Matrix Computation and its applications
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Lecture - 27
Examples of different subspaces of a vector space of polynomials having degree less than or equal to 3

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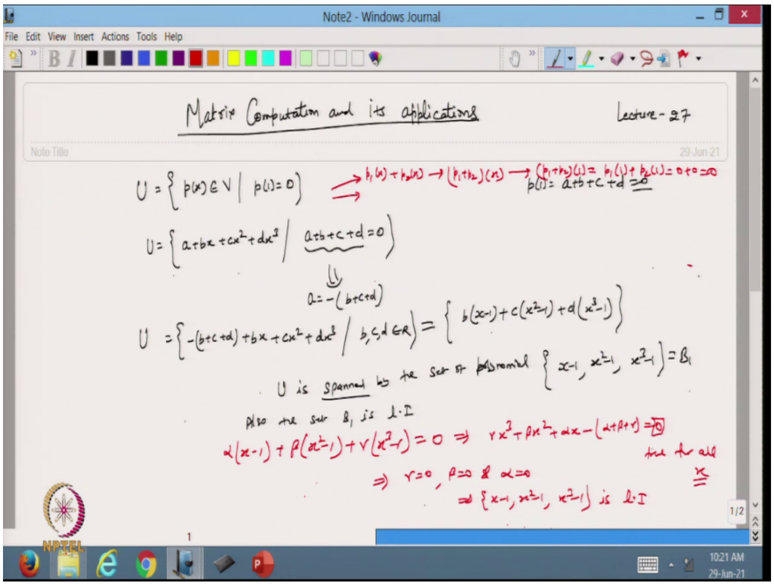

Hello viewers, welcome back to the course on Matrix Computation and its application. So, this is lecture number 27; in the previous lecture, we have started with one example. So, today we are going to start with that example. So, in the last lecture, we have started with the example that, suppose we have a set of polynomials of degree less than equal to 3 and this is a vector space I know that dimension 4.

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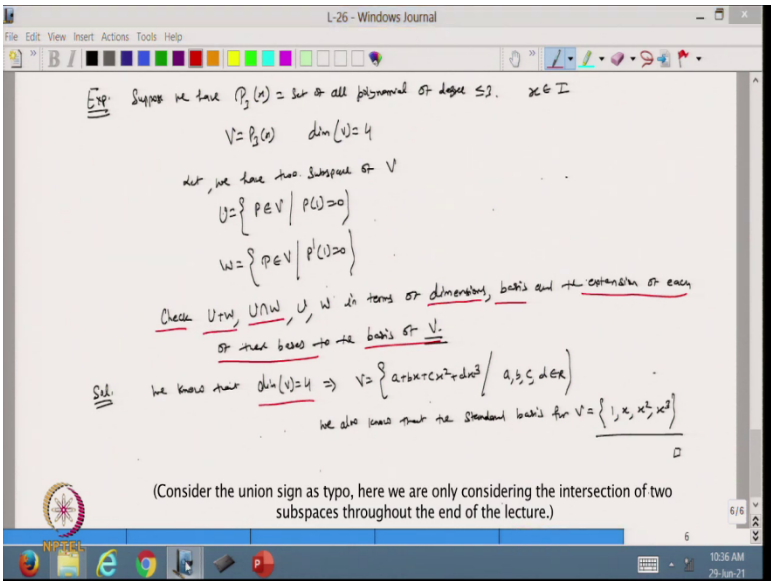

So, from there I choose two subspaces, one subspace is U with all the polynomials such thatU={p$\in V|p(1) = 0$}; it means that 1 is the root of this polynomial and W={p $\in V|p'(1) = 0$}, means the derivative of the polynomial and taking the value at x equal to 1 is 0.

Then we want to check that what will be about U +W, U ∩W, U, and W in terms of dimensions, basis and the extension of each of these basis to the basis of V. So, this thing we need to do for this one. Now, we know that it is a the vector space set of polynomial of degree less than equal to 3, this is of dimension 4 and this can be written as a polynomial V={a+bx+c$x^2 + dx^3 \in V|a, b, c, d$ ∈R}.

So, we are talking about the real polynomial and the standard basis for this is {1, x, x^2, x^3 }So, this thing is there. Now, from here, now I take the set U. So, U is all the polynomials belonging to the vector space V, such that p(1)=0; it means that I am taking U as the polynomial. So, polynomials have started. So, I will take a+bx+c$x^2 + dx^3$. So, if I choose p(1). So, this is my p, such that a+b+c+d=0 because p (1)=0.

So, this is the set which contains all the polynomials of such type. Now, from this condition I can, because this is a four variable and with one only one condition. So, I can write my a as -b+c+d. And from here I can write that, my U will be; so instead of a, I can put -b+c+d+ b

$x+cx^2+dx^3$. So, all these sets of polynomials, where b, c, d belong to the real line. And from here if you see, then I can have from here. Now, what I can do is, I can take the b, then I will get x-1, then I take the c, so c can be written as $c=x^2-1$, and $d=x^3-1$. So, this is a set of polynomials. Now, from here you can see that this set of polynomials contains only three coefficients to be found out, that is b, c and d. So, now, that is my U. So, from here I can say that, my U is spanned by. So, I can say that U is spanned by the set of polynomials. So, that is the polynomial is $x-1, x^2-1, x^3-1$; because they are spanning the whole U. Now, this is a spanning set. Now, also, so suppose I take this set as I may be, I can call it B_1.

So, also the set B_1 is linearly independent. How can we say that this is linearly independent? So, because I just take the linear combination. So, suppose I take the linear combination

$$\alpha(x - 1) + \beta(x^2 - 1) + \gamma(x^3 - 1) = 0$$

So, this is the linear combination I have taken. And from here I can write the corresponding equation $\alpha x + \beta x^2 + \gamma x^3 - \alpha - \beta - \gamma = 0$. So, this is true for all x. So, if it is true for all x, it means we have to compare the coefficient of the same term on both sides and the right hand side is the 0 polynomial.

So, this is basically a 0 polynomial and I am taking this true for all x; so which implies that in this case $\gamma = 0$, $\beta = 0$, and $\alpha = 0$, because constant term 0 and the coefficient of x, x^2, x^3 all are becoming .0 And from here we can say that the set or the polynomials $x-1$, x^2-1, and x^3-1 this set is linearly independent.

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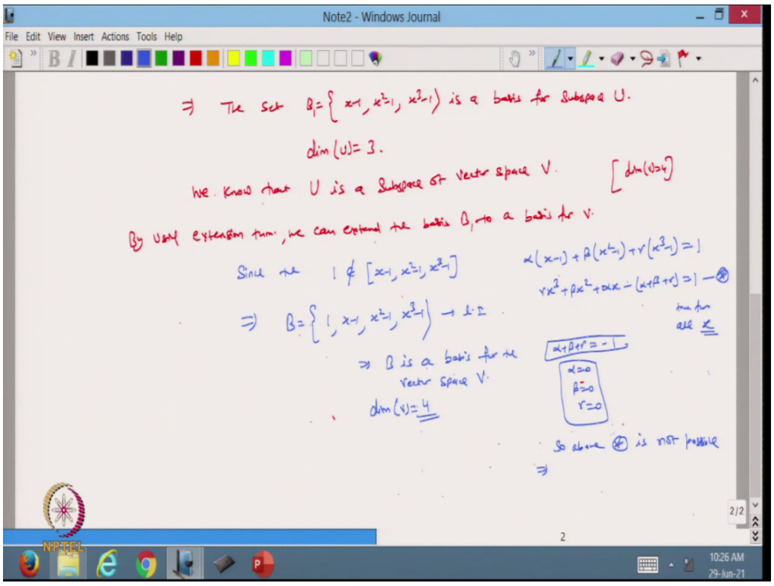

So, now if it is linearly independent, from here I can say that the set $B=\{x-1,x^2-1,x^3-1\}$ is a basis for subspace U. Now, one thing is very much clear to us that U is a subspace. Because in this case I have not proved this one that, whether U is a subspace or not; but it is a U is a subspace, because I can take from here, because two conditions we need to satisfy.

So, I can take the polynomials $p_1(x)+p_2(x)$; then from here I know that if I write $(p_1+p_2)(x)$ and this will give me, it will satisfy the condition that p_1+p_2 at 1 that will become

$(p_1 + p_2)'(1)=p_1'(1) + p_2'(1)=0+0=0$; so that will be 0 and similarly the scalar multiplication. So, it is a subspace and its dimension.

So, dim(U) = 3. Now, we know that U is a subspace of vector space V and also I know that the dim(V) = 4. So, what I can do is that I can extend this basis of U to the basis of V by the extension theorem. So, by using extension theorem, we can extend the basis B.

So, this is the basis B I have taken, B_1 basically, so I just write it B_1. So, we know so, by using this extension theorem, we can extend the basis B_1 to the to a basis, not the to a basis for V; because for V the standard basis already we know, but this can also be extended. Now, from here I can write that, since the.

So, what I can write is that, 1; if you see from here, 1 does not belong to the spanning set of x-1, x^2-1, x^{3-}1, because anything constant multiple cannot delay or cannot cancel out the x^3, x^2 or x. So, it means that, 1 cannot be written as a linear combination of this one.

So, this things you can verify by the same way that, I write $\alpha(x - 1) + \beta(x^2 - 1) + \gamma(x^3 - 1) = 1$. Then from here I will get that $\alpha x + \beta x^2 + \gamma x^3 - \alpha - \beta - \gamma = 1$ and suppose this is true for all x,.

Now, from here, you can see from here that, if I take this one; then from here I can write that $\alpha + \beta + \gamma = -1$ and $\alpha = 0$, $\beta = 0$, $\gamma = 0$. And this is not possible that, if α, β, γ are 0s; then their summation will be -1, it means that from here, I can say that this system is inconsistent.

So, above so, this is basically it will become a system of equations. So, the above star is not possible. So, from here which implies that, that 1 cannot be written as a linear combination of this. So, from here I write the basis B as 1, x -1, x^2- 1, x^3- 1. So, it is. So, it has four vectors.

So, from here I can say that, B is a basis for the vector space V and dimension in this case of V will be 4 and that is already we know that the dimension is equal to 4. So, from here you can check that this is true. So, after doing this one, the same way I will go for the subspace V.

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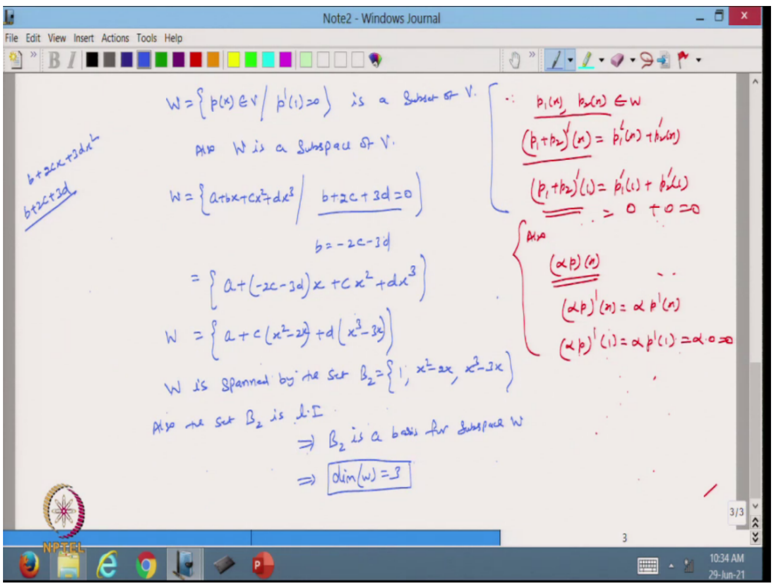

Now, I will discuss the W subspace. So, W $=\{p(x) \in V | p'(1) = 0\}$ is a subset of V.So, now, of course, is a subset of the vector space V also W is a subspace of V. How is it a subspace of V? Because let I take $p_1(x)$, $p_2(x) \in W$; then $(p_1 + p_2)'(x) = p_1'(x) + p_2'(x)$. Now, what about $(p_1 + p_2)'(1)$? So, that will be equal to $p_1'(1) + p_2'(1)$ and this is 0 + 0, that will be 0; because these two polynomials belong to the W and in the W. So, I should take the derivative. So, there is no problem, we will take the derivative.

So, derivative also we know this one and we also know that this is equal to this. So, this is $(p_1 + p_2)'(1)=0$; it means that the sum of two polynomials also belongs to the set W and also alpha p(x) is the one I want to do. So, I just want to take them. So, this one I want to check. So, I just take $(\alpha p)'(x) = \alpha(p)'(x)$, this will be $(\alpha p)'(1) = \alpha(p)'(1) = \alpha.0 = 0$. So, it means that the scalar multiplication also belongs to the set W and vector addition also belongs to the W; so it means that W is a subspace of V.

So, from here I can write W$=\{a + bx + cx^2 + dx^3 | b + 2c + 3d = 0\}$. So, W$=\{a + (-2c - 3d)x + cx^2 + dx^3\} = \{a + c(x^2 - 2x) + d(x^3 - 3x)\}$.

Now, from here you can say that, the W is spanned by the set; so I call it B_2, spanned by the set. So, here I can have $\{1, x^2-2x, x^3-3x\}$. So, this set is there. So, it is spanning the whole W also, the set B2 is linearly independent. So, this one is linearly independent; because it is a constant term, it is a x^2, x^3.

So, different, different order of the polynomial. So, definitely these are linearly independent. And from here we can say that W; so I can say from here that B_2 is a basis for subspace. So, it is a basis for the subspace W and which implies that the dimension of W will be 3. So, it is of dimension 3.

Now, again the same way that, since W is a subspace of V; so we can extend B_2 to a basis for vector space V. So, what we can do that, we can extend this one and from there. So, since now if you see from here that, this contain the second degree polynomial, third degree polynomial and the constant term, so x is not there.

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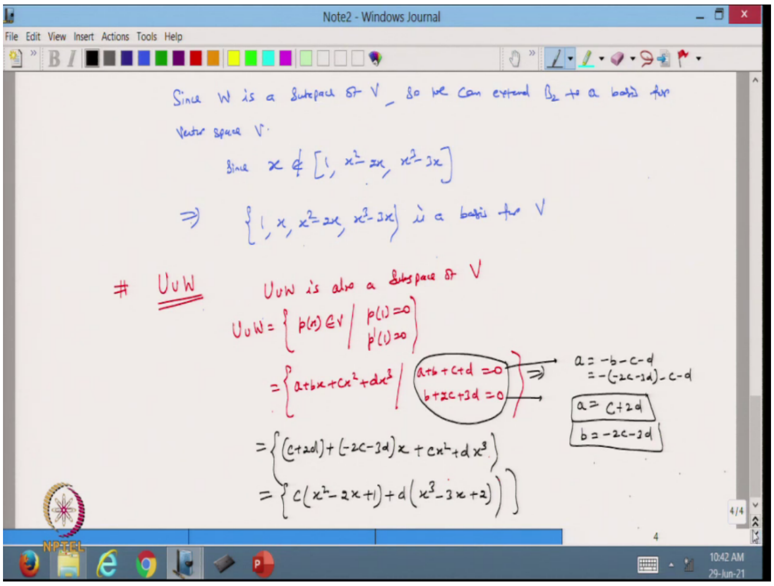

So, I know that x does not belong to the span $\{1,x^2-2x,x^3-3x\}$. So, it does not belong to this span, this one. So, from here we can say that this set $\{1,x^2-2x,x^3-3x\}$ is a basis for V. And now it is a 4-dimensional space. So, by extension theorem, we are able to extend the basis to the basis of V. So, this thing is over.

Now, we go for the next subspace and that subspace is; what about U ∩ W? Because this one is already we need to find out; what will be U ∩ W and U + W? So, these things we have to find out. Now, the space we from the previous knowledge we know that, that U ∩ W is also a subspace of V, because it is intersection.

And now from here I can find that, $U \cup W = \{p(x) \in V \mid p(1)=0, p'(1)=0\}$; So, it contains all the polynomials which belong to the set U. So, if it is a p(1), which belongs to the set W, which is $p'(1)=0$; $U \cup W=\{a+bx+cx^2+dx^3 \mid a+b+c+d=0; b+2c+3d=0\}$. So, $a=-b-c-d= -(-2a-3d)-c-d$; then $a=c+2d$. And, b we have written in the terms of $-2c-2d$. So, So, from here I can write that, this can be written as the set of polynomials that is $\{(c+2d)+(-2c-3d)x+cx^2+dx^3\}=\{c(x^2-2x+1)+d(x^3-3x+2)\}$.

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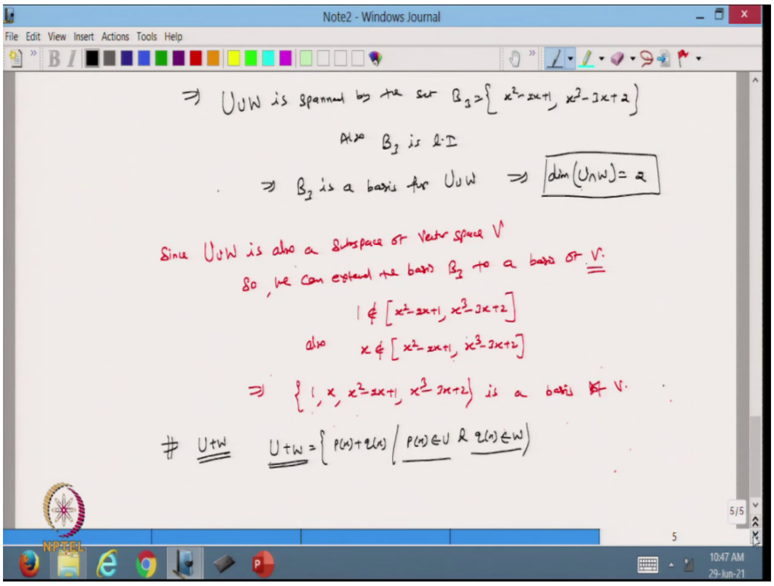

Now, from here which implies that, that U∩ W is spanned by the set I call it B_3. So, this is $\{x^2-2x+1, x^3-3x+2\}$. So, this is a polynomial we got from here. Now, also B_3 is linearly independent; because it is a third degree polynomial, it is a second degree polynomial. So, the B_3 is linearly independent; from here I can say that, B_3 is a basis for U ∩ W. And from here I can say that the dimension of U ∩ W is 2.

So, now I am able to find the dimensions and that is equal to 2. So, from here you can say that, this is of dimension 2. Now, since U ∩ W is also a subspace of vector space V; so we can extend the basis, extend the basis B_3 to a basis of V. So, it is of dimension 2; now I can extend this to the basis of V.

So, that we can do the same way we have done. So, this is the, this is very easily we can do; we can induct 1 from there, that because 1 is not there and x we can include and from there you will see that, this will extend to the basis ratio, because you can say from here that.

So, I can 1, because 1 does not belong to the span of $\{x^2-2x+1, x^3-3x+2\}$; also x does not belong to the span of$\{x^2-2x+1, x^3-3x+2\}$. So, from here I can extend the set as $\{1, x, x^2-2x+1, x^3-3x+2\}$.

So, now, I can there is a basis of V. So, from here you can check that, we are able to find three type of basis, different basis for the V. First we have found this one, then we have found the another set this one and now we are able to find this set. So, it is also a basis of V using the extension theorem.

Now, based on this one, I am going to do the find out, going to find out the other subspace that is we call it U + W. Then how can we find out U + W? So, in this case, this is the last one I need to do. So, U + W basically, U + W is contained in the polynomials p(x) +q(x) set of all the polynomials, such that p(x) belongs to U and q(x) belongs to W.

So, this is a set of polynomials; because I know that U +W that is also a subspace of the vector space V and it is made up of the polynomial vectors coming from U + the vector coming from the W. So, this is set to the polynomials we are going to take that, p(x) is coming from U and q(x) is coming from W. Now, from here based on this one we can say that.

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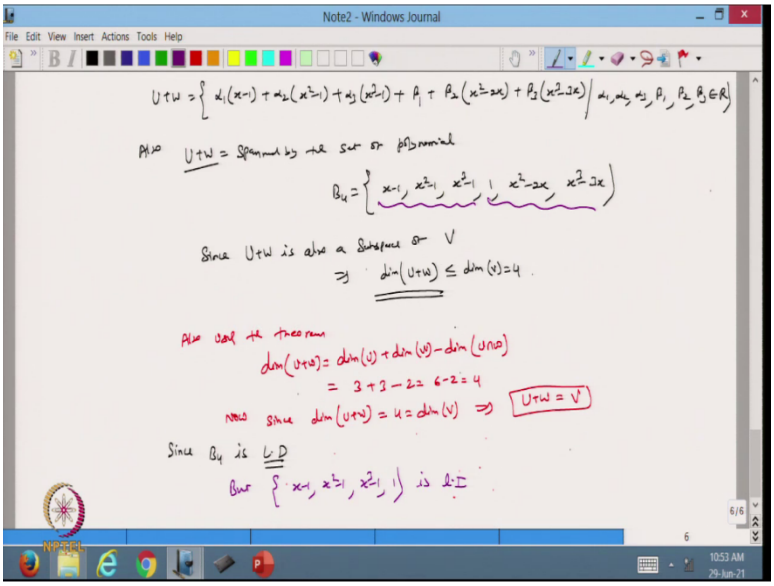

So, U+V or U + W is basically a set of polynomials. So, I can write the set of polynomials. So, the p(x) if I take. So, I am taking the p(x). So, p(x) should belong to the U. And if it belongs to the U, then I know that its basis is this one. So, these are the basis of U; So, these are the basis of U; $\{x-1, x^2-1, x^3-1\}$. So, I can write down that, so it contains the one.

So, I can write that suppose $\alpha_1(x-1)+\alpha_2(x^2-1)+\alpha_3(x^3-1)$. So, this is the polynomial coming from W. So, in the W; I know that, the basis of W is 1, x^2. So, this is the basis for W. So, $\{1,x^2-2x,x^3-3x\}$.

So, it is 1. So, I can call it $\beta_1(1+\beta_2)=x^2-2x$. So, $x^2-2x+\beta_3(x^3-3x)$. So, this is the polynomial we are going to have, such that α_1, α_2, α_3,β_1 , β_2, and β_3 that belongs to the real number. So, this is the set of polynomials that is coming in U + W and also. So, U+ W we are able to write; now we need to find the dimension of U + W.

So, I can say that U + W we already know, that is spanned by the set of polynomials. So, I am taking all the vectors that belong to the spanning of U and spanning of W. So, it will be $x-1,x^2-1,x^3-1,1,x^2-2x$ and x^3-3x. So, this is the basis I call it B_4.

So, this is a set that is spanning U + W and, but it contains the six elements. Now, from here I can write that since U+W is also a subspace of V. So, which implies we already know that the dimension of U+ W is less than equal to the dimension of V that is equal to 4.

It means its dimensions at max can be 4. Now, from here also using the theorem, we know that the $\dim(U+W)=\dim(U)+\dim(W)-\dim(U\cap W)$. So, from here I know the $\dim(U)=3$, $\dim(W)=3$ and $\dim(U\cap W)=2$. So,$\dim(U+W)=4$, that is equal to the dimension of V; which implies that U + W itself is equal to V. And if it is equal to V, then I can choose the dimensions. Now, automatically I can say that since B_4 is LD, linearly dependent; because we know that if the vector space is dimension 4, then any vector more than 4 will always be linearly dependent.

So, I can from here, now I know that the first three are linearly independent that we already know and the last three also linearly independent. So, maybe you can check from here that, if I take the set; but if I take a set $x-1$, $x^{2-}1$, x^3-1 and 1, so this is linearly independent. And if it is linearly independent, it means I can take this as a basis of this one.

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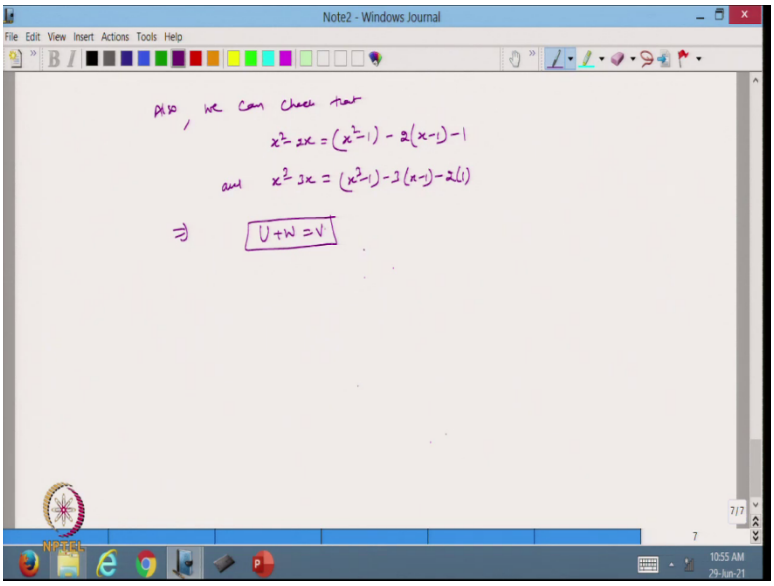

And also we can check that. So, I can check from here that the set x^2-2x can be written as (x^2-1)-2(x-1)-1, because from here you can say that x^2 and then it is -2x and - 1 + 2 - 1, so that will be 0. And x^3-3x can be written as (x^3-1)-3(x-1)-2(1);

So, from here you can check that this and these vectors can be written as the linear combination of the vectors before that one. So, it means that these are the LD and we can remove this one. So, from here we can say that, U + W = V and its dimension is, if the basis is another basis.

So, based on this one, we can say like this one. So, let me stop here. So, in the today's lecture, we have discussed about that how we can define the subspaces U+W, U∩W and how we can extend their bases to the dimension, to the basis of the vector space V. So, thanks for watching.

Thanks very much.