Matrix Computation and its applications
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Lecture - 31
Range space and null space of a linear transformation

Hello viewers. Welcome back to the course on Matrix Computation and application. So, today we are going to define some other terms related to linear transformation.

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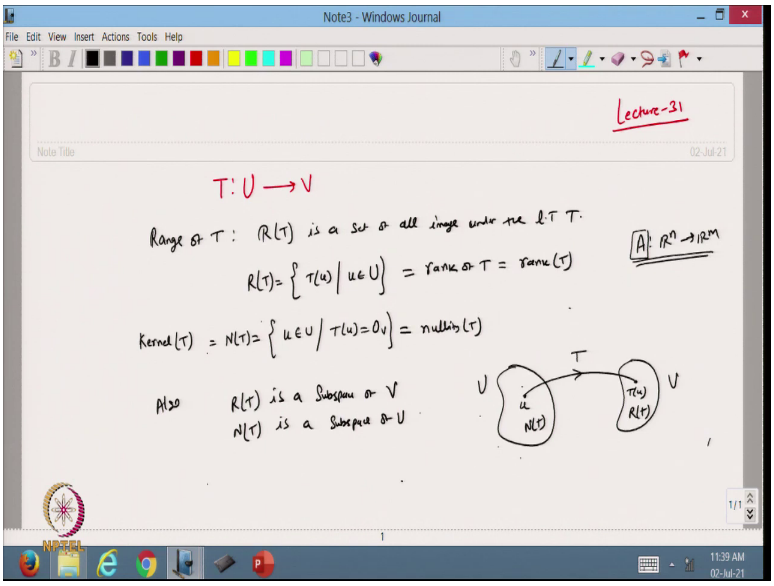

So, suppose we have a linear transformation from a vector space U to a vector space V; then in the previous lecture we have discussed how we can find the linear transformation from U to V, that is uniquely determined.

Now, we define some other terms related to the linear transformation, first one is range. Range of T is denoted as R(T) and we have already discussed the linear map from R^n to R^m. So, now, we know that what is range space of T. So, range space of T is the set of all images under the linear transformation T , i.e we can write it as

$$R(T) = \{T(u) \mid u \in U\} = \text{rank of } T = \text{rank}(T)$$

Next, we will define the null space or we can say kernel of T. We represent it by N(T) and is defined as

$$\text{Kernel}(T) = N(T) = \{u \in U \mid T(u) = 0_v\} = \text{nullity}(T)$$

So, here we have defined the rank and nullity of the matrix. Also we know that

R(T) is a subspace of V and

N(T) is a subspace of U.

Now consider a linear transformation $T : U \rightarrow V$ and we are defining the linear transformation T. So, this is my map u to T(u). Also, R(T) lies in V that maps to N(T) that will belong to the subspace U. So, we have already proved that these are the subspaces of U and this is the subspace of V.

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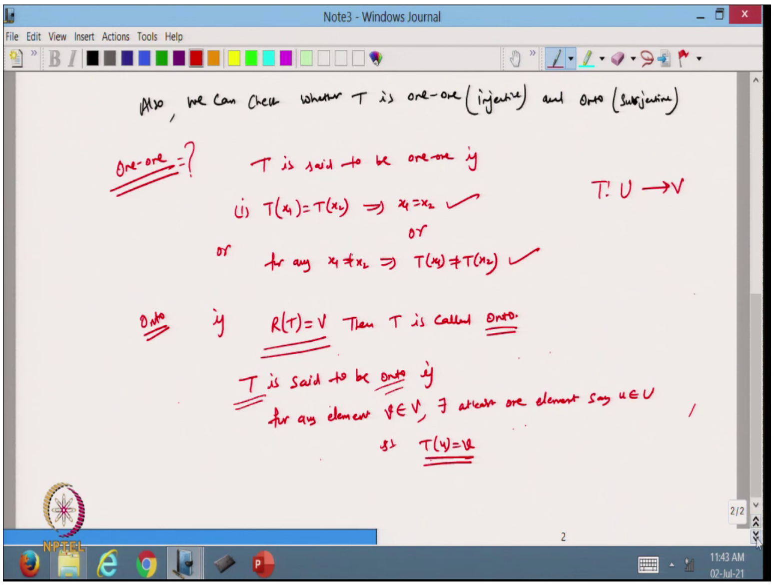

Now, we have to check whether T is one to one, that is injective and onto that is subjective.

So, we also know how to check one one. So, for this one we will say that the transformation T is said to be one one or injective, if $T: U \rightarrow V$

(i) $T(x_1) = T(x_2) \Rightarrow x_1 = x_2$

Or

For any $x_1 \neq x_2 \Rightarrow T(x_1) \neq T(x_2)$

So, these two conditions, either of the conditions I need to satisfy to check whether this is one one or not.

(ii) The second one is onto. So, for onto, if $R(T) = V$ then T is called onto. So, if I take the range space of T and that is equal to the vector space V; because we are defining T from U to V and if my range space is equal to V, it means that for any element there is a preimage in U, such that $T(u) = v$.

So, that is called the T onto. So, how to check? Now, we can say that T is said to be onto if for any element $v \in V$, $\exists$ at least one element say $u \in U$, such that $T(u) = v$.

Therefore, we say that this linear transformation T is onto. Now, I want to check how we can find this range space and null space when a transformation is given to us.

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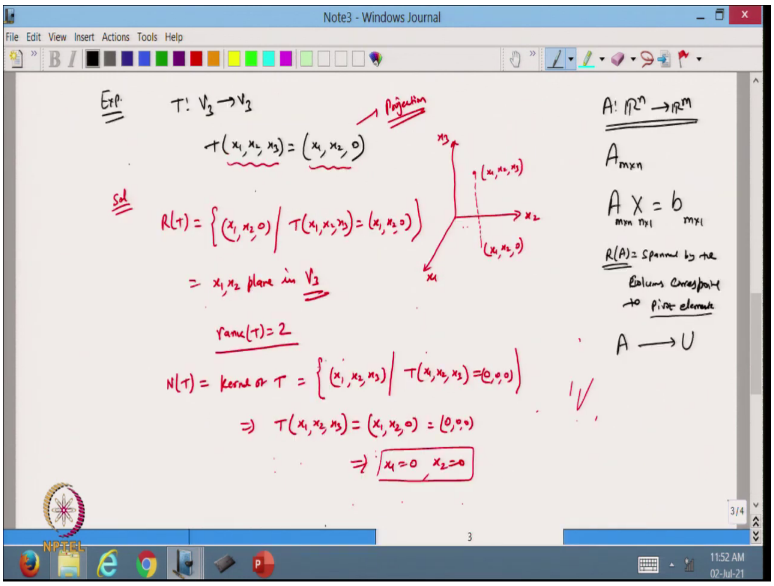

So, let us do one example because we already know that if we have a map $A: R^n \to R^m$. In that case we know how to find R(T). So, if you remember then my matrix will be $A_{m\times n}$. Because I need $A X = b$ so, we can write it as

$$A_{m\times n} X_{n\times 1} = b_{m\times 1}$$

Now, if you remember, we know that :

R(A) = spanned by the columns corresponding to pivot elements.

So, the columns which have the pivot elements, we take the corresponding columns in the given matrix, so that will span the range space of A.

So, this way we can define. And also we can define the null space the, we take the matrix A into the echelon form say, U and then the zero rows corresponding to the zero rows that rows span the null space, the n of the given matrix A. So, that is the way we can define these two .

Now, let us take one example. I have a linear transformation T: $V_3 \rightarrow V_3$ defined as

$$T(x_1, x_2, x_3) = (x_1, x_2, 0)$$

So, this is just the projection, because I am projecting the V_3 whole space into the plane x_1 and x_2 plane; this means I have an element suppose I take space V_3 space.

So, I can call it (x_1, x_2, x_3) . So, this is my xyz plane. Now, in this case I choose any vector element from here. So, that is (x_1, x_2, x_3) and I just take the projection on the xy plane and this element will be $(x_1, x_2, 0)$.

So, this is our projection transformation. We have already defined this as linear transformation. So, I want to find what is the range space of T. So, if you see from here, the range space of T is

$$R(T) = \{ (x_1, x_2, 0) \mid T(x_1, x_2, x_3) = (x_1, x_2, 0) \}$$

it means in this case my range space will complete the xy plane. So, here we can easily define that,

$$R(T) = x_1, x_2 \text{ plane in } V_3.$$

So, that will be my whole range space, because if you see for any value of x_1 , x_2 , x_3 ; it maps the same elements, except the x_3 =0. So, in this case I can say that, this is the plane in x_1, x_2 plane in V_3. So, that is my range space.

And I can say that the rank(T) = 2; because it is x_1, x_2 plane and which is of dimension 2, so its

$$\text{Rank}(T) = 2$$

Now, I will define the kernel of T,

$$N(T) = \text{kernel of } T = \{(x_1 , x_2 , x_3) \mid T(x_1 , x_2 , x_3) = (0 , 0 , 0)\}$$

Therefore, $T(x_1 , x_2 , x_3) = (x_1 , x_2 , 0) = (0,0,0)$

$$\Rightarrow \quad x_1 = 0 , x_2 = 0$$

and the third element is 0 = 0. So, I will get only two conditions from this one.

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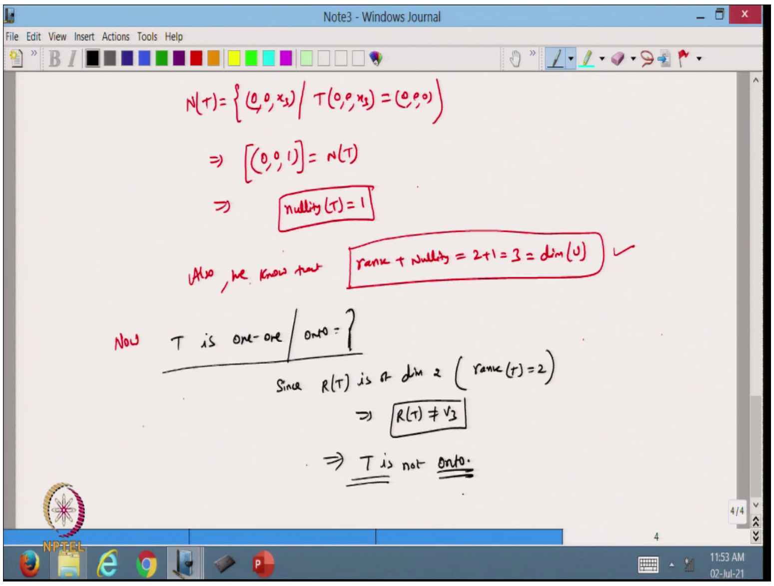

So, I can define my kernel of T as

$$N(T) = \{(0 , 0 , x_3) \mid T(0 , 0 , x_3) = (0 , 0 , 0)\}$$

Now, the basis for this will be (0,0,1) . So, I can say that the span of this (0, 0, 1) vector, that will span the whole kernel of T.

⇨ $[(0,0,1)] = N(T)$

⇨ $\text{Nullity}(T) = 1$

And also, we know that,

rank + nullity = 2 + 1 = 3 = dim(U).

I have not proved this theorem yet, the rank nullity theorem. We will prove in the coming lectures. So, from here I can say that : rank + nullity = 3 ,in this case. Now, I want to check whether this T is one-one or onto or not.

Since, R(T) is of dimension 2 (rank(T)= 2)

and the transformation $T : V_3 \rightarrow V_3$ so which implies that

⇨ $R(T) \neq V_3$

because that is a subspace of V_3 is not equal to V_3 in this case. It means, there are few elements in V_3 which are not in the range space of the given linear transformation T.

⇨ T is not onto.

So, from here I can say that my transformation T is not onto; because there are few elements in the V_3, which does not have the pre image in u, because my range space is just only this element.

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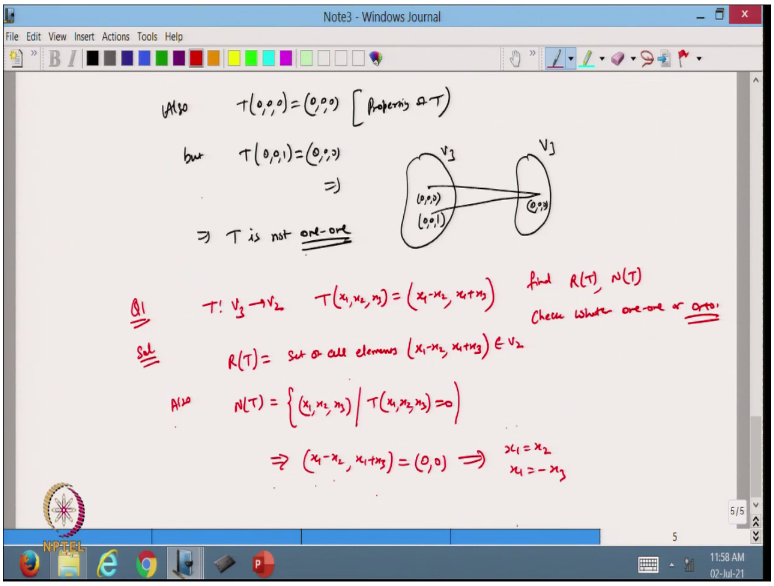

So, from here I can say that my T is not onto. Also, also I know that

$$T(0,0) = (0, 0, 0) \text{ [property of T]}$$

But,

⇨ $T(0,0,1) = (0,0,0)$

because it is coming under the null space.

It means if I take the mapping from V_3 to V_3. So, two elements are there that are (0, 0, 0) and (0, 0, 1), though they map to the same element that is (0, 0) element. As from here I can say that, T is not one to one. So, this way we can find out whether this linear transformation is one one, onto and then we can define its range and null space.

So, let us take one more example; suppose I define a linear transformation T: $V_3 \rightarrow V_2$ defined as

$$T(x_1, x_2, x_3) = (x_1 - x_2 , x_1 + x_3)$$

Now, I want to find the range space of T, kernel of T and check whether one one or onto. So, these things we need to check for this linear transformation. Now, how can I define the range space? So, range space is

$$R(T) = \text{set of all elements of the form } (x_1 - x_2\,,\, x_1 + x_3) \in V_2.$$

So, range will contain this type of element. Also kernel of T is

$$N(T) = \{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = 0\}$$

- ⇨ $(x_1 - x_2\,,\, x_1 + x_3) = (0,0)$
- ⇨ $x_1 = x_2\,,\;\; x_1 = -x_3$

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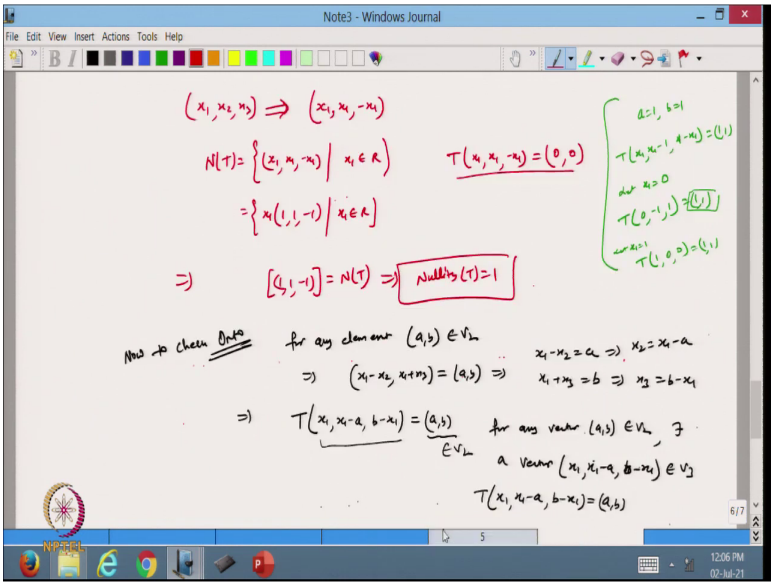

So, I can define my elements as

$$(x_1, x_2, x_3) \;\; \Rightarrow (x_1, x_1, -x_1)$$

So, from here I can define kernel of T as

$$N(T) = \{(x_1, x_1, -x_1) \mid x_1 \in R\}$$

$$= \{x_1\,(1,1,-1) \mid x_1 \in R\}$$

So, from here I can say that this vector (1, 1,-1) spans the whole kernel of T.

- ⇨ $[(1, 1,-1)] = N(T)$
- ⇨ Nullity(T) =1

So, this way we are able to define the range space and nullity. The rank will be of course 2. Now, the question is that I want to check whether it will be one to one and onto or not? How can we check this?

Now, onto means for any element, suppose I take the element I call it $(a, b) \in V_2$;

- ⇨ $(x_1 - x_2 , x_1 + x_3) = (a,b)$

because I just took one element (a, b) from V_2 and the elements of V_2 are of this form. Now, from here I get

- ⇨ $x_1 - x_2 = a \Rightarrow x_2 = x_1 - a$

  $x_1 + x_3 = b \Rightarrow x_3 = b - x_1$

Now, from here I can write that for T.

- ⇨ $T(x_1 , x_1 - a, b - x_1) = (a,b) \in V_2$

it means for any vector belonging to V_2, there is a vector V_3, such that T of that vector is equal to this one, ok. So, it implies that, for any vector $(a, b) \in V_2$, there exists a vector, at least one vector; that is $(x_1 , x_1 - a, b - x_1) \in V_3$, such that

$$T(x_1 , x_1 - a, b - x_1) = (a,b)$$

- ⇨ T is onto

Now, you can choose any element from here; for example, choose a = 1 , b=1. So, what is going to happen in this case? In this case

$$T(x_1 , x_1 - 1, 1 - x_1) = (1,1)$$

Let $x_1=0$, $T(0,-1,1) = (1,1)$

let $x_1=1$, $T(1,0,0)= (1,1)$

So, from here you can check that for any element in this case, I am able to get at least one element that mapping.

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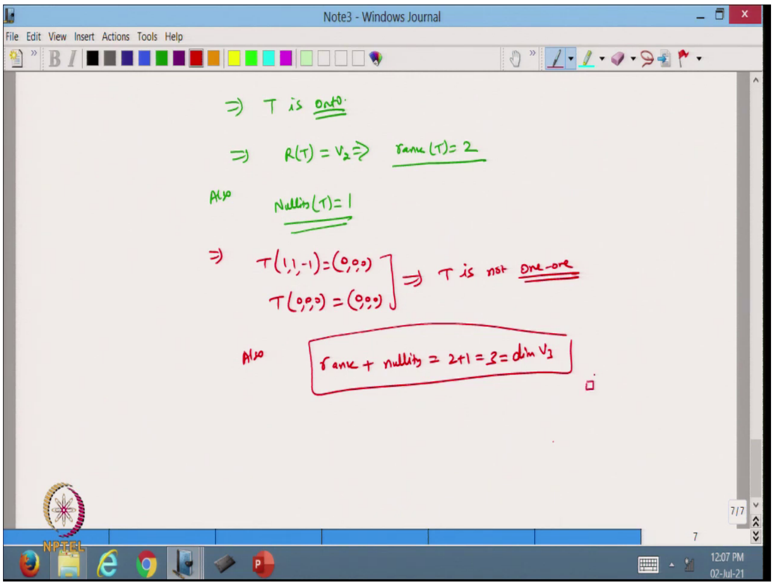

So, from here we can say that,

- ⇨ T is onto.
- ⇨ $R(T) = V_2$
- ⇨ Rank(T) = 2
- ⇨ Also , Nullity(T) = 1
- ⇨ $T(1,1,-1) = (0,0,0)$

  $T(0,0,0) = (0,0,0)$
- ⇨ T is not one-one

Because two different elements map to the same element in this case. So, from here I can say that T is not one to one. So, T is onto, but not one to one. From here you can check

$$\text{rank} + \text{nullity} = 2 + 1 = 3 = \dim (V_3)$$

So, if you see from here; we are defining the linear map from V_3 to V_2, so it is equal to the dimension of V_3. So, there is some relation coming up about the rank and the nullity and the dimension of the space U, from where we are defining this linear transformation.

Also, we have shown how we can check whether the linear transformation is one one ,onto and now we define the range space and the nullity of the given transformation.. So, let me stop here today. So, today's lecture, we discussed the range space and the null space, the kernel of the given transformation T and then also how we can check whether the given transformation is a one one or onto or not and we have discussed a few examples based on this one. So, in the next lecture, we will continue with this one. So, thanks for watching

Thanks very much.