Matrix Computation and its applications
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Lecture - 28
Linear transformation

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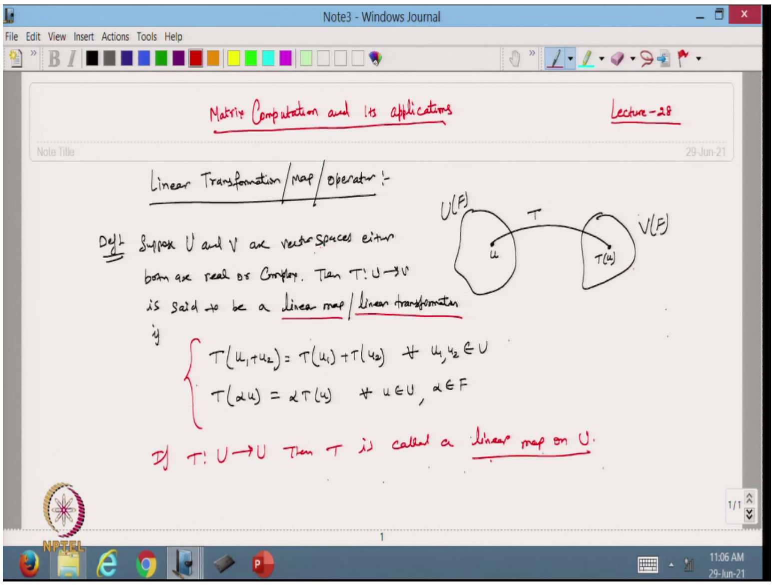

So, hello viewers. So, welcome back to the course on Matrix Computation and its Application. So, in today's lecture, we are going to start with the topic that is called the Linear Transformation, because earlier also we have discussed the linear transformation or linear map in the case of a matrix of order m cross n. So, now, we are going to start with the linear transformation from one vector space to another vector space.

So, we are going to start with the topic linear transformation or linear map or sometimes it is also called operator. So, in this case, what are we going to do? Suppose, we have one vector space U and I have another vector space V, we are assuming that they are having the same field F, because if F is real then it is real in both the case or if F is complex that is a complex. So, U and V both are vector spaces having the same field.

Now, from here suppose I define a transformation from $T: U \to V$. And this is the element I have taken, suppose u, then this is the image of u and that will be $T(u)$. So, this is a you can say that a linear map, a map not the linear, I am just taking that $T: U \to V$, then by the definition.

So, just write that we write the definition here that suppose U and V are vector spaces either both are real or complex, they should be same, then the map from U to V is said to be a linear map or linear transformation if; so, $T(u_1+u_2)=T(u_1)+T(u_2)$ $\forall u_1,u_2 \in U.$ $T(\alpha x)=\alpha T(x)$ $\forall \alpha \in F, u \in U.$ So, if these two conditions are satisfied, then we say that this map we are taking from U to V is a linear map or a linear transformation. So, that is the definition of the linear map.

Now, if we take the $T: U \to U$, U to U means it is coming starting from the U and image is also lying in U. Then, T is called a linear map on U. So, it is called the linear map on U. So, now, from here I can generalize these things.

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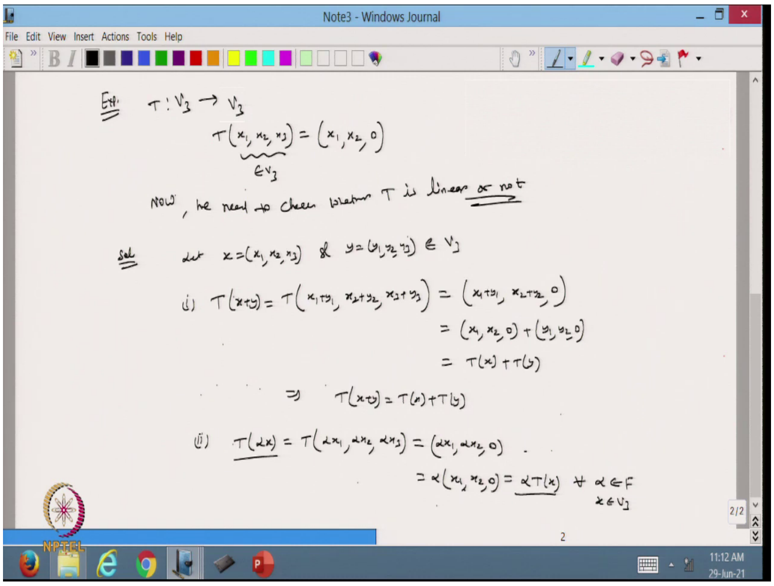

So, for example, so when I take the vector space V_3 to V_3 by the transformation T. So, I am taking a transformation $T: V_3 \to V_3$. So, I will take the vectors from V_3 and V_3 will contain the vector like $\{x_1, x_2, x_3\}$, because it is coming from the belongs to V_3. So, I am defining this one as, suppose I define V_2, so it should be from V_2, so I define it like this one. Because V_2

will contain all the elements which have two components. So, in this we keep taking the third component 0. So, this is the transformation I have taken.

Now, we need to check whether T is linear or not. Now, let I take the two vectors let

x=($x_{1,}x_{2,}x_3$) and y=($y_{1,}y_{2,}y_3$), that belongs to the space V_3. Then, T(x+y)=T(($x_{1,}x_{2,}x_3$)+($y_{1,}y_{2,}y_3$))=T($x_1 + y_{1,} x_2 + y_2, x_3 + y_3$) because these two vectors belong to the V_3 and that is a vector space. So, the this will belong to the vector space and this will be equal to ($x_1 + y_{1,} x_2 + y_2$,0). So, that is the image of this one.

Now, this image I can write as (x_1, x_2, 0) +(y_1, y_2, 0 because just if I add these together in V_3, it becomes this. Now, from here you can see that this is equal to T(x)+T(y). So, from here you can say that T(x+y) in this case is equal to T(x) +T(y). So, the first condition is satisfied. So, from here I can say that this is the first condition.

The second condition will be T(αx). So, again, it is can be written as T(αx), I can write as T (αx_1+αx_2+αx_3). So, this one I can write as, so now, I will apply my transformation. So, this will be equal to (αx_1, αx_2,0). Now, from here I can take the α common and then I can write x_1, x_2, 0. So, this will be equal to α and this I can write as T(x).

So, from here I can write that the T(αx) =αT(x), and this is true for all α belongs to the field and for all x, and x belongs to V_3.

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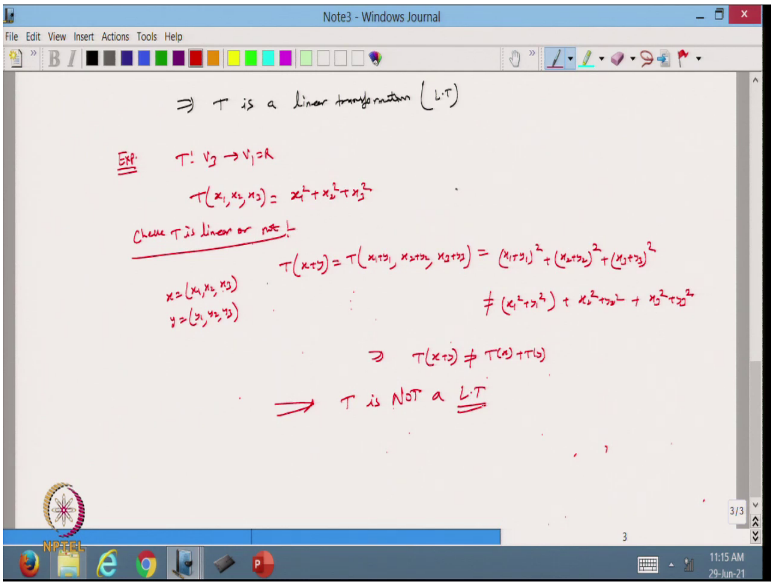

So, if both the conditions are satisfying, then from there we can say that the T is a linear transformation and in the short form we call it LT, the linear transformation. Similarly, we can define another example. For example, I take T from V_3 to V_1 under the operation, suppose I take $T(x_1, x_2, x_3)=x_1^2 + x_2^2+x_3^2$. So, V_1 means real number, so that is a real number basically. So, this is the transformation I have taken.

Now, I want to check if T is linear or not. And from here it is clearly I can see that if I take T(x+y) as I have defined my $x=(x_1, x_2, x_3)$ and $y=(y_1, y_2, y_3)$, then I can write this as x_1+y_1, x_2+y_2 and x_3+y_3. Now, from here this will be equal to, if I apply the transformation this should be equal to $(x_1 + y_1)^2+(x_2 + y_2)^2 + (x_3 + y_3)^2$. Now, definitely we know that this is not equal because it will be $x_1^2 + x_2^2+x_3^2+y_1^2 + y_2^2+y_3^2$. So, from here you can say that $T(x+y)\neq T(x)+T(y)$. So, from here that T is not a linear transformation. So, it is not a linear transformation.

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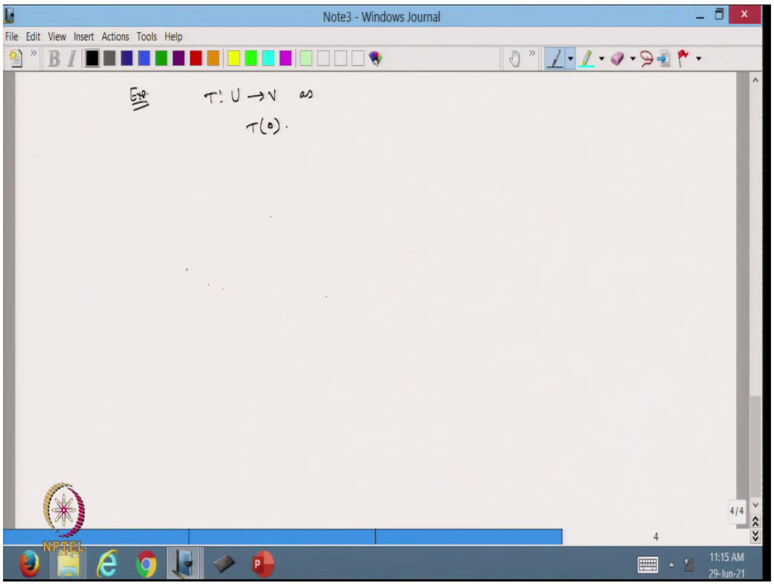

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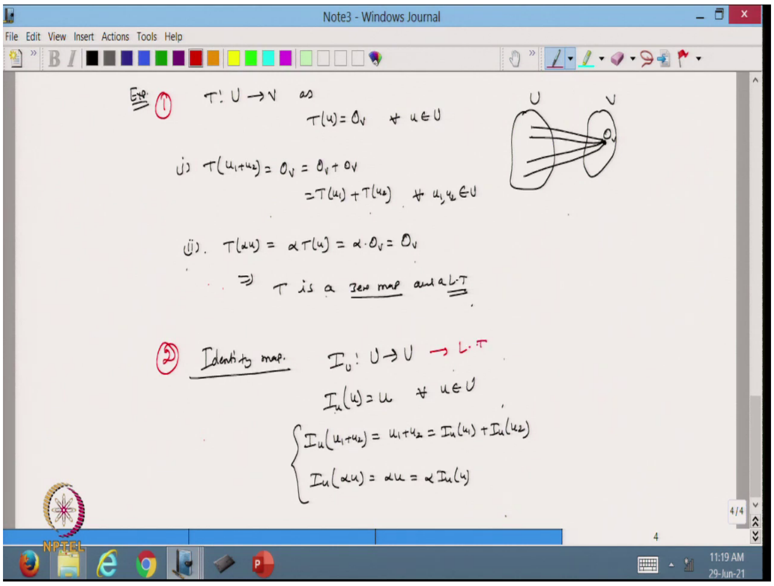

Similarly, I can define another example. Similarly, I can define the another example is as I take the transformation T: U →V ; or maybe I can just first define $T(0)=v_0$. It means that suppose my U is there and this is my V.

So, I am taking all the elements in the U, so that is also going to this element, this is also going to this element, all is going to this element. So, that is my basically 0 element in V because V is a vector space. So, it will contain the identity, the additive identity that is 0 element. So, I am putting all this element of u towards 0. So, this is true for all u belongs to U.

Now, from here I can say that $T(u_{1+}u_2)$. So, this one can be written as because this element belongs to U and which in map to this, so it is equal to 0_v. And this one I can write as $T(u_{1+}u_2)=0_v=0_v+0_v=T(u_1)+T(u_2)$, this is true for all u 1 and u 2 belongs to U. So, the first condition is satisfied.

And the second one is $T(\alpha x)=\alpha T(x) = \alpha.0_v=0_v$. So, both the conditions are satisfying. So, from here I can say that this is linear transmission. So, T is basically call it 0 map and linear transformation and a linear transformation LT. So, it is called the 0 map and also it is a linear transformation. So, this is example number 1, I can say.

Then, from here I can take another example that we call an identity map. So, identity map I am taking I_u: U → U, such that I(u) it applies on u, it gives you the u, and this is true for all u belongs to U. So, now from here you can check that this is linear transformation because $I_u(u_1+u_2)=u_1+u_2$ by the definition and $I_u(\alpha u)= \alpha u=\alpha T(u)$ and that can be written as I_u. So, both the conditions are satisfied. So, from here I can also say that this is also a linear transformation LT. So, this is another linear transformation we can define.

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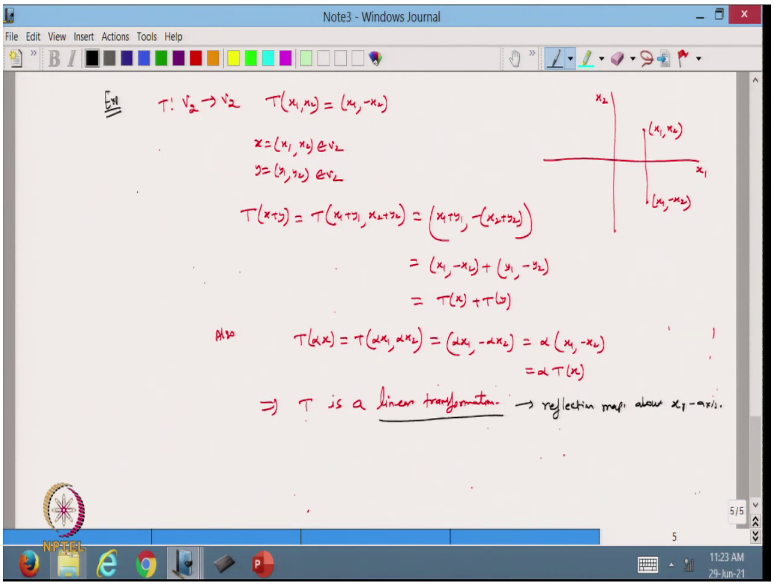

Also, I can define another example. I take the transformation $T:V_{2} \rightarrow V_{2}$. By the transformation that $T(x_1,x_2)=(x_1,-x_2)$. So, basically what we are doing here? That suppose we have a two-dimensional and at some point there is x_1, x_2, now, I take just the image of this one along the x axis.

So, this is basically the x_1 axis and this is the x_2 axis, I just take the image of this one here, and this will be my x_1, $-x_2$. So, I am just taking the reflection about the x_1 axis. So, I am defining the transformation here. Now, I want to check whether this is a linear transformation or not. So, now, from here I just take $x=(x_1,x_2)$; $y=(y_1,y_2)$. So, both belong to V_2. Now, $T(x+y)=T((x_1,x_2)+(y_1,y_2))=T(x_1+y_1,x_2+y_2)=(x_1+y_1,-x_2-y_2)=(x_1,-x_2)+(y_1,-y_2)$. So,the vector addition is satisfying. Also, $T(\alpha x)=T(\alpha(x_1,x_2))=T(\alpha x_1,\alpha x_2)=(\alpha x_1,-\alpha x_2)=\alpha(x_1,-x_2).=\alpha T(x)$. So, it is a linear transformation.

So, which implies that T is a linear transformation and this is basically this transformation, here if you see, so this is called the reflection. So, it is a reflection map about the x_1 axis or x axis you can take. So, this is a linear transformation we are defining.

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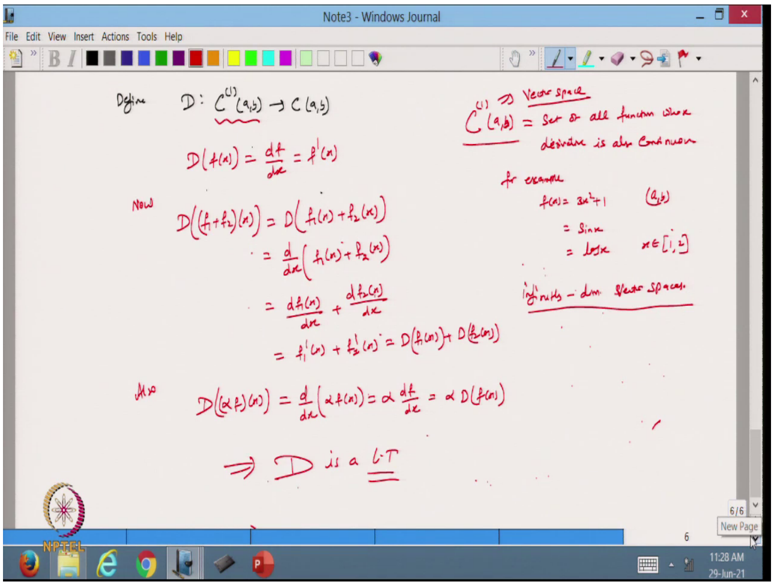

So, the after that we are going to define a very important linear transformation and that is called the, so let us we define a D, that is a map $D:C^{1}(a,b) \rightarrow C(a,b)$. So, what we are going to do is that this is a space, so C^1,I am defining, C means continuous function and 1 means one derivative.

So, it is a set of all functions whose derivative is also continuous. So, the function, the set of all functions whose first derivative is also continuous. So, this is a set of all the functions defining from a to b, whose set whose derivative is also continuous.

For example, you take the function f (x)=$3x^2+1$. So, if I take the derivative then the function is again continuous and suppose I take (a, b) the interval, so in this interval that is true. Or I can define may be sin x, I can define may be log x, where x $\in [1, 2]$. So, this type of all the functions belongs to this set. And C[a,b] is a set of continuous functions. So, I am defining the a transformation D:$C^1(a, b) \rightarrow C(a, b)$. And these are the in fact, these are vector space, but it is infinitely dimensional spaces, infinitely dimension vector spaces because we have done only the finite finite dimensional vector spaces, but these are the example of the infinite dimensional vector spaces, ok.

Now, what do I do? So, I define the D over a function that is coming from here, suppose it is a function of x then this is equal to D(f(x))=$\frac{df}{dx}$=f'(x).

So, it is a transformation that is a different derivative transformation defining from this space to this space, this vector space to this vector space.

Now, so I want to define what is the D (f_1+f_2)(x)=D $(f_1(x)+f_2(x))$. Suppose, I take the two functions here that is equal to $f_1(x)+f_2(x)$ and that I know already that if I take the $\frac{d}{dx}(f_1(x) + f_2(x))$ like two function addition of two function, then from the theory of derivatives we know that this is equal to $\frac{df_1(x)}{dx}+\frac{df_2(x)}{dx}$. And this is again $\frac{df_1(x)}{dx}+\frac{df_2(x)}{dx}$=D ($f_1(x)+f_2(x)$). It means that this addition is true. Also, D((αf)(x))=$\frac{d}{dx}(\alpha f(x))$=α$\frac{df}{dx}$= $\alpha D(f(x))$. So, from here I can say that D is a linear transformation. So, the derivative is basically a linear transformation. So, this is the one example of linear transformation.

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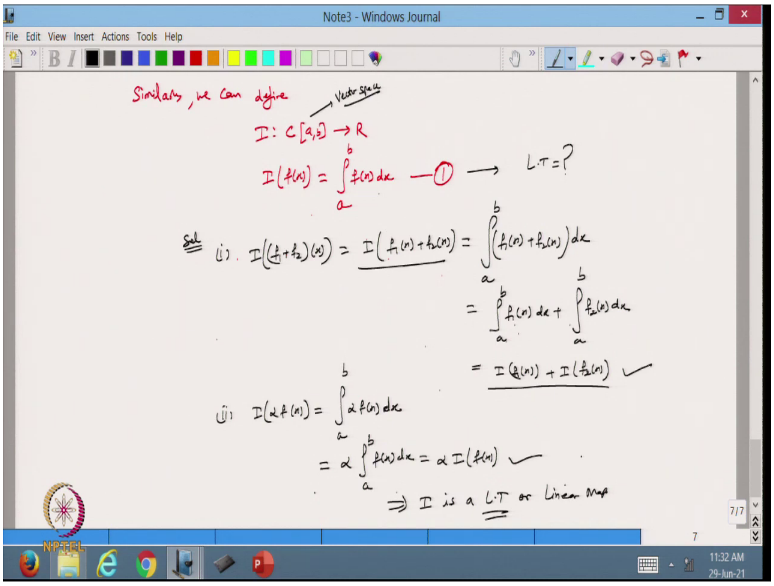

Similarly, we can define. So, I will define the another type of transformation

$I: C[a, b] \rightarrow R$ such that $I(f(x)) = \int_a^b f(x)dx$. So, this is my transformation I have defined.

Now, I want to check whether this transformation is linear or not. So, this one I want to check whether it is a linear transformation or not. So, for this one, I choose two vectors. So, I of two vector I am taking f_1+f_2 belongs to the set of continuous functions and this is equal to $I(f_1(x)+f_2(x))$ that we define, and this by the transformation it becomes $\int_a^b (f_1(x) + f_2(x))dx$.

So, now, from the theory of integration I can write this as $\int_a^b f_1(x)dx + \int_a^b f_2(x)dx$ because I know that the sum of two continuous functions is a continuous function. So, addition is well-defined and this is a vector space. And this we already know from the theory of integration that this is possible.

And this is equal to, this is equal to $I(f_1(x))+I(f_2(x))$. So, this is equal to this one, it means that the addition vector addition is valid, so this is the first property and the second property is $I(\alpha f(x))=\alpha I(f(x))$. So, this I can define $I(\alpha f(x))=\int_a^b \alpha f(x)dx$

And if I define this value, then from here I can take my α common outside from this integral, so it is from $\alpha\int_a^b f(x)dx$. And from here you can see that the first property is satisfying, this is satisfying. So, from here I can say that integration I is a linear transformation or linear map. So, it is a linear transformation, the integration process. So, this is one of the very important examples we can define.

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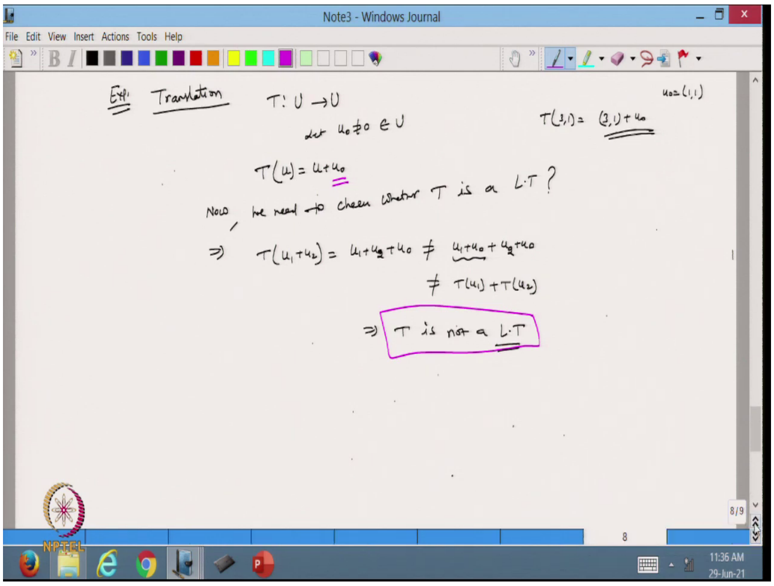

Now, I will define one more example that is translation. Suppose, I define the map let suppose we have a map from U to V, and let I take some $u_0 \neq 0$; belongs to the U. So, I define a map T from so, let us say we define it from the same vector space, so U to U, i.e., $T:U\rightarrow U$.

Now, we define $T(u)=u+u_0$. So, it is just the translation by the element u_0. If I take; suppose I am taking T or maybe I take a vector, suppose I take a vector in V_2 like (3, 1), then it becomes $(3, 1)+u_0$; I can take u_0 maybe (1, 1). So, that will be (1, 1). So, this type of thing we can define. So, this is true. We are defining this one then. So, this is my transformation.

Now, we need to check whether T is a linear transformation or not. Now, from here, now from here we can say, what about $T(u_1+u_2)$? So, by the definition it should be equal to $(u_1+u_2)+u_0$. I am doing that transformation. And from here you know that this is not equal to

$u_1+u_0+u_2+u_0$ because it means this is not equal to it will be $T(u_1)+T(u_2)$. So, the addition is not defined.

So, from here I can say that the T is not a linear transformation. Although, it looks like the linear transformation because what we are doing is just taking the translation by the element u naught. But if we check from here then we found that this is not the linear transformation. So, the translation by some nonzero element is not a linear transformation. So, these things we can check from here. So, we will stop here today.

So, in today's lecture we have started with the generalization of the linear transformation. And we have shown that derivatives or the integration process or the identity map all come under the category of linear transformation. So, in the coming lectures, we will continue with more examples or maybe we will define how a linear transformation can be defined from a one vector space to another vector space. So, thanks for watching.

Thanks very much.