Matrix Computation and its applications

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Lecture - 32
Rank and nullity of a linear transformation

Hello viewers, welcome back to the course on Matrix Computation and its application. As in the previous lecture, we have discussed how we can check the linear transformation is one-one onto or not. So, we will continue with that concept with lecture number 32.

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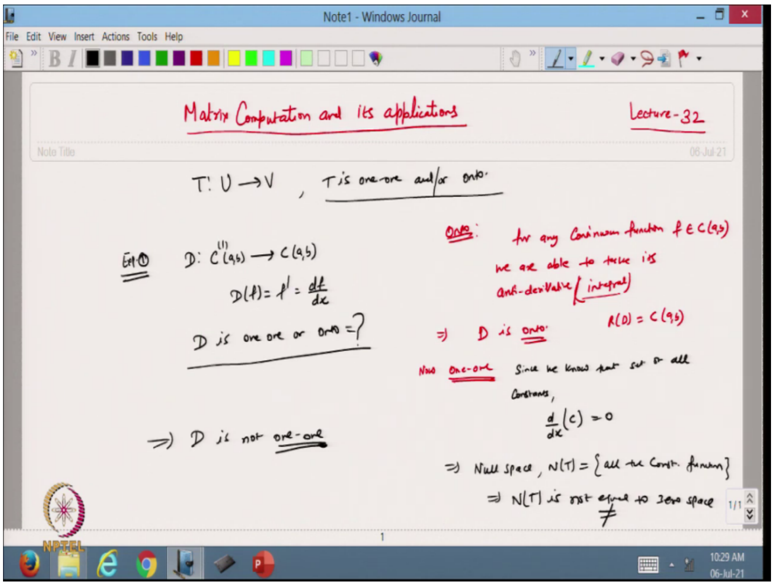

In the previous lecture, we discussed how we can check that the transformation T: U → V (a vector space from U to vector space V) that T is one to one and or onto . So, we have discussed a few examples. Now, we will take another example.

So, previously we have discussed the linear operator D there is a differential operator from the C^1 space to C the set of all first derivative continuous function to the set of continuous function in the interval from a to b i.e

$$D : C^1(a,b) \rightarrow C(a,b)$$

Defined as $D(f) = f'$ or df/dx.

Now, I want to check whether D is one-one or onto. So, for onto we know that for any continuous function f ∈ C(a,b), we can take its anti-derivative. So, anti-derivative we know that this is called the integral.

So, we know that for every continuous function in the given interval (a, b) we can take its integration. So, for any continuous function I can define its integral. So, from here I can say that this linear transformation D is onto because I am going from C any continuous function, I can take the antiderivative and if I take the antiderivative then this will reach to $C^1(a,b)$.

And in this way we can show that if for any vector in the range space = C(a, b) we can have a vector in the domain such that D of that function is equal to this. So, we can say that D is onto. So, here D is onto means the range space of D is equal to the whole space C (a,b). So, this is defined that D is onto.

Now, how to check one-one? Since we know that the set of all constant, If I take the set of all constants and I take the derivative

$$\frac{d(c)}{dx} = 0$$

So, from here I can say that the null space that is N(T) definitely will contain a set containing all the constant functions and some other functions that we do not know then it may happen that we will get some vectors whose derivative is going to 0.

So, one thing is sure that all the constant functions that we are going to have the derivative equal to 0. So, in this case, from here I can say that my null space is not equal to zero space. So, this is not equal to zero space. So, the operator D is the differential operator. I can say that this differential operator is not one to one. Because I have constant functions which are going to give you the derivative 0.

So, in this case, it is not going to be one-one.

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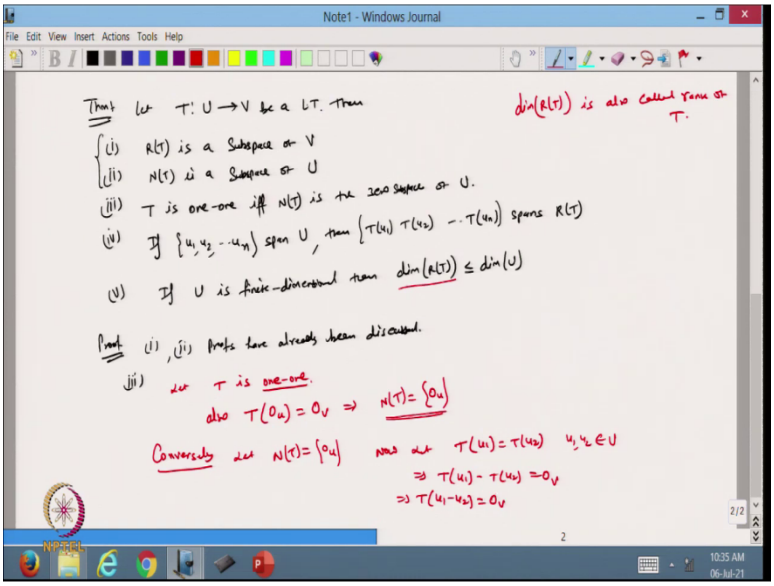

Similarly, now based on this one I just want prove a theorem,

let me take a linear transformation T: U → V. Then

(i) R(T) is a subspace of V

(ii) N(T) is a subspace of U

(iii) T is one-one if and only if N(T) is the zero space of U

(iv) If $\{u_1, u_2, \ldots u_n\}$ spans U, then $\{T(u_1), T(u_2), \ldots, T(u_n)\}$ spans to R(T).

(v) If U is finite-dimensional then the $\dim(R(T)) \leq \dim(U)$.

and the dimension of R(T) is also called the rank. So, you can write from here that the dimension of range space T is also called the rank of T. So, these are the few results we want to prove.

(i),(ii) So, the first two results we have already proved when we have discussed the matrices from A_n to A_m. So, you can check these from the lecture-based on that one and it is very easy to show also. Now, I can write the 1st and 2nd proof as I have already discussed.

(iii) So, it says that my transformation T is one-one if and only if the null space is the 0 subspace of U. So, let us prove this one. So, in this case, what I do is that first I will check that my T is one-one. It means if the T is one-one then I can say that T and also, I know that $T(0_u) = (0_v)$.

So, I can write that T is one-one also this is true for all the linear transformations, so, if it is a one-one and this is also true. So, which implies that the $N(T) = \{0_u\}$ and from here we can show that N(T) is the only 0 subspace of U.

Conversely, let $N(T) = \{0_u\}$

Now let $T(u_1) = T(u_2)$ for u_1 and $u_2 \in U$.

⇨ $T(u_1) - T(u_2) = 0_v$.

⇨ $T(u_1) - T(u_2) = 0_v$ because this is the linear transformation.

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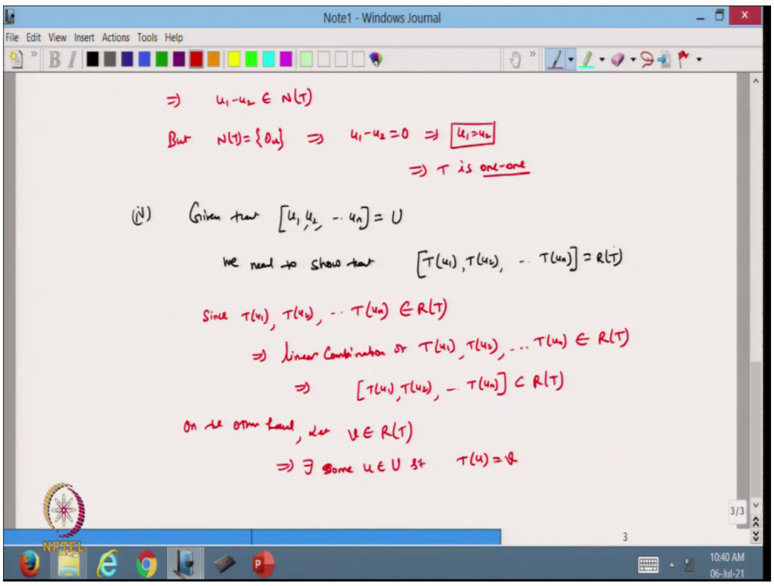

⇨ $u_1 - u_2 \in N(T)$

But, $N(T) = \{0_u\}$ => $u_1 - u_2 = 0$ => $u_1 = u_2$

⇨ T is one to one.

So, from here I can say that T is one to one. So, if I need to check if the linear transformation is one-to-one or not, we should concentrate on the null space. If we can find an element other than the 0 elements which are mapping to the 0 elements under this transformation then from there, I can say that the given transformation is not one-one. So, this is what we can show.

(iv) Now it is given that $[u_1, u_{2,} \ldots, u_n] = U$.

Now, from here we need to show that $[T(u_1), T(u), \ldots T(u_{n)})] = R(T)$

if I take the span of this one that is equal to R(T). So, this we need to show. Now, from here what I need to do is that. So, I will take the one way. So, let. So, this is the I am taking that let I take an element of v belongs to now I can one thing is true that since, $T(u_1), T(u_2), \ldots, T(u_n) \in R(T)$.

And R(T), I know that is a subspace of the vector space v then if this belongs to R(T) their linear combination also belongs to R(T) and from here, I can say that the linear combination then I can write that

- ⇨ linear combination of $T(u_1), T(u_2), \ldots, T(u_n) \in R(T)$
- ⇨ $[T(u_{1)}, T(u_{2)}, \ldots, T(u_{n)}]$ also belong to R (T).

So, we have shown that if I take the element from the left-hand side then we can show that this belongs to the right-hand side. On the other hand,

let $v \in R(T)$

Now from here if v belongs to R(T) then I can say that

$\exists$ some $u \in U$ such that $T(u) = v$

because it is a definition of the linear transformation from U to V.

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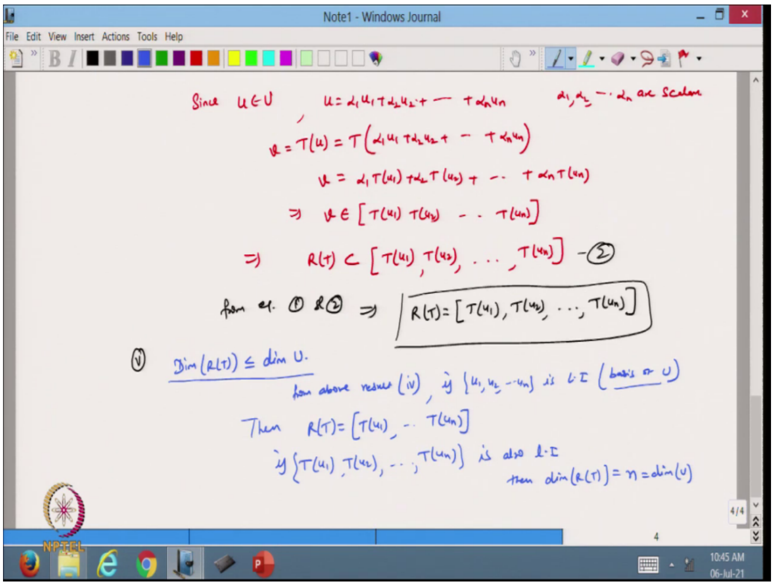

Now, since $u \in U$. So, I can write that u can be written as

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots.. + \alpha_n u_n \;,\; \alpha_1, \alpha_2, \ldots\ldots \alpha_n \text{ are scalars}$$

because I have it has been given that this u_1, u_2, u_n this spans U, it means any vector from U can be written as a linear combination of this one. Now from here I can write

$$v = T(u) = T(\alpha_1 u_1 + \alpha_2 u_2 + \ldots.. + \alpha_n u_n)$$

$$v = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \ldots.. + \alpha_n T(u_n)$$

$$v = [\, T(u_1) \quad T(u_2) \ldots\ldots T(u_n)]$$

$$R(T) \text{ subset of } [\, T(u_1) \;, T(u_2) \;, \ldots\ldots, T(u_n)] \ldots\ldots\ldots..(2)$$

So, from equations 1 and 2 we can write that

$$R(T) = [\, T(u_1) \,, T(u_2), \ldots\ldots, T(u_n)]$$

So, that is the way we can show. So, here you just check that it has been given that u_1, u_2, u_n they are spanning the whole U it is a note given that they are linearly dependent or not. So, if I take the set of vectors and vectors which span U then I take the corresponding image of these vectors that will span the whole range space R (T).

The 5th one is we can check that the dim(R (T)) $\leq$ dim(U). So, this is clear from here that now I can write from the above result that is the (iv) if I say that if u_1, u_2, u_n this set is LI. Then we know that the given transformation is uniquely determined by the value of the transformation at its basis.

So, it will be then it will be a basis of U because they are spanning the whole U, they are linear dependent. So, it becomes the basis of U and from here I know that

$R(T) = [T(u_1), T(u_2), \ldots\ldots, T(u_n)]$

Now it may happen that if $T(u_{1)}$, $T(u_{2)}$, $T(u_{n)}$ if this set is also LI. Then

$\text{Dim}(R(T)) = n = \dim(U)$.

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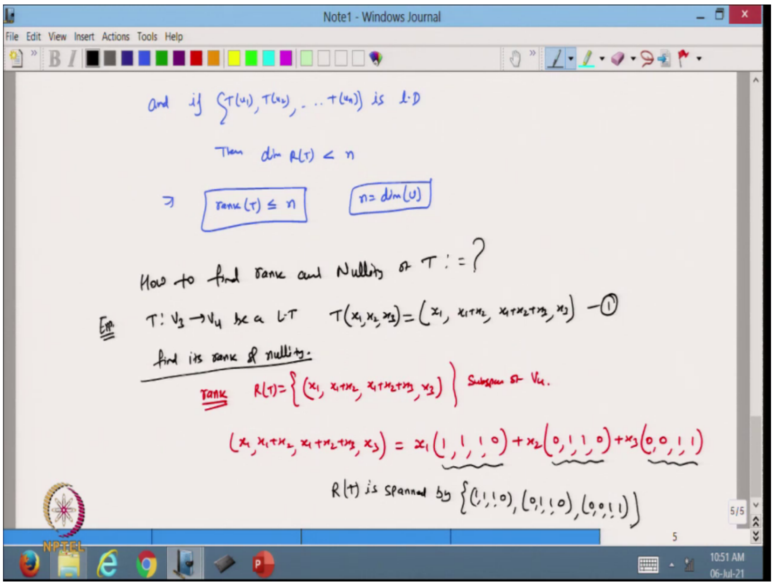

And if the set $\{T(u_{1)}, T(u_{2)}, T(u_{n)}\}$is linearly dependent then if it is a linear dependent, I can remove those vectors which are the linear combination of the previous one. So, then I can say that the dim(R (T)) $<$ n.

So, from here I can say that dimension or the rank(T) $\leq$ n where n = dim(U), I have taken that because this is a linearly dependent set of vectors n in number.

So, let us do that: how to find rank and nullity, how to find the rank and nullity of a linear transformation T. So, we are going to discuss this. Now, for example, suppose there is a transformation T: $V_3 \rightarrow V_4$.

So, this is my linear transformation be a linear transformation and defined as

$$T(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, x_3) \quad \ldots\ldots(1)$$

So, this is a linear transformation it is given to us.

Now the question is, How to find its rank and nullity? So, it is given to us that this is a linear transformation. Now from here I want to check its rank means the dimension of his range space and the dimension of its null space.

Now let us discuss the rank, now I know that this belongs to R(T). So,

R(T) is a set of all the vectors of this form $\{x_1, x_1 + x_2, x_1 + x_2 + x_3, x_3\}$ subset of V_4. Now I can write this as x_1, $x_1 + x_2$, $x_1 + x_2 + x_3$, and x_3. So, this one I can write in the form of I can take an x_1 common.

$$(x_1, x_1 + x_2, x_1 + x_2 + x_3, x_3) = x_1(1,1,1,0) + x_2(0,1,1,0) + x_3(0,0,1,1)$$

So, I can find x_1, x_2, x_3 from here. So, that is a way we can find, it means that this vector is a linear combination of these three vectors.

So, I can get these three vectors. So, this is my first vector, second vector, and the third vector, now from here I can say that R (T) is spanned by this vector. So, I can call it this one.

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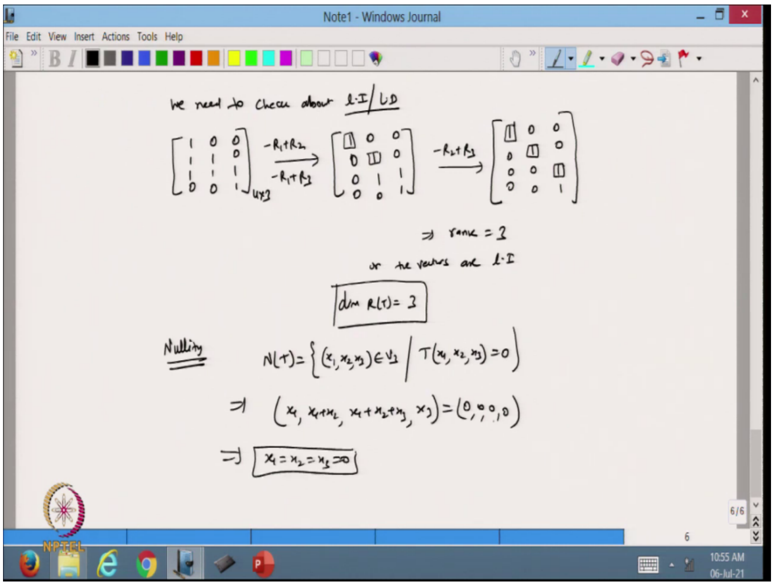

Now, for the rank, we need to check about linear independence or linear dependence. So, these vectors are given to me I can write this vector in the terms of a matrix as a column vector of a matrix. So, I can write here

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, I convert this matrix into the row echelon form using elementary row operations

$-R_1 + R_2$, $-R_1 + R_3$, $-R_2 + R_3$. So, I can get these vectors. So, these three vectors from here I can say now based on this one I can say here that the rank of this matrix rank = 3 because all these three vectors are linearly independent or the vectors are linearly independent.

So, that is things we have already discussed when we have discussed the matrices. So, from here I can say that the dimension of R(T) is 3. Now we will discuss the null space

$N(T) = \{(x_1, x_2, x_3) \in V_3 \mid T(x_1 x_2, x_3) = 0\}$

⇨ $(x_1, x_1 + x_2, x_1 + x_2 + x_3, x_3) = (0,0,0,0)$

⇨ $x_1 = x_2 = x_3 = 0$

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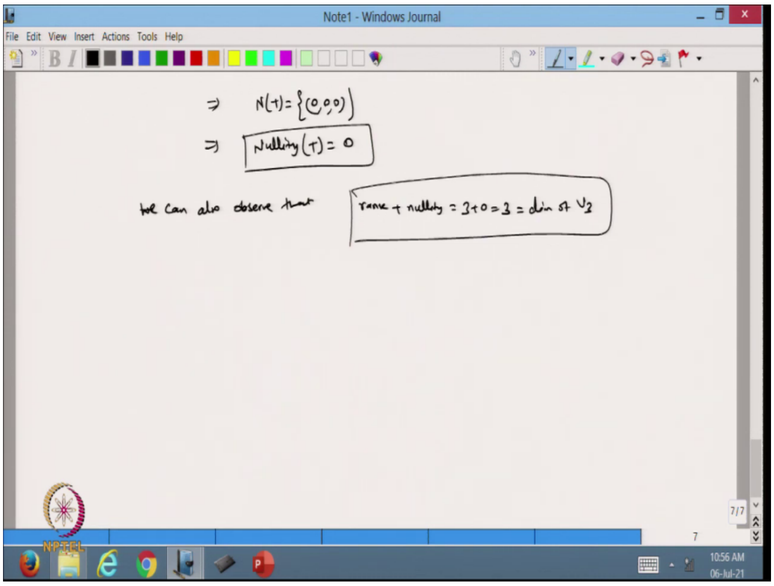

So, if all the three are coming 0 from here I can say that the null space of T will contain only 0 elements of V_3 i.e

⇨ $N(T) = \{(0,0,0)\}$

⇨ $\text{Nullity}(T) = 0$

So, nullity is 0 and the rank is 3 and from here we can also observe that

$\text{Rank} + \text{Nullity} = 3+0 = 3 = \text{dim of } V_3$

So, this is how we can find the rank and the nullity of the given linear transformation. So, we will stop here. So, in today's lecture, we have discussed some facts about the linear transformation and then we have shown that from the given linear transformation how we can find its rank that is a rank of the linear transformation and nullity of the linear transformation. So, in the next lecture, we will also continue with this one. So, thanks for watching.

Thanks very much.