Matrix Computation and its applications

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Lecture - 29

Properties of linear transformation

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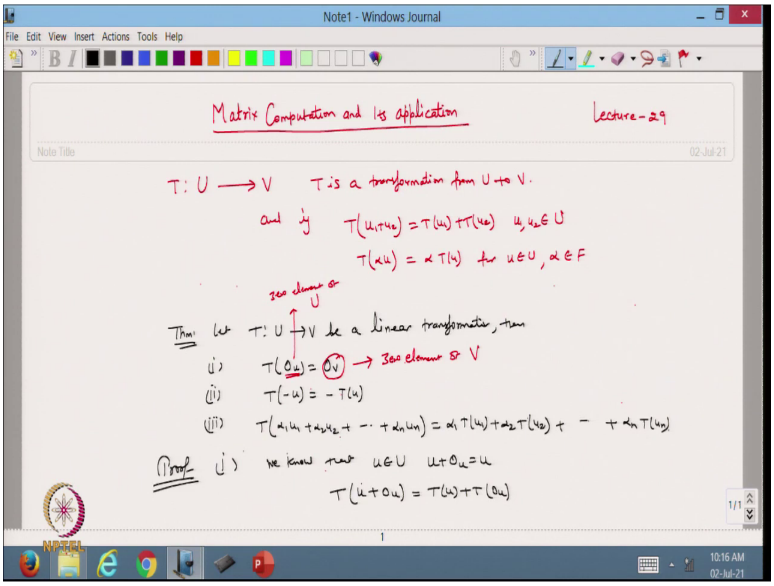

Hello viewers, welcome back to the course on Matrix Computation and Application. So, in the previous lecture we have started with the linear transformations. So, now, in this lecture we will continue with that one. So, in the previous lecture we have started with the linear transformation, that suppose I have a vector space U and another vector space V. So, I define a transformation T: U → V. And, this transformation T is a transformation from U to V.

And, if it satisfy the condition that $T(u_1 + u_2) = T(u_1)+T(u_2)$ and $T(\alpha u_1) = \alpha T(u_1)$. where u_1 and u_2 belong to U. So, it is true for all u_1 and u_2 and this is also true for all u belongs to U, and α belongs to the scalar field F. So, this transformation is called the linear transformation, if it satisfies these conditions.

Now, we want to use some properties or maybe I call it a theorem that lets me have a transformation from U to V. So, suppose it is a linear transformation if so, then 1st one I want to show that $T(0_u)=0_v$. The 2nd one is $T(-u)=-T(u)$. And, the 3rd one is that if I take the transformation of the linear combination. So, suppose I have $T(\alpha_1 u_1+\alpha_2 u_2+\alpha_3 u_3+\ldots+\alpha_n u_n)= \alpha_1 T(u_1)+\alpha_2 T(u_2)+\alpha_3 T(u_3)+\ldots+\alpha_n T(u_n)$. So, these are the three results about the transformation. Now, we can prove this very easily. So, proof so, the first one is that now in this case you can check that, this I have written 0_v means, this is the 0 element of vector space V. And, this is the 0 element or maybe I can write that 0 element of vector space U.

So, now, from here we know that so, we know that for any u belongs to U, if I put u+0 = u. Now, if I take the transformation T(u+0) then by the property of linear transformation it will be equal to T(u+0)= T(u)+T(0),and now this can be written as.

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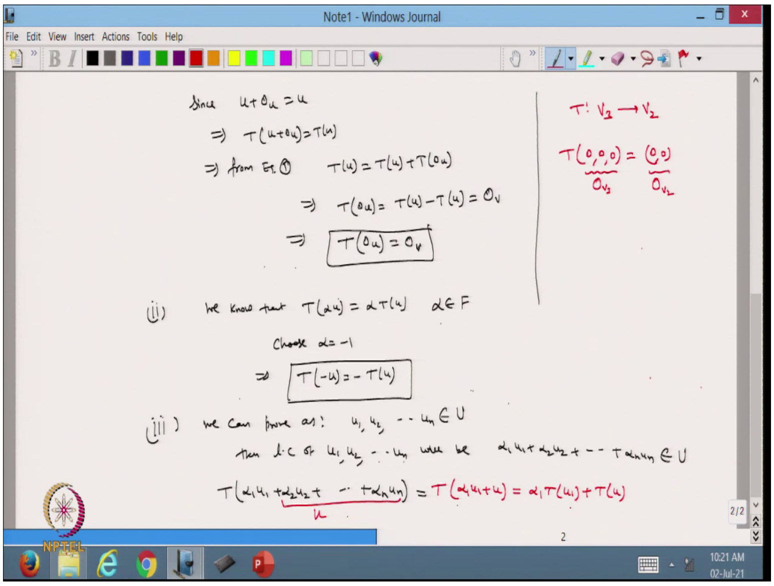

Now, since u+0 = u which implies that so, now, from this one the left hand side will become T(u+0)=T(u). So, now, I can write from here, that $T(u)=T(u)+T(0_u)$. So, maybe I can give this as equation number 1.

Now, from equation 1 we can write T(u) on the left hand side and the right side is already T(u) this one. And, now from here I can write my $T(0_u)=T(u)-T(u)=0_v$. And, now T(u) is in the v and -T(u) is also in the v, that is just the inverse additive inverse of T(u). So, then it will be equal to 0_v. So, from here we can say that $T(0_u)=0_v$. So, from here the thing is that suppose we have a transformation from maybe I will call it V_3 to maybe V_2, then it is understood from here that T(0, 0, 0), because (0, 0, 0) is the 0 of v_3. So, this element you can call it 0 of v_3 and this will be equal to (0, 0) and this is 0 in v_2.

So by this one this is always true, that 0 element of the space u will map into the 0 element in the space V. So, this is the first one. So, the 2nd one is now we know that T (-u). So, I want to prove this one. So, let us say that I will take $T(\alpha u)=\alpha T(u)$, and α is coming from the field.

So, in the field now if I choose α it is equal to minus 1, because 1 is there in the field. So, the additive inverse minus will definitely be in the field so, in the given field. So from here we can have that T(-u), then it can be written as T(u). And, this is true for all elements of u.

And, the 3rd one is we can prove so; we can prove as like I have a some linear combination. So, what I am going to do is that. So, let we have $\{u_1, u_2, \ldots, u_n\}$ all belong to u. Then, the linear combination of $\{u_2, \ldots, u_n\}$will be like $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n$ and this also belongs to the U that we already know.

So, now what do I do? So, this is the one. So, I can write this as $T(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n)$. So, this one I can write. I can choose this part as some u. Because, this is also linear combination of $\{u_1, u_2, \ldots, u_n\}$ And, from here I can write this as a $T(\alpha_1 u_1)$+u. And, this one is I can write from the property of the linear transformation, this can be written as $\alpha_1 T(u_1)$+u. And, then T(u) can further be written in this form and from here I can write.

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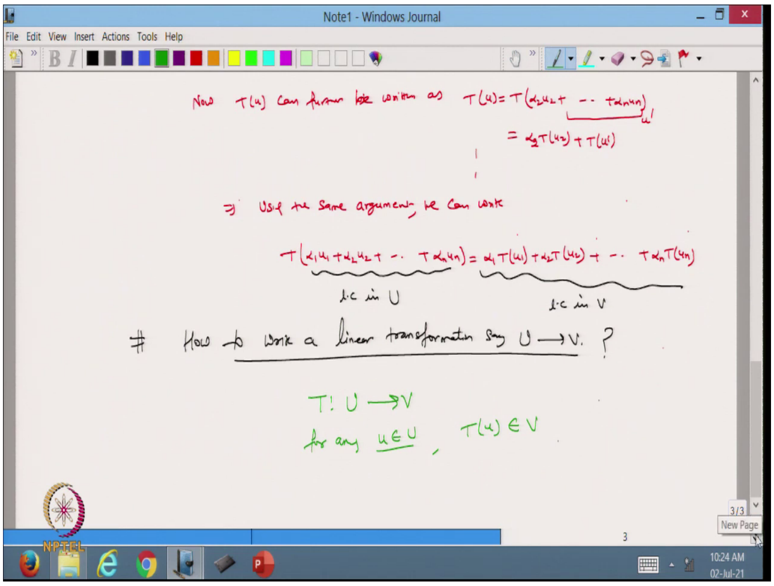

Now, T(u) can further be written as T(u) = $T(\alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n)$. So, I can write this as again $\alpha_2 T(u_2)$ + the remaining part of this one, I can call it maybe T(u'). And, similarly I can go and from here going using the same argument, we can write that

$T(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \alpha_3 T(u_3) + \ldots + \alpha_n T(u_n)$. So, the same alphas are working here also and here also. So, this is the linear combination. So, this is a linear combination in U. And, you can check from here that this is a linear combination in v. So, $\alpha's$ are the same and these things are different. So, that means that the linear transformation of the linear combination U, that is equal to the linear combination in V. And, the α of the coordinates will be the same. So, this is what we can prove from here.

Now, so, after doing this one, now, what I need to do is that. So, the next thing is that question is how to write a linear transformation, say from vector space U to the vector space V. So, the question is how can I write the linear transformation? Now, because since because I want to find the linear transition T : U→V; this one I want to find.

It means and my T is that for any u belongs to U, T of u should belongs to V. So, it means I for finding the linear transformation, I need to find the value of that transformation for each u belongs to U, and it means and we already know that u is a vector space. So, if we use a vector space it is going to have an infinite number of points. So, it is not possible for us to find the image of each member of the vector space u.

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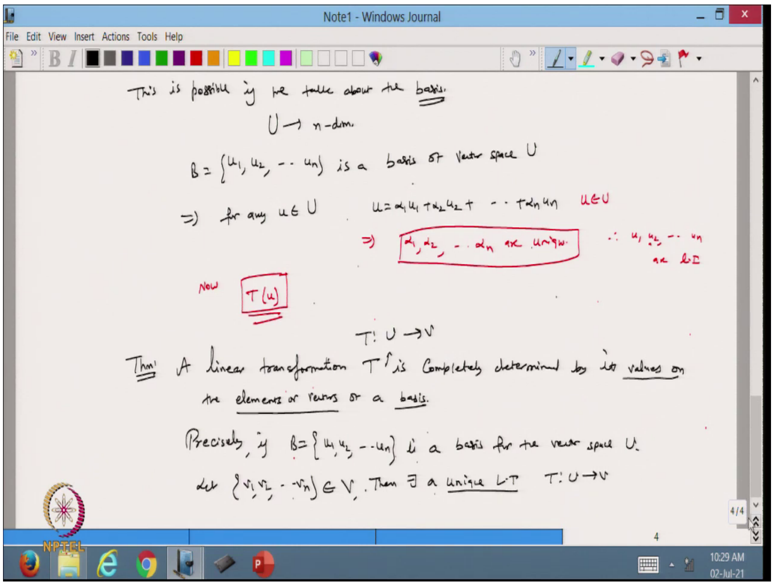

So, how can we find the linear transformation from one vector space to another vector space? So, these things we need to find out. So, this is possible, if we talk about the basis. Because, we know that suppose I have a vector space U and that is nth dimension n dimensional. So, dim(U)=n.

Then, suppose I have a basis $\{u_1,u_2,\ldots,u_n\}$. So, this set is the basis of a vector space U. Now, from here we know that, for any u belongs to U. I can write u as some linear combination $\alpha_1 u_1+\alpha_2 u_2+\alpha_3 u_3+\ldots+\alpha_n u_n$. And, since this is the basis so, if it is a basis then it is L I. So, from here we know that, this α_1, α_2, α_3... α_n are unique.

So, I can find the system of equations and that system of equations is going to have a unique solution. Because, $\{u_1,u_2,\ldots,u_n\}$ are linearly independent. So, this representation will be unique. Now, so, it is any u so, u belongs to U. So, this is true for any u. Now, if I am able to find the T(u), then we are done.

So, T(u) we can find out. So, how can we find out how to define the T(u)? So, that is what we do with the theorem. So, let us write one important theorem or the definition you can write.

So, a linear transformation T is completely determined by its values on the elements or vectors of a basis. So, it is completely determined by its value on the elements or the vectors of a basis. So, basis of T, basically basis of u so, how will you define that one?

So, I can write that precisely, if suppose I have a B is a set of vectors $\{u_1,u_2,\ldots,u_n\}$. So, I am defining the linear transformation T. So, T maybe you can define it from U to V. So let I take the set B. So, the basis is a basis for the vector space U. Now, there exist now let the vectors $v_1, v_2,\ldots,v_n$ that belongs to the vector space V. Then, there exists a unique linear transformation there exists a unique linear transformation T from U to V.

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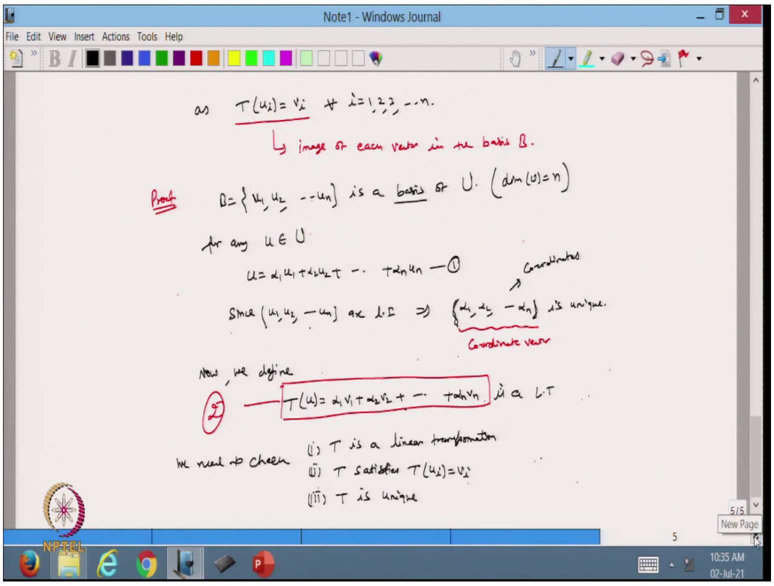

Such that, T of u i is equal to v i and this is true for all i 1, 2, 3 up to n. So, from here we can say that I have taken a basis of the vector space u and suppose there exists some vectors $\{v_1, v_2,\ldots,v_n\}$ and the number vector that belongs to V. Then, I can find a unique linear transformation from T(u) to T(v) from u to v as $T(u_i)= v_i$. It means, from here you can see that, we are able to find the image of each vector in the basis.

So, if we are able to find the image of each element in the basis, then we are able to find the linear transformation. So, these things are so, this theorem gives you that how we can find out the linear transformation? Now, so, this proof is very simple, that now I have a basis B is equal to the basis $\{u_1, u_2,\ldots,u_n\}$ So, this is the basis of vector space U. And, suppose it is a basis it means that the dim(U) = n, that we already know, is a basis means it spans u and they are linearly independent.

Now, for any u belongs to the space vector space U, I can write U as $\alpha_1 u_1+\alpha_2 u_2+\alpha_3 u_3+\ldots+\alpha_n u_n$. So, first I defined that, how can we find out the linear transformation? So, I can write like this one. So, maybe I call it 1. So, this is a basis or all are linearly independent, which implies that α_1, α_2,..., α_n. this set of or coordinates. So, this is a coordinate vector, coordinates and this vector I call it coordinate vector.

So, this coordinate vector is unique. Means, this system is not going to have an infinite solution; it is going to have a unique solution. Because it is linearly independent, I know that the rank will be equal to the n here. So, it will be the unique solution. So, now, from here I can write, then now we define $T(u)=\alpha_1 v_1+\alpha_2 v_2+\ldots+\alpha_n v_n$. And, you know that v_1, v_2,...,v_n all are coming from the vector space V. So, these are this one.

So, now we define a transformation $T(u)=\alpha_1 v_1+\alpha_2 v_2$. So, I claim that this is a linear transformation. So, I call it maybe 2. So, I define this as a linear transformation. So, I am taking $T(u)=\alpha_1$. So, alpha is coming from here and v_1 is the vector we have taken from the vector space V.

So, let us do this one. So, I need to check so, we need to check. So, of course, T is from U to V. So, what we need to check first is that T is a linear transformation. So, this we need to check. And, the 2nd one is we need to check that T satisfies the condition $T(u_i)=v_i$.

Because, this is what we have shown here, that it can be determined if it has the image of each of the basis elements, of a basis of a vector space U. So, satisfy this condition and the 3rd one is that T is unique. So, it should be unique. So, these things we need to check.

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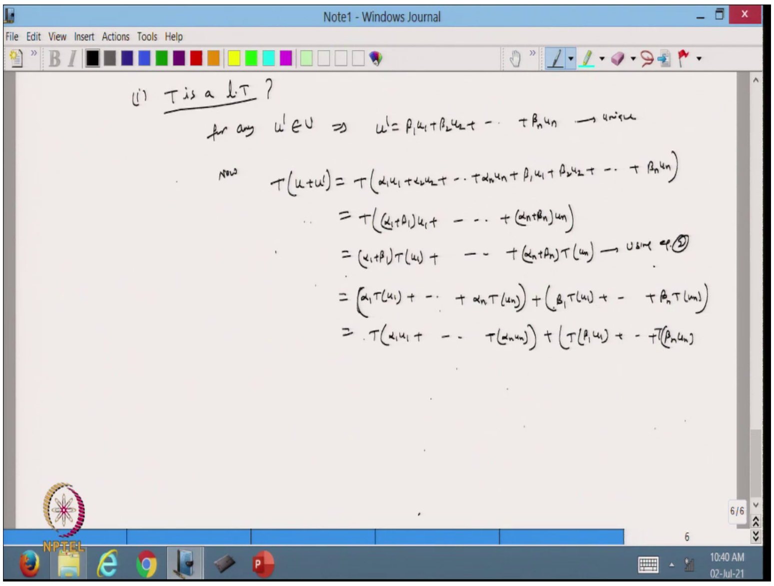

Now, the first one we can check, that T is a linear transformation . So, this one we need to check. Now, for u_i know that, for any maybe I can take u'. So, for any u' belongs to U, we can write u' is equal to suppose it is coming $\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \ldots + \beta_n u_n$. This is true. So, this can be written as this one and this representation is unique, that we know.

Now, T(u)+u' ; let us see what is going to happen. So, this can be written as T(u), u is $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \ldots + \beta_n u_n$, and since this belongs to the vector space. So, I can write this as a $(\alpha_1 + \beta_1)u_1 + \ldots + (\alpha_n + \beta_n)u_n$. Now, these things we already seen that this is equal to $(\alpha_1 + \beta_1)T(u_1) + \ldots + (\alpha_n + \beta_n)T(u_n)$.

And, from here I can separately write like this one. So, it can be written as I can write it as $(\alpha_1)T(u_1) + \ldots + (\alpha_n)T(u_n)$. So, I can write like this one plus $(\beta_1)T(u_1) + \ldots + (\beta_n)T(u_n)$.. And, this one is again so, I can write this as T $(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n)$ plus the same way I can write $T(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \ldots + \beta_n u_n)$, because of the linear transformation I have taken this one.

So, I am taking T(u+u') . This is coming like this one and this one I can write in this way from. So, this we have taken from equation 2, this one using equation 2. Now, I am doing this

1 as $\alpha T(u)$because that belongs to the vector space V. So, this one and now I am going back from there. So, I can write this as a $(\beta_n)T(u_n)$. So, here also I am taking the help of the equation number 2,

because $(\alpha_1)T(u_1)$ *and this* $(\alpha_n)T(u_n)$. So, I am writing in this way and maybe from here I can write directly.

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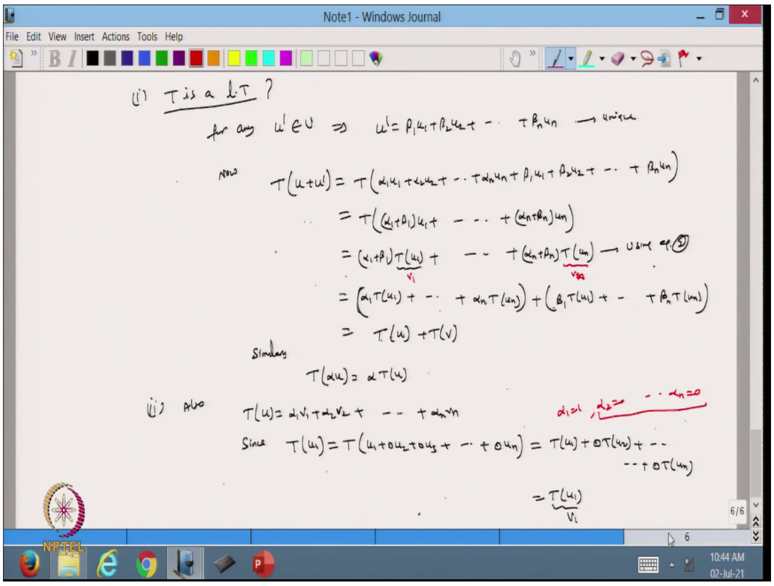

So, after writing this one, because to show that, it is equal to T (u) + T(v), so, from here I can write this one can be written as. So, if you see from here then this is equal to T(u)+T(v) . Because, I am writing here that $\alpha_1 v_1+\alpha_2 v_2$ some vector is there and this is coming. So, $T(u_1)$; I am taking as an element from as a v_1 and this is I am taking as a v_2, v_n and then we are putting this value here. So, this becomes T(u)+ T(v).

Similarly, we can define T of α some u that can be easily written as αT(u). Now, and also so, it is a linear transformation, also now we have a $T(u)=T(u)=\alpha_1 v_1+\alpha_2 v_2+\ldots+\alpha_n v_n$. So, this thing is there. Now, we are defining this as a T(u). So, this one I can write as T of maybe I can write as $T(u_1)=T(u_1+0u_2+0u_3+\ldots+0u_n)= T(u_{1)}+0T(u_{2)}+0T(u_{3)}+\ldots+0T(u_n)$. that we already know and this is equal to T of u $_n$.

Now, from here I can write this as because this is a 0 element. So, it will be $T(u_1)$. And, if you see from here, then this is the same as v_1. Because, we have defined the linear transformation in this way, that $T(u)=\alpha_1 v_1+\alpha_2 v_2+\ldots+\alpha_n v_n$. So, from here I can say that my α_1 is 1, α_2 is 0, and all alpha these are the 0. So, from here it will I will left only with $\alpha_1 v_1$. And, that will be equal to $T(u_1)$. So, it means that this is equal to v_1.

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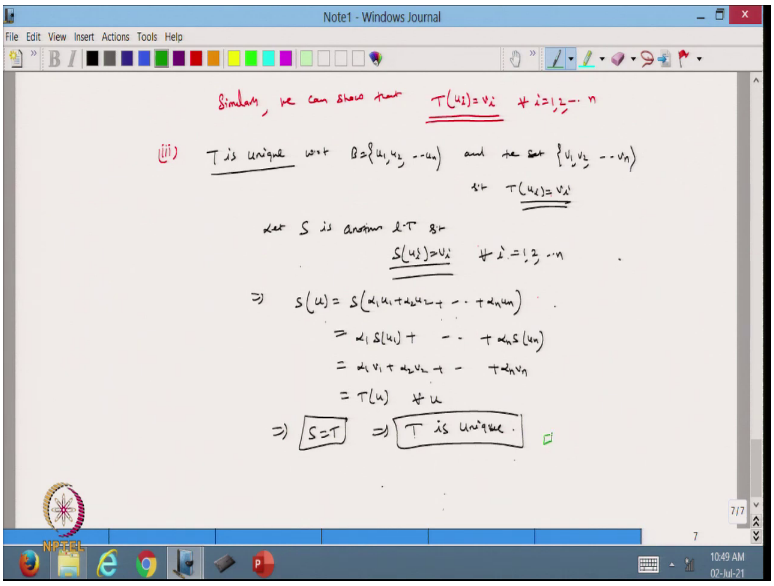

So, the same concept will go. Similarly, we can show that $T(u_i)=v_i$. And, this is true for all i from 1 to up to n, the same argument will work there. Similarly, I can define $T(u_2)$. So,$T(u_2)=T(0u_1+u_2+0u_3+\ldots+0u_n)$ and then from here it will be that $T(u_2)=v_2$. So, this one we can define like this from here. So, here we have defined this property and then we got this value.

Now, this condition is that each of the basis elements should have an image in the vector space V, like this one $T(u_1) = v_i$. So, the 3rd one is so, I want to show the 3rd one is that, the T is unique means if I have a set of basis and I have a set of element that is $\{v_1, v_2,\ldots,v_n\}$, then we can have only a unique linear transformation.

So, T is unique means unique with respect to basis that is $\{u_1, u_2,\ldots,u_n\}$ and the set$\{v_1, v_2,\ldots, v_n\}$, such that $T(u_1) = v_i$.. So, if you change any of the thin conditions then you will get another transformation that we need to show that it is unique. So, these things we can prove by contradiction that let S is another linear transformation, such that $S(u_1) = v_i$. So, this I am taking another linear transformation, true for all i 1,2,… ,n. So, this is what I am taking. Now, from here, I know that S of so, for any u that will be S of I know that u can be written as $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n$

Now, this can be written as $\alpha_1 S(u_1) + \ldots + \alpha_n S(u_n)$. Now, $\alpha S(u_1) = v_i$. So, I can write this as $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots + \alpha_n v_n$, this is what we have defined here. Now, if you see from here then this is equal to T(u). And, this is true for all of you. So, which implies that S will be equal to T, because this is true for all u whatever we have taken that shows that S(u) is equal to T(u) for all u. So, from here I can say that S is equal to T and that gives you that T is unique. So, T will be unique.

So, to define the linear transformation, we need a set of bases in the vector space u and then we need some vectors in v so that we can define a map like this one. So, suppose we have the knowledge of $T(u_i)=v_i$ and based on this knowledge I can define the linear transformation like this one. And, that linear transformation will be linear transformation and it will satisfy this condition and it will be unique. So, that way we can define the linear transformation.

So, today let me stop here today. So, in the today's lecture we have started that how we can define a transformation, linear transformation from a vector space U to a vector space V. So, in the next lecture we will perform some examples, that how we can find out the linear transformation, based on the given basis given to us for a given vector space U. So, thanks for watching and.

Thanks very much.