Matrix Computation and its applications

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Lecture - 30

Determining linear transformation on a vector space by its value on the basis element

Hello viewers. So, welcome back to the course on Matrix Computation and its application. So, in the previous lecture we have defined how we can find a linear transformation from a vector space to another vector space. So, in this lecture we will continue with that one.

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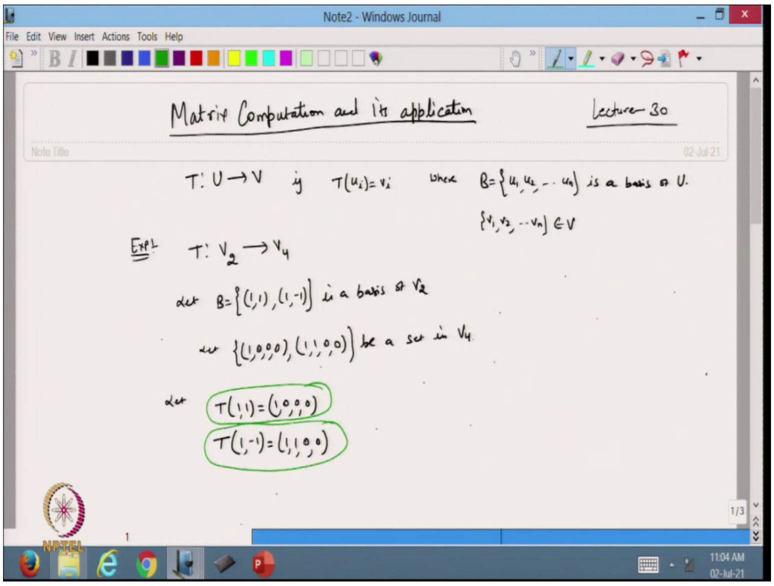

So, now in the previous lecture we have discussed that we can define a linear transformation from a vector space to another vector space V, if we are able to define $T(u_i)=v_i$ where I have a basis B that is made up of $\{u_1,u_2,\ldots,u_n\}$ and another set is $\{v_1,v_2,\ldots,v_n\}$ that belongs to the V and this is a basis of the vector space U.

So, let us do one example of how we can find out. So, let us take one example suppose we have a we wanted to find the linear transformation T from maybe suppose I want to take from $T:V_2 \rightarrow V_4$. I know that V_2 is the vector space of two dimensions and it is a four dimension and I want to define the linear transformation from this.

Now, to know the linear transformation we need to find we need to check the basis. So, suppose. So, for this one now let the set B I take as the basis 1, 1 suppose I take and 1 minus 1. So, these are the basis of V_2. Of course, these are linearly independent and 2 in number. So, definitely it is a basis. Now once I know the basis I need to find a set of vectors also. So, let me take the vectors. So, maybe suppose I just take $\{1,0,0,0\}, \{1,1,0,0\}$.

Now, I need to find the linear transformation. So, what I need to do is that. So, for finding the linear transformation I will say that let my be a set in V_4. So, let's assume that $T(1,1)=(1,0,0,0)$ and $T(1,-1)=(1,1,0,0)$. So, from here now I need to find a linear transformation.

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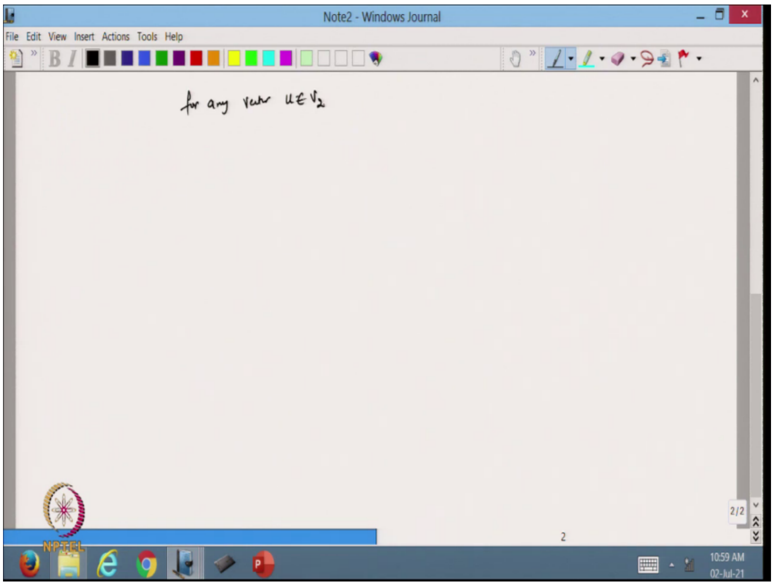

So, for this one now for any vector suppose I take the vector u belongs to for any vector belongs to V_2 or maybe I just write any vector maybe I will define x belongs to V_2 we can write.

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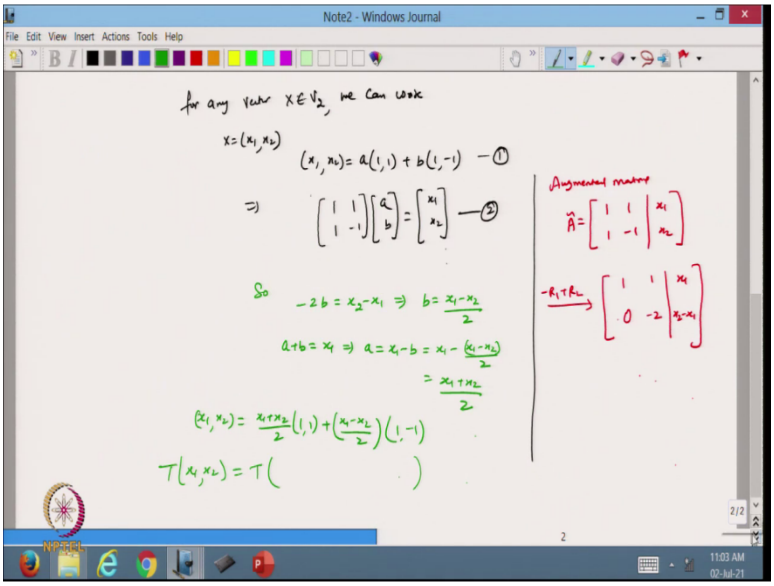

So, suppose x is basically supposed to be $x=(x_1, x_2)$. Now I can write this as (x_1, x_2) is equal to some linear combination of the basis a(1,1)+b(1,-1). Now from here I need to find the value of a and b. So, this is my linear combination. So, you know from here that I will get a system of equations with this vector as the column vector and this one a b and that is going to be equal to (x_1, x_2).

And this is a system of equations and this system is going to give us a unique solution because it is the basis. So, I know I can solve this one. So, this is the way generally we can write that its augmented matrix is represented by A tilde.

So, this will be basically $[1 \quad 1\,1 \; - \; 1\,][a\; b\,] = \left[x_1\; x_2\right]$-----(2). Now I need to convert this matrix into the row echelon form. So, I can write this matrix as $-R_1+R_2$. So, I will get (1 1) x_1 now I multiply by -1 and add. So, it will be (- 1 2)(x_2-x_1). Now, from here you can see that based on this one we can write. So, we can write that $-2b=x_2-x_1$. So, that gives me that b= $\frac{x_1-x_2}{2}$. Also, $a+b=x_1$ which gives me that $a=x_1-b$ that gives me a=$\frac{x_1+x_2}{2}$. Now, I am able to find the value of a and b. So, from here I can write vector (x_1,x_2)=$\frac{x_1+x_2}{2}$(1,1)+$\frac{x_1-x_2}{2}$(1,-1). Now after doing this, this is the linear transformation or linear combination equal to this one and this is a unique one.

So, now, I can define my what will be the T of any vector belonging to V_2. So, it will be written as from the. So, it can be written as T of the whole values, the whole vector or the whole linear combination and this can be written as T (x_1,x_2).

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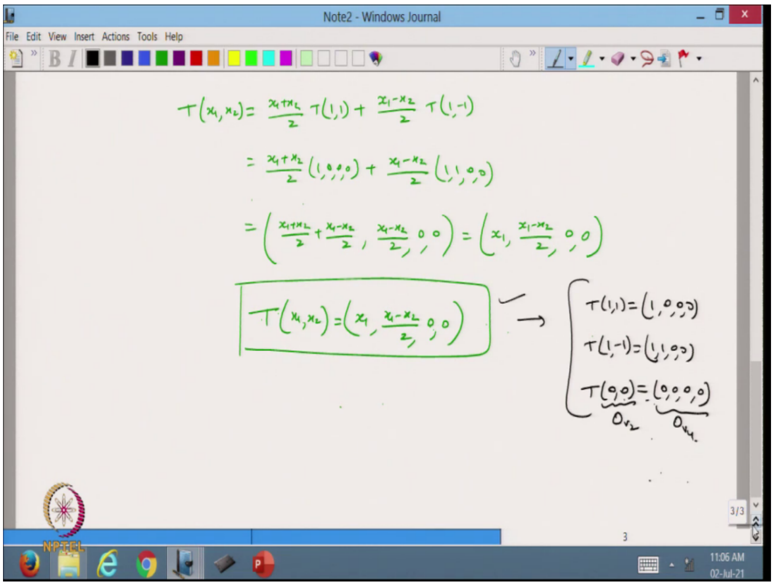

So, I can write this as $T(x_1,x_2)=\frac{x_1+x_2}{2}T(1,1)+\frac{x_1-x_2}{2}T(1,-1)$.. So, T(1,1)=(1,0,0,0) and T(1,-1)=(1,1,0,0) and T(0,0)=(0,0,0,0). So, now we need to use these two values.

So, suppose I have the vector (1, 0, 0,0) and (1, 1, 0, 0) now I can write this as it is $\frac{x_1+x_2}{2}$(1, 0, 0,0) +$\frac{x_1-x_2}{2}$(1, 1, 0, 0)=(x_1,$\frac{x_1-x_2}{2}$,0,0). So, this is my linear transformation. So, from here I can write that $T(x_1,x_2)=(x_1,\frac{x_1-x_2}{2},0,0)$. So, this is my linear transformation and this transformation if you see this will be unique also because we are taking the basis and the vectors. So, this is my image I have defined like this one.

So, this transformation will be unique in this case and now from here you can verify whether this transformation is satisfying the given condition or not. So, from here you can check that T(1,1) will be what. So, if you see from here it will be 1 and then it will be T(1,-1)=(1,1,0,0). So, you can check T(0, 0) where it is going?

So, if you see from here it is going to (0, 0, 0, 0). So, it is the 0 element of V_2 and this is the 0 element of V_4. So, it is my linear transformation that will be unique based on this one. So, the same way we can define now based on this one I can define another linear transformation.

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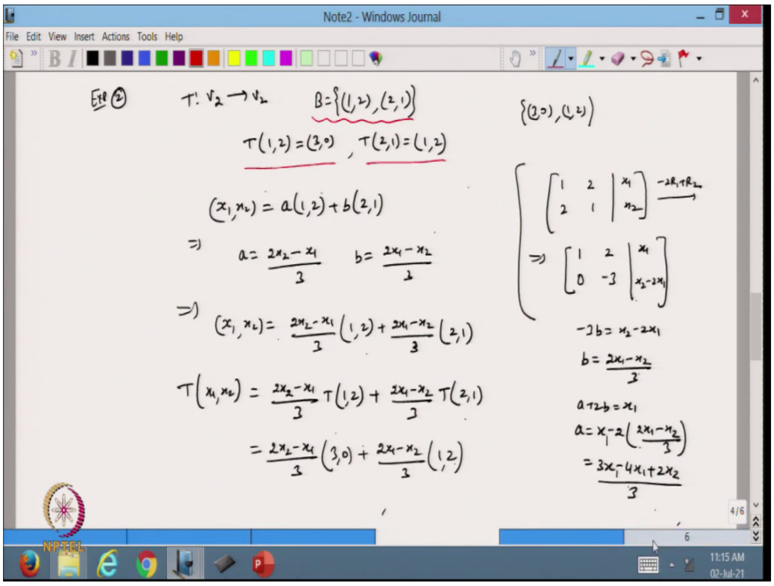

So, example 2 now suppose I wanted to find the linear transformation T from V_2 to V_2. Now the same way I need a basis. So, suppose I take {(1, 2), (2, 1)} and the vector I am defining. So, let us define T(1, 2)= (3, 0) and T (2, 1)=(1,2). Suppose I take this one. So, I am taking another set of vectors that is (3, 0) and (1, 2).

So, this is the vector I have taken. Now the same way I want to define the linear transformation T. So, now, from here I can define any vector from V_2. So, this is basically a linear independent because it is the basis. So, I can define from here that $(x_1,x_2)=a(1,2)+b(2,1)$. Now from here again I can write in this form.

So, maybe from here I can define that this can be written as (1, 2), (2, 1) and this is x_1 and x_2 this is my augmented matrix and after that I can use $-2R_1+R_2$. So, it is $\{(1,2),(0,-3)\}(x_1,x_2-2x_1)$. So, this will be my echelon form and from here you can check that my b. So, $-3b=x_2-2x_1$. So, b $=\frac{x_2-2x_1}{-3}$. Further, I can write from here and my $a+2b=x_1$. Since, b $=\frac{x_2-2x_1}{-3}$ and that will give you here a= $\frac{3x_1-4x_1+2x_3}{3}$.

So, this way we can write now from here I can. So, from this value now I can write from here that my $(x_1, x_2) = \frac{(2x_2 - x_1)}{3}(1,2) + \frac{2x_1 - x_2}{3}(2,1)$. So,T $(x_1, x_2) = \frac{(2x_2 - x_1)}{3}T(1,2) + \frac{2x_1 - x_2}{3}T(2,1)$.

Then, So,T $(x_1, x_2) = \frac{(2x_2 - x_1)}{3}(3,0) + \frac{2x_1 - x_2}{3}T(1,2)$. So, if you see from here that a and b are just the opposite side whatever the a I have taken the b will be just the opposite of that one. Now based on this one I can find out the complete linear transformation.

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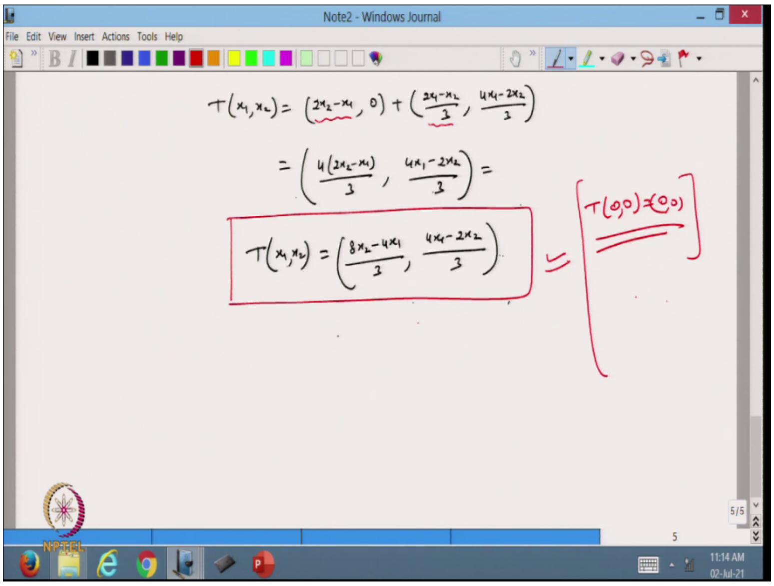

So, I will write this and it will become $T(x_1, x_2) = (2x_2 - x_1, 0) + (\frac{2x_1 - x_2}{3}, \frac{4x_1 - 2x_2}{3}) = (\frac{4(2x_2 - x_1)}{3}, \frac{4x_1 - 2x_2}{3})$ that I can write that $T(x_1, x_2) = (\frac{8x_2 - 4x_1}{3}, \frac{4x_1 - 2x_2}{3})$. So, now, this is the way we can define our linear transformation and after defining this linear transformation you can verify with the given values you can verify whether it is coming or not or maybe also you can verify that (0, 0) now it is going to be (0, 0).

So, that is verification; one of the verification I have done. So, this is my linear transformation from V_2 to V_2. Now after doing this one what will happen if we do not have the complete basis? As we are already given in this case, these basis are given to me and then we have defined the linear map.

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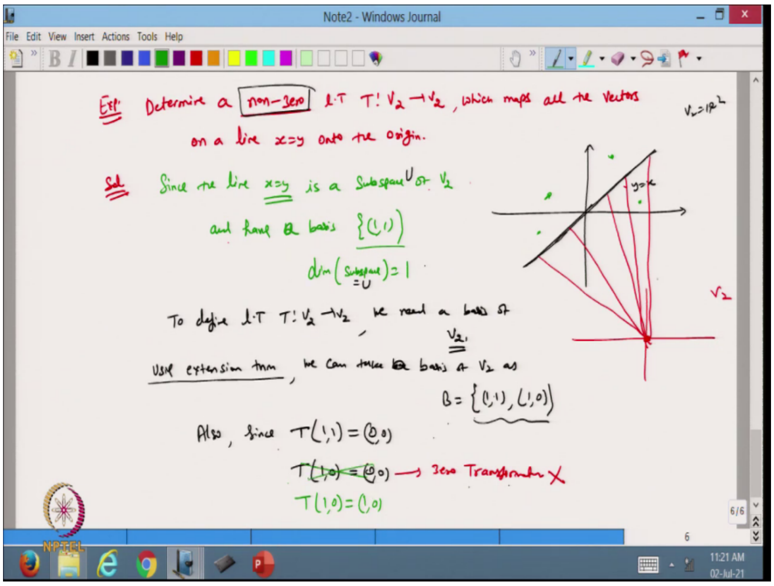

Now, suppose I have a question about the example that determines a non-zero linear transformation T from V_2 to V_2 which maps all the vectors on a line. So, line I am taking x = y on to the origin. So, this is my question now: what is going on here in this case. So, if you see from here it is V_2 to V_2.

So, suppose I have a V_2 , basically R^2. So, V_2 is. So, in this case I have defined the line. So, this is my line y =x. So, it wants a transformation which maps all the points on this line to suppose this is my another V_2. So, I want this in my origin in this case. So, all the points should map to here on this line.

So, this transformation is needed and it also wants to be non-zero. I cannot define the 0 transformation that all the points from V_2 to V_2 map to 0 element. So, the question is that now I have a point here. So, this is one other point, one another point maybe another point another point what about those points? Because to find the linear transformation I need all the elements from V_2 to V_2 only then we are able to find the linear transformation.

So, in this case now I can write that since the line x = y. So, this is a line I know that is a subspace of V_2 and we know that the basis of V_2 and having a basis. So, I can take the basis as (1, 1) because if you take the (1, 1) basis it will map and it will generate all the elements on this line. So, this is the basis and I know that the dimension of this subspace whatever the space subspace is is 1.

Now, since we need to find the linear transformation from V_2 to V_2. So, to define linear transformation T from V_2 to V_2 we need a basis of V_2. So, for this 1 what do I do? I use the extension theorem. So, using the extension theorem. So, subspace I call it maybe I will call it subspace u.

So, the subspace is I am taking a u. So, by the extension theorem we can take a basis of V_2 . So, what should I do? I will take the basis (1 1) that is already there of this vector space vector subspace u and then maybe I can define another vector that should be linearly independent. So, I just take the simplicity I will just take (1, 0) that is it.

Because I know that this is linearly independent of each other. So, I can define this basis. So, we can take this basis of V_2 and I call it basis B. Also now we need to define the image of each element in the basis.

So, also since T(1,1)=(0,0) because that is already there we need to find out the transformation which maps all the vectors on the line to the origin. So, that should be there now suppose if I define T(1, 0) =(0, 0) then this will give you 0 transformation but we need non zero transformation. So, this is not possible.

So, what I need to define is that. So, I will define my T (1,0) as equal to any other element. So, maybe I can (1, 0) go to the same element. So, T (1,0), I just take 1, 0. So, this is not possible.

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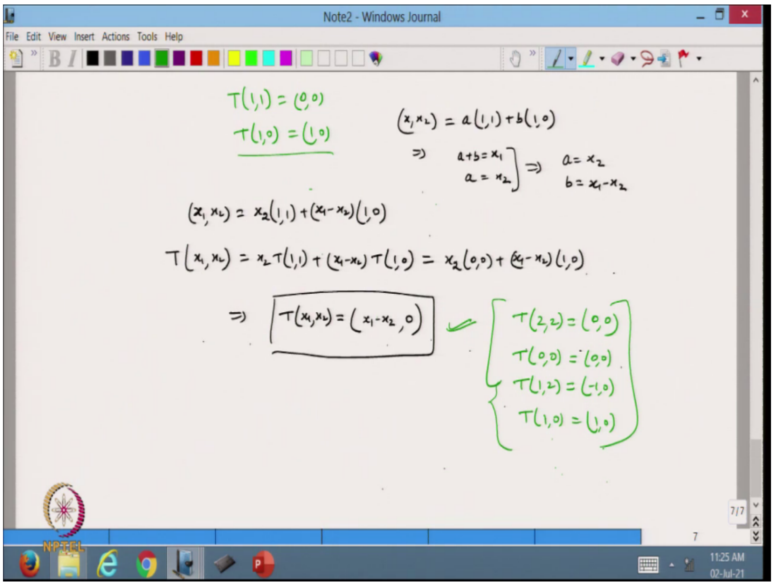

So, after defining this one now I have a T(1, 1)=(0,0) and T(1, 0) that maps to (1 0). You can take any other element. Maybe I can define T(1, 0) goes to (0, 1) or maybe T (1, 0)goes to (1, 1) any vector I can take. So, I have defined this value. Now from this I can define my $(x_1,x_2)=x_2(1,1)+(x_1-x_2)(1,0)$. Now I define the transformation $T(x_1,x_2)=x_2T(1,1)+(x_1-x_2)T(1,0)$. and T (1, I) have defined (0, 0) and T(1,0)=(1,0). So, it will be $x_2(0,0)+(x_1-x_2)(1,0)$. So, if I take the first component it will be x_1-x_2 and another component will be 0. So, this is my linear transformation.

Now, we want to check whether it is satisfying the given condition or not. So, from here you can verify that now from here I can verify that T of any element I suppose I take (2, 2). So, it will go to (0, 0) all the elements on this line going to this one. What about if I take T(0,0)? Then definitely here it is putting 0. So, it is going to (0,0) what about if I take T(1, 2)? So, it will be T(1, 2)=(-1,0) and T(1,0)=(1,0). So, this is the way we have defined it.

So, from here you can check that all the elements on this subspace this line will map to 0 and all other elements we have defined in this way. So, that is satisfying. So, this is a linear transformation which is moving from V_2 to V_2 and this is a nonzero transformation which maps the whole subspace into the 0 subspace. So, this way we can define the linear transformation. So, that is the way we can define the linear transformation here. Now suppose I take another example.

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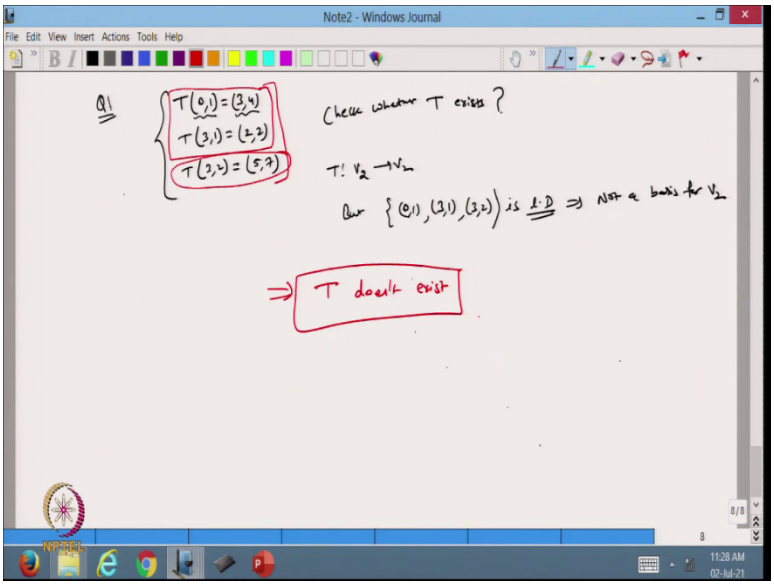

Suppose I have a transformation like this one T(0, 1) that goes to the supposed element (3, 4) T(3,1)=(2, 2); T(3, 2)= (5, 7). So, the question is whether any linear transformation exists in this case or not.

So, now from here check whether T exists. So, now, from here you can see that I need a transformation basically from V_2 to V_2 because this is an element coming from the V_2 and this is also an element of V_2. So, I need to find that transformation this one, but we know that the set {(0, 1), (3, 1), (3, 2)} is a basis is this set is linearly dependent because we know that if it V_2 then the vectors more than 2 will be always be need dependent. So, not a basis for V_2.

So, in this case I can say from here that from here I can say that T in this case does not exist. So, I cannot define a linear transformation from V_2 to V_2 which satisfies this condition. So, this way we cannot define because in this case what is going to happen? Suppose somebody says that we can take the first two elements here and then we can define the linear transformation that may be possible, but in that linear transformation it may happen that this is not satisfying; the third condition is not satisfying.

So, if the third condition is not satisfying, then we will say that the limit of this linear transformation does not exist in this case. So, because here it is needed that T should satisfy all this condition and that is not possible. So, T does not exist in this case. Now, after doing this one, let me stop here.

So, in today's lecture we discussed how we can define a linear transformation based upon the given set of basis and set of vectors in the image space that is from u to v and we have done a few examples based on that one. So, the next lecture will also continue with this one and define some other properties of the linear transformation. So, thanks for watching and.

Thanks very much.