Matrix Computation and its applications

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Lecture - 03
Some examples of vector spaces

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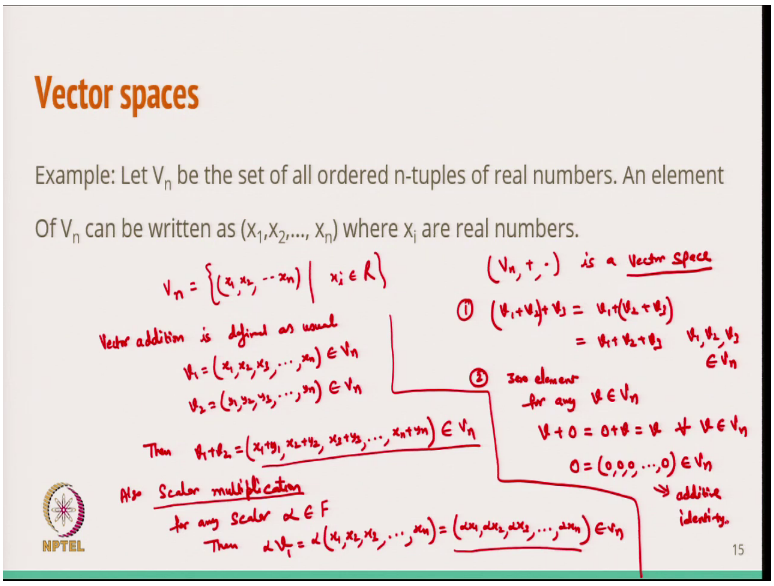

Hello viewers, welcome back to the course on Matrix Computations and its applications. So, in the previous lecture, we have discussed about that the definition of vector spaces, and we have started with the n-tuples Vn, so, we will continue from that one.

So, in the previous lecture, we have started with the fact that Vn is the n-tuple.

So, Vn ={x1, x2, …, xn} where each xi belongs to a real line. Now, we want to discuss or show that this makes the vector spaces, it means that we have to satisfy all the properties are related to the vector spaces.

So, first of all, I will define that what is the vector addition and scalar multiplication in this case, so, we define the vector addition. So, vector addition is defined as usual, as usual means suppose I take a element v 1 suppose this v 1 is a element and I represent that element by x 1, x 2, x 3 and x n and I take the another element v 2 and I represent by y 1, y 2, y 3, y n, so, this I take from V n this is also I am taking from V n, then I define v 1 plus v 2.

So, this is the same way we used to add two vectors. So, we are adding component wise, so, that is x 1 plus y 1, x 2 plus y 2, x 3 plus y 3 and so on x n plus y n. So, this is we call it the vector addition we are taking in the case of V n and we that is why we call it the usual because this is the way we used to take for addition of two vectors.

Also, the scalar multiplication, so, scalar multiplication I am defining that if I take a scalar, so, for any scalar alpha and we know that alpha comes from the field, whatever the field we are taking in this case, so, in this case, we are taking the real numbers. So, alpha is coming from the field then I define alpha v or suppose alpha v 1.

So, in this case, I am taking alpha and then, v 1 I have taken x 1, x 2, x 3, x n and we know that for a given vector, if we multiply by the scalar, then we can define take the scalar inside and that becomes alpha x 1, alpha x 2, alpha x 3, alpha x n and this also belongs to V n, in this case also, this also belongs to V n.

So, from here that this is the way we are taking the scalar multiplication, and this is the way I am taking the vector addition. So, based on this one, we want to check that V n so, with addition and scalar multiplication just I have defined so, I, we want to show that is a vector space.

So, for this one, I need to satisfy all the properties, so, I will show you that few properties that how we can define, but the 1st property is that the associativity. So, that is very easily we can define because it is just the vector addition I am taking. So, we can take v 1 one vector, v 2 and v 3 and then, I am writing as this one. So, I can define this one is equal to v 1 plus v 2 plus v 3 this one and from here, we can write that v 1, v 2 and v 3.

So, this is we are defining for all v 1, v 2, v 3 belongs to V n where v 1 I have defined like x 1, x 2 like this one and v 2 is y 1, y 2 and v 3 I can take another one like z 1, z 2, z 3 up to z n, so, that is also we can take in the another form. So, this is very easily we can satisfy. So, this property is satisfied.

The second one is that now, we want to define that after the associativity, the zero element. So, zero element means for any vector v belongs to the V n, if there exist v so, I will take only v plus 0 element, then it should be equal to 0 plus v and then it should be equal to v, so, this is should be true for all v belongs to the space.

Now, the question is what is the 0 element here. So, we can define the 0 element. So, 0 element here is can be taken as element with all the component 0 and we know that this belongs to V n because 0 is already there in the real number and we are taking all the set which take all the real numbers.

So, this is belongs to the V n. So, this is the 0 element and by the vector addition, we can see that v plus 0 or 0 plus v that will be equal to the vector v again for all v belongs to V n. So, in this case, this is called the 0 element and we also call it additive identity.

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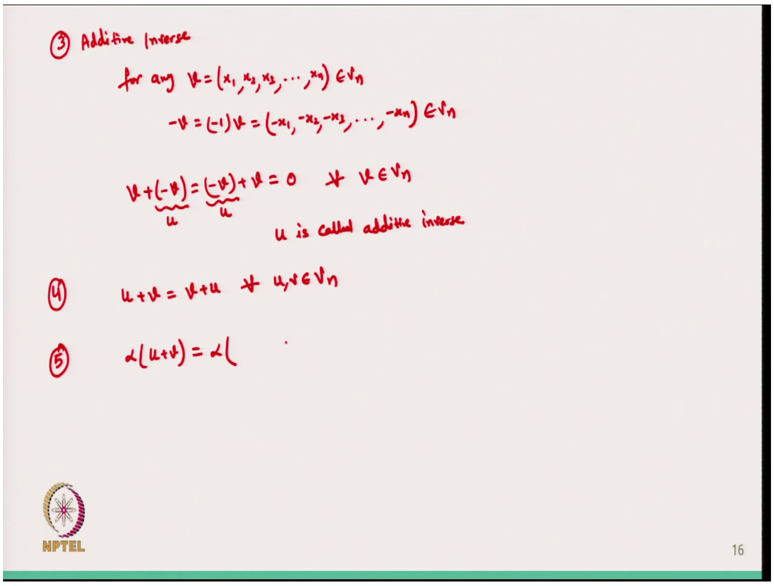

So, after this additive identity, now next property I want to define for the additive inverse. So, in the additive inverse, for any vector v, so, v I am just now taking x 1, x 2, x 3, x n. So, we could because we can give the name, so, just taking v as this one and that belongs to V n because earlier, I have taken v 1, but now I am just taking that for any v belongs to this one or.

Now, we can define a vector minus v. So, in the minus v, I am just what I am doing is that I am taking minus 1 multiply by v and this will be equal to because I have already defined the scalar multiplication, so, I can write this as a minus x 1, minus x 2, minus x 3, minus x n that is also belongs to V n because this way we have defined the scalar multiplication.

So, from here, I can say that v plus this vector or maybe minus v plus v it can be written as 0 where 0 is additive identity. So, this is true for all v belongs to V n whatever the element you take, we can take its minus of v the additive inverse and then, this will come equal to 0. So, in this case, this element we call it u, u, so, u is called additive inverse ok.

So, after this, the 4th property we can define is that for any element u plus v, it can be shown that this is equal to v plus u, this is true for all u, v belongs to V n. The same way I can choose a element u and v and then, it is made up of real numbers so, compound wise addition I can do and then, we can just change the position of the elements so, it become the commutative. So, these 4 properties are satisfied from here, I can say that this is a commutative group under addition.

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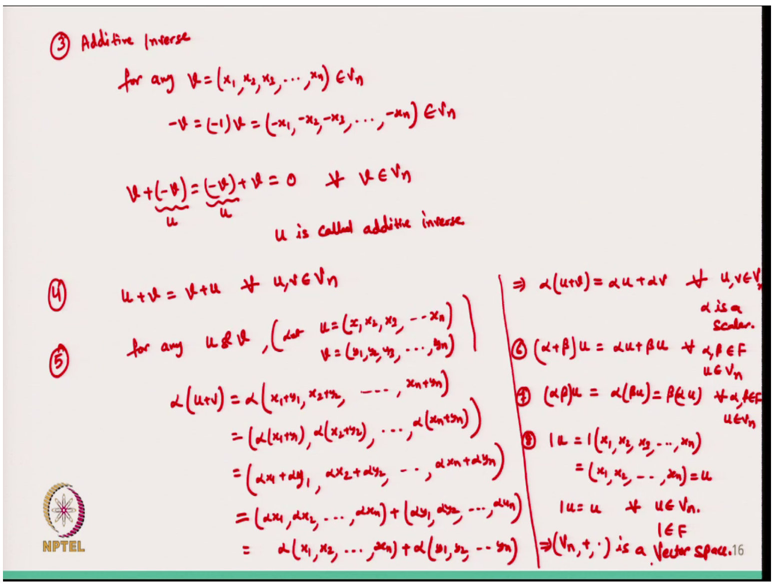

Then, the 5th property comes is distributive property. So, I can from here, I can show that for any alpha and u plus v so, I can define my alpha I am taking here. So, suppose u is some element there, so, I just take the element u for any u and v. So, let us say we take u is equal to maybe x 1, x 2, x 3, x n it can be any element, v I am taking y 1, y 2, y 3, y n ok so, these two elements I am taking, then alpha u plus v.

So, this can be written as alpha and u plus v I can write as x 1 plus y 1, x 2 plus y 2 and x n plus y n, this I can take with the vector addition and then, now I apply the property of scalar multiplication. So, this will become x alpha x 1 plus y 1, alpha x 2 plus y 2 and alpha x n plus y n.

Now, this alpha into x 1 plus y 1 these all are the real numbers and in the real numbers, I know that we can write this as alpha x 1 plus alpha y 1, alpha x 2 plus alpha y 2, alpha x n plus alpha y n. So, this one I can write and now, I can write this as alpha x 1, alpha x 2, alpha x n plus alpha y 1, alpha y 2, alpha y n.

So, I am just separating the vectors because if we take the addition, it will I will get going to get the same thing and this can be written as alpha. Now, I can take the alpha common and from here, I can write like x 1, x 2, x n plus alpha y 1, y 2, y n and from here, I can write from here that alpha u plus v now can be written as alpha u plus alpha v and this is true for all u and v belongs to V n and alpha is a scalar and scalar is always coming from a field.

So, this property, the 5th property is satisfied, the same way the 6th property I can satisfy that alpha I am taking the scalar plus beta another scalar and if I write u, then the same way I can show that this is equal to alpha u plus beta u for all alpha, beta belongs to the field and u belongs to the vector space V n. So, this property the same way we can define.

Then the 7th one we can define as alpha beta u, so, here I can write this form as alpha beta u or I can write beta alpha u this is true for all alpha, beta belongs to the field and u belongs to the V n. So, this properties is also just the scalar multiplication we have defined by that way we can verify.

And then, now last one is that if I take a 1 element from the field and multiply by u, so, 1 and u suppose it is I am taking x 1, x 2, x 3, x n and this can be written as by the scalar multiplication, it can written as x 1, x 2, x n and that is equal to u. So, from here, I can write that 1 q is equal to u and that is true for all u belongs to the and 1 is coming from the field F. So, this property is also satisfied.

So, from here I can say that V n under the usual addition and scalar multiplication is a vector space because it satisfy all the properties, so, based on this one, I can say that this is a vector space.

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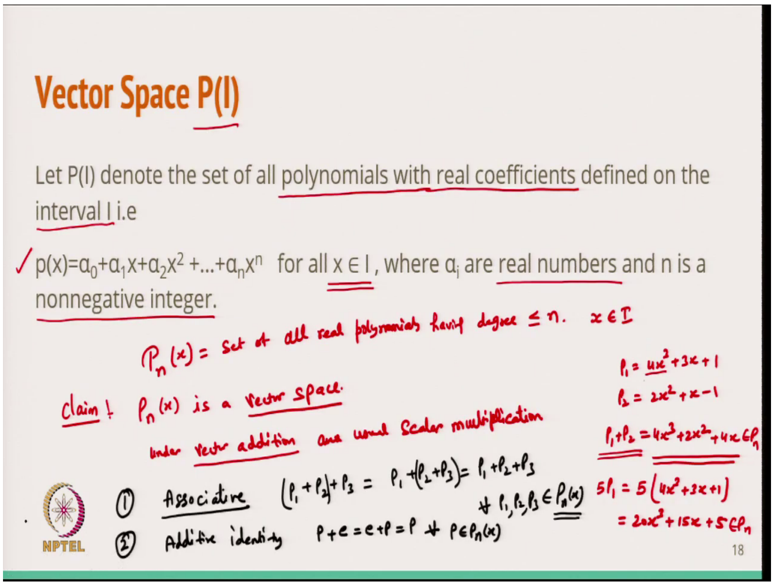

Now, so, after doing this one, we can define another vector space.

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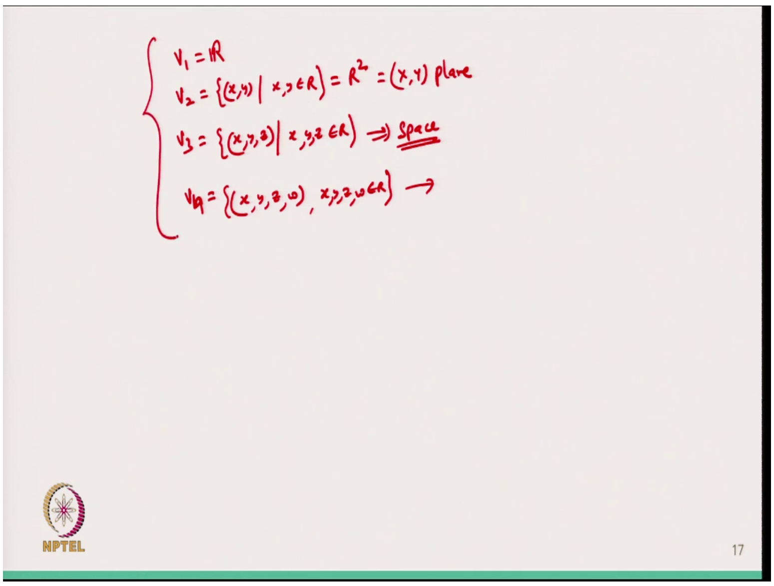

So, like before that, I just want to give example that V n you must have seen that in this case, V 1 will be a real line, the whole real line, V 2 will be just containing the two element so, it will be a type of x 1 sorry you can write this as ok this will be x, y all the element x, y where x and y belongs to real number.

So, this will be equal to basically R square, the x, y plane and that is x, y plane, V 3 is again the element x, y, z all the elements triplet and x, y, z; x, y, z belongs to the a real line, so, it is the space 3D space. So, this three we know, then we can define V 4 or V 4 I can define as the element x, y, z and w the same way ok.

So, this is the 4th dimension, 5th dimension and then, we can define the nth dimension. So, this way, these are the few examples we have taken for V n. Now, after that we define the another type of vector space and we call it the P I. So, what is P I? So, let P I denote the set of all polynomials with real coefficients defined on the interval I.

So, now, we are taking all the polynomials with real coefficients and that is defined on the interval I so, that is a set we are taking. For example, I take a polynomial like this one p x so, alpha naught plus alpha 1 x plus alpha 2 x 2 x square and so on up to alpha n x n. So, in this case, this is my polynomial, and I am taking my x coming from the interval I where this coefficient alpha i's are the real numbers we are taking, and n is a nonnegative integers.

So, in this case, I can say that this polynomial belongs to the set P I. So, now, from here, I can define a set P n x and this one I can take the set of all real polynomials. Real polynomials means the coefficients are coming the real, set of all the polynomials having degree less than or equal to n.

In this case, we have a degree n and here, I am defining P n x a set of all the polynomials whose degree is always less than equal to n and I am taking this one and x we take that belongs to some interval I. So, this is the set I have taken. Now, my claim is that so, this is my claim that P n x is a vector space ok.

So, the question is that if it is a vector space, under which operation they are vector space. So, now, I am defining that it is vector space under vector addition. So, vector addition means, it is a usual vector addition that how the two polynomials are added and usual scalar multiplication.

For example, I know that if I take a polynomial suppose I take polynomial P 1, just take 4x cube plus 3x plus 1 and I take P 2 is 2x square plus x minus 1. Now, we know that if I add the polynomial P 1 and P 2, then the terms with the same power will add, so, it will become 4x 3 cube I have taken, then x square is coming from here.

So, 2x square and then, I am taking the, collecting all the terms corresponding to the x so, it will be 4x and this will cancel out so, that will be the addition, usual addition of the polynomials. Also, I want to define suppose 5P 1. So, 5P 1 is means that scalar multiplication so, 5 is coming from the real number and now, I multiplying by the polynomial 4x cube plus 3x plus 1 and from here, it becomes 20x cube plus 15x plus 5. Now, the question is that vector this addition is valid or not?

So, you can see that this also belongs to the polynomial P n, this also belongs to the polynomial P n ok. So, this is the way two polynomials can be added and this is the way we can have the scalar multiplication with the given polynomial. So, from here, I have defined this operation so, that is usual addition and usual scalar multiplication.

So, the question is that whether it is a vector space or not? So, now, I will we can verify, or we can check all the properties whether it is satisfying or not. So, in this case so, let us check the properties. Now, so, the binary operation is already there now, I just want to check that it is a vector space or not so, the 1st property is associative property.

So, in the associative property, we know that if I take a polynomial P 1, I just represent by P 1 so, P 1 can be this so, it is a function of x so, just I am writing p 1 that is it, it is understood that it is a function of x I am taking. So, P 1 I am taking one polynomial and another polynomial I am taking P 2 and another polynomial I am taking P 3 and I am doing the vector addition.

So, this one if I add here or P 1 plus P 2 plus P 3 or P 1 plus P 2 plus P 3 so, if I add here first and then, add it to P 3 or add here and then, P 1 and then, add it to this one so, all are same for all P 1, P 2, P 3 belongs to the P n x ok. So, this is true because we can verify from here that does not matter whether we add P 1, P 2 and another P 3 or P 2, P 3 first and then, P 1.

This this we can verify just by the taking three different type of polynomial of degree less than equal to n and then because the coefficients are real so, the same power of x will be added. The 2nd one is that I want to take the additive identity. So, the question is again comes that what is the additive identity here?

It means I need a element e such that if I add to some P or e plus P, then I should get P and it should be true for all P belongs to the polynomial set this one.

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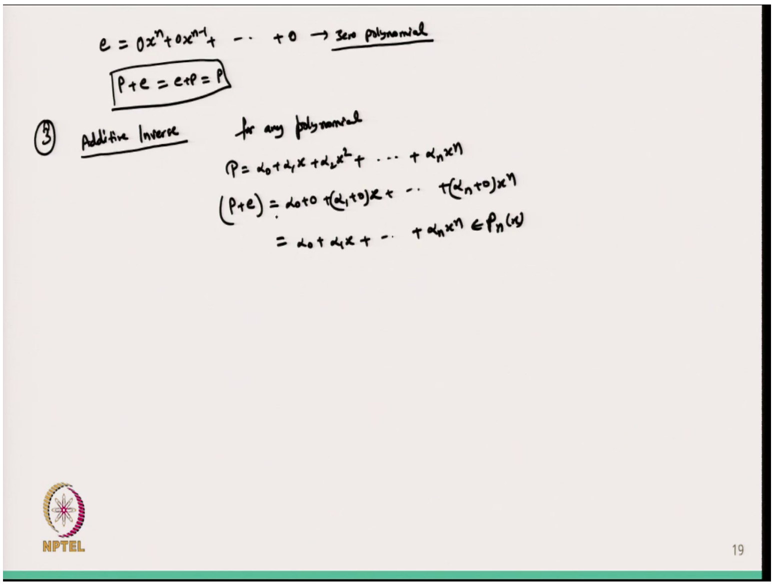

So, now, from here, I need to find what is my e. So, now, you can from the previous knowledge of polynomials, I can define a polynomial 0 x n 0 x n minus 1 and so on up to 0. So, it is a polynomial with all the coefficient 0. So, this polynomial is called zero polynomial. Zero polynomial means all the coefficient in this case are 0 and we can write that this is my additive identity.

Because if I choose any polynomial P plus e, then the coefficient will add up and then, you can verify that this will be equal to P again. So, in this case, this is my zero polynomial and I have my additive identity. So, the 3rd one is that additive inverse. So, in the additive inverse also, I know that for any polynomial P, so, suppose I take the polynomial P as so, we what we have defined alpha naught.

So, suppose I take a polynomial alpha naught plus alpha 1 x plus alpha 2 x square alpha n x n. So, this is the nth degree polynomial and in this case, if I suppose I add P plus e, then it will be alpha 0 plus 0 alpha 1 plus 0 and so on alpha n plus 0 x n and this should be again the alpha 0 plus alpha 1 x alpha n x n and that is again the polynomial belonging to the set of polynomials P n x ok. So, this is also true.

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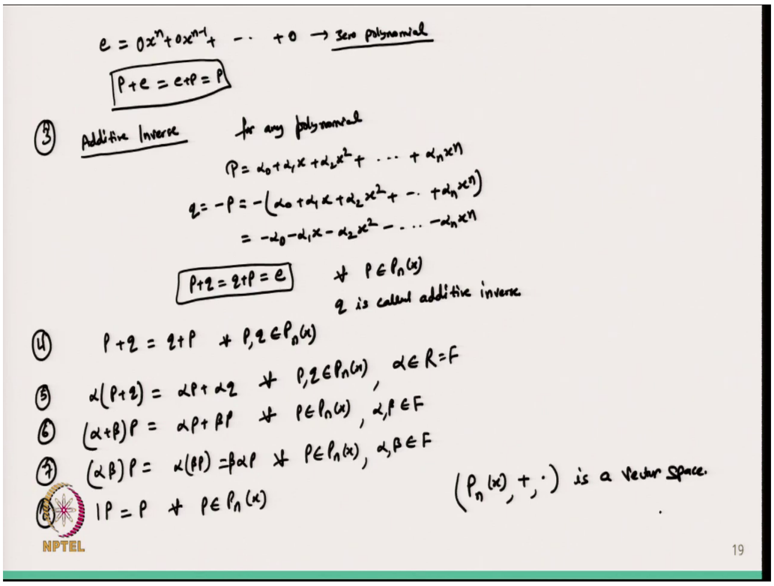

So, from here, I can say that no, no this I need a inverse, so, I have to take the inverse basically, I will choose the inverse. So, I will choose the inverse another polynomial minus P. So, it will be minus into alpha 1, this one and from here, this will be minus alpha 0 minus alpha 1 x minus alpha 2 x square and then, minus alpha n x n.

So, if I take this element as I just called this element as q and this is my P, so, from here, you can see that p plus q or q plus p that will be equal to the identity that is a 0 element this one and this is true for all P belongs to the P n x. So, in this case, the q is called additive inverse, so, this is I can define.

The 4th one is that commutative. So, from here I can see that I can choose a polynomial P plus q, I can show that that is equal to q plus p or all P and q belongs to P n x. So, this one I can show very easily so, it is a commutative group under the usual addition or the polynomial addition.

The same way I can define another property 5th one is the distributive property. So, for any alpha, the scalar P plus q I can show or just verify we can very easily that it will be equal to alpha P plus alpha q and that is true for all P and q belongs to P n x and alpha is coming from real line R because we are defining the coefficients are real so, it is my field basically we are taking the field in this case.

The 6th one is I can define alpha plus beta p that I can show that is it will be equal to alpha P plus beta P and that is also true for all P belongs to P n x field and the 7th one is alpha beta P can be written as alpha beta P or may be beta alpha P, this is true for all, and alpha beta belongs to the field.

And then, the last one is that the 1 coming from the real line and 1 into p if I write so, any polynomial if it is multiplied by 1 the scalar multiplication, then definitely equal to P and this is true for all P belongs to the given set P n x. So, it means that it satisfy all the given property, then from here, I can always say that P n x, the set of all the polynomials of degree less than equal to n under the usual addition and scalar multiplication is a vector space.

So, now we stop here. So, today we have discussed two main examples the vector space that V n and the set of all the polynomials of degree less than equal to n and we have shown that how we can check that whether the set is a vector space or not. So, in the next lecture, we will continue with the concept of vector spaces. Thanks for watching.

Thanks very much.