then, there exists some u belonging to U; such that T (u) is V. Now, V belongs to R(T). Now. So, what I need to show is that. I want to show that for any V $_I$ have taken from R(T), there is u from the U. So, that T(u) =V. Now, I want to show that V belongs to the span of this one. So, this one I need to show. So, if u belong to U.

Since set B1 is a basis of U, then I can write my U as a linear combination of $u = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_k u_k + \alpha_{k+1} u_{k+1} + ... + \alpha_n u_n$. So, I can write like this one. And this is linearly independent. So, all alpha I s are uniquely determined, because they are linearly independent. So, this is the unique representation of U. I can write from here, and all alpha is are scalars.

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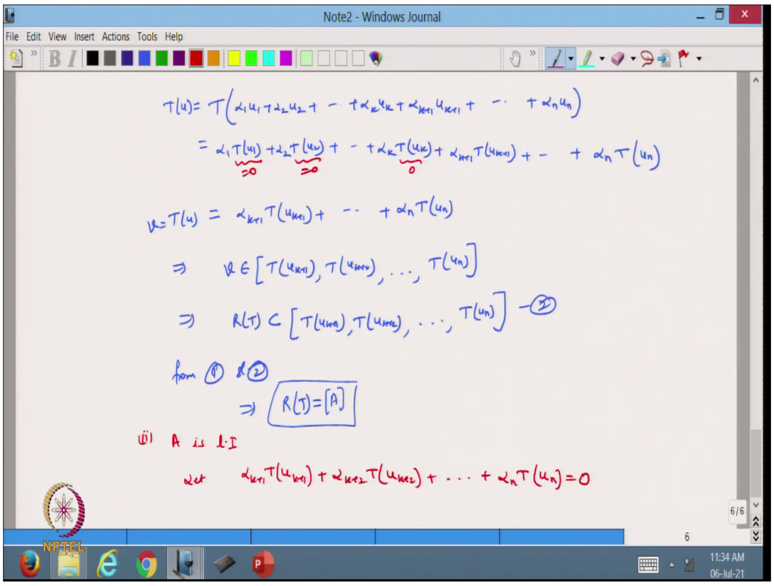

Now, from here, I can write my T(u). So,

$$T(u) = T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_k u_k = \alpha_{k+1} u_{k+1} + ... + \alpha_n u_n)$$
$$\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_k u_k - \alpha_{k+1} u_{k+1} - ..... - \alpha_n u_n = 0_u$$
$$B_1 = \{u_1, u_2,, ..., u_k, u_{k+1}, ..., u_n\}$$
$$\alpha_1 = \alpha_2 = ... = \alpha_k = \alpha_{k+1} = ... = \alpha_n = 0$$
$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} t-2z \\ 2t-z \\ z \\ t \end{bmatrix}$$
$$t\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

.

$$T(u) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_k T(u_k) + \alpha_{k+1} T(u_{k+1}) + ... + \alpha_n T(u_n).$$

So, I can write like this one. Now, from here, you see that this is equal to 0, because I know that u_1 up to u_k are the basis of N (T). So, its image will be 0.

So, this will be 0. So, from here, I can write that this is $v = T(u) = \alpha_{k+1} T(u_{k+1}) + ... + \alpha_n T(u_n)$

It means that the T(u) and that is my v can be written as a linear combination of this one. So, from here, I can say that the v belongs to the span of $T(u_{k+1})$, $T(u_{k+2})$,…$T(u_n)$. And from here, I can say that my R(T) is a subset of this $T(u_n)$. So, from 1 and 2, we can say that the range space of T is equal to the span of A because this is A this one. So, we can show.

Now, we need to show the second one that A is linearly independent. So, that we need to show. So, for this one, let me take a linear combination. So, I just take linear combination as

$$\alpha_{k+1} T(u_{k+1}) + ... + \alpha_n T(u_n) = 0.$$

So, this linear combination I take and put equal to 0 now. So, I need to show that this A is linearly independent. It means we need to show that all this $\alpha_{k+1} T(u_{k+1}) + ... + \alpha_n T(u_n) = 0$.

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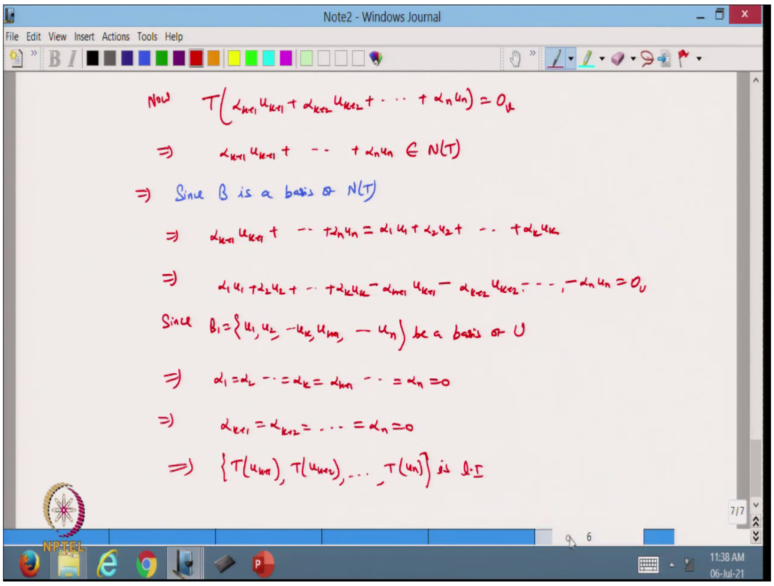

Now, from here I can write that $T(\alpha_{k+1}u_{k+1} + ... + \alpha_n u_n) = 0_k$. So, if it is equal to 0, which implies that $\alpha_{k+1}u_{k+1} + ... + \alpha_n u_n \in N(T)$, because their image is going to 0 and this 0 is a 0 of v. So, we already know that this belongs to the 0 of v.

Now, I know that if it belongs to N (T) then, the null space has the basis B. So, from here that since B is a basis of N (T). So, now, we can write that

$$\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_k u_k = \alpha_{k+1}u_{k+1} + ... + \alpha_n u_n$$

$$\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_k u_k - \alpha_{k+1}u_{k+1} - ..... - \alpha_n u_n = 0_u$$

So, if you check from here, then this is a basis of u. Now, since $B_1 = \{u_1, u_2,, ..., u_k, u_{k+1}, ..., u_n\}$ be the basis of U. So, now from here, which implies that $\alpha_1 = \alpha_2 = ... = \alpha_k = \alpha_{k+1} = ... = \alpha_n = 0$. So, this is all 0, because these are the basic elements.

And from here, which implies that $\alpha_{k+1} = ... = \alpha_n = 0$ and from here, I can say that the set $\{T(u_{k+1}), T(u_{k+2}), ..., T(u_n)\}$ is linearly independent. So, now we can show that this is linearly independent and it spans R (T).

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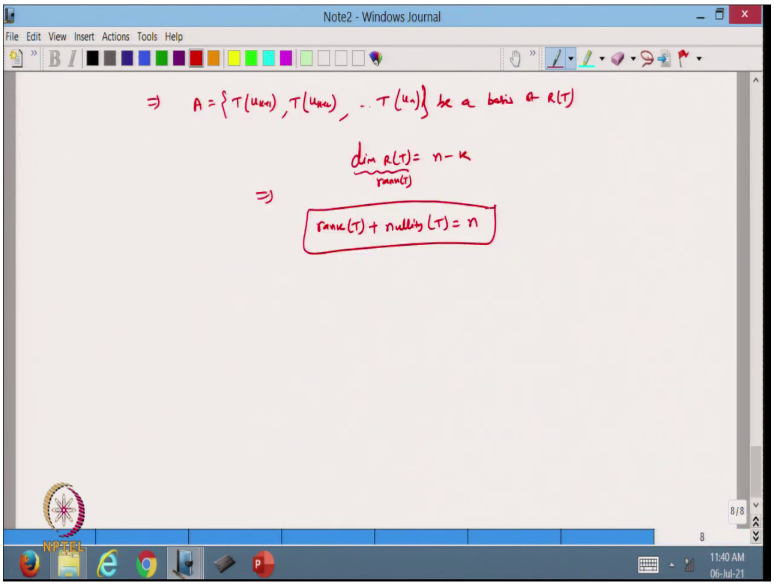

So, from here, I can say that A that is $\{T(u_{k+1}), T(u_{k+2}), \ldots, T(u_n)\}$ be a basis of R(T), because this is what I wanted to show that this set A is the basis of R(T). And now, from here, you can say that the basis of R(T).

So, the dimension of R (T) will be n - K and from here. So, the dimension of R (T) is the rank of T. So, from here, I can write that rank of T + K is the nullity of T and that is equal to n, where n we have taken that n is the basis of the dimension of u. So, that is equal to this one.

So, from here we are able to show that the rank plus nullity is equal to the dimension of the space U whenever we have a linear transformation from U to V. So, this is a very important theorem in the case of rank nullity. So, we stop here now. So, in today's lecture, we have discussed a very important theorem that is the rank nullity theorem. So, that shows that if we have a linear transformation T from any vector space finite-dimensional vector space U to V.

Then, the rank of T plus the nullity of T is always equal to the dimension of the vector space U and it is a very useful theorem. We can use this theorem in the future to solve many problems. So, I hope that you have enjoyed this lecture thanks for watching.

Thanks very much.