Matrix Computation and its applications
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Lecture - 25
Characterization of basis of a vector space and its subspaces

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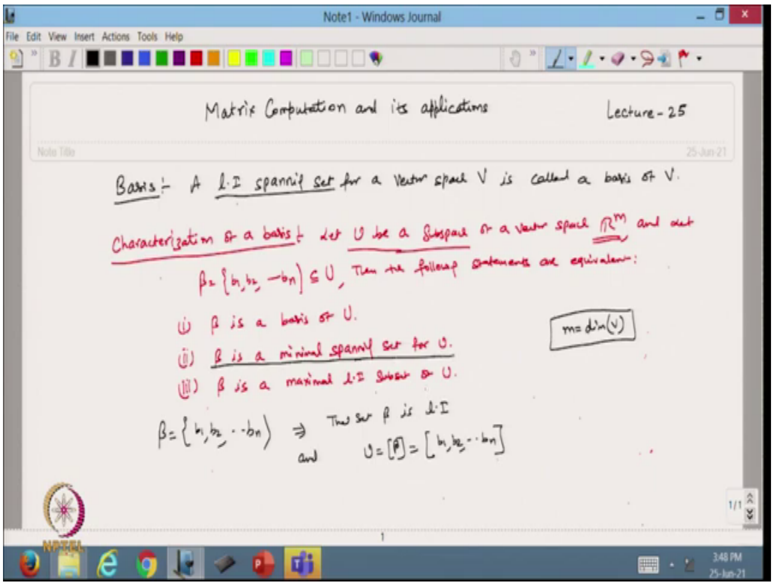

Hello viewers. So, welcome back to the course on Matrix Computation and its application. So, in the previous lecture we have discussed some applications of linear algebra. So, today we are going further for that one. So, let us start with this. So, today I want to give you one definition of the basis. So, as we already know, if suppose A linearly independent spanning set for a vector space V is called a basis of the vector space V.

So, it is the linearly independent spanning set, it should be linear independent and it should be a spanning set also. So, that definition we already know. So, now, based on this one I want to write the characterization of a basis. So, what are the characteristics of a basis? Now suppose let U be a subspace of a vector space say R^m and let I take a set beta that contains the elements $b_1, b_2, \ldots, b_n$ and they belong to the subspace U, then the following statements are equivalents.

So, in this case we are taking the U as a subspace and we are taking the vector space R^m the m dimensional vector space and let I take the set β which contain the vectors $b_{1,}\ b_{2,}\ldots,\ b_n$. So, this is the n number of vectors that are contained in the given subspace U, then the following statements are equivalent. So, the 1st one is that β is a basis of U, 2nd one β is a minimal spanning set for U and the 3rd one is that β is a maximal linearly independent subset of U.

So, all these three statements are equivalent. So, suppose I just take this one. So, now, this can be discussed now we have a β. So, beta is containing the elements $b_{1,}\ b_{2,}\ldots,\ b_n$. So, that we are taking n number of vectors and I know that the v is the dimension of v that is equal to m in this case.

Now if I say that the bβ is a basis of v implies that that the set beta is linearly independent because these are the basis and U is spanned by β or I can say that this is span by the vectors $b_{1,}\ b_{2,}\ldots,\ b_n$. So, then it becomes the basis. Now, the next thing says that beta is a minimal spanning set for U.

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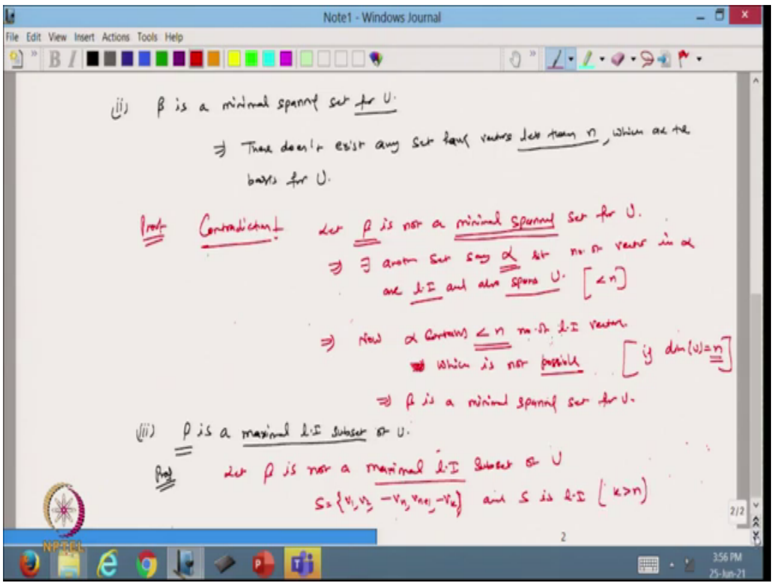

It means, if I say that beta is a minimal spanning set for U, it means that there cannot be a smaller set which contains the vectors less than n can be a spanning set for U which implies that there does not exist any set having vectors less than n, which are the basis for U less than n means suppose n is 5.

So, I cannot have a set which is less than 5 and which is also the basis for the vector U. So, this is the statement about the minimal spanning set. So, this can be just can be seen that suppose let us go by contradiction and we can prove this one that. So, let us go by contradiction.

So, let β is not a minimal spanning set for U which implies they exist another set say alpha such that. So, if it is a not a minimal spanning set it means, there is another set alpha such that α such that the number of vectors in α are l I and also spans U and of course, if it is a minimal spanning set.

Alpha is a minimal spanning set then also that this number of vectors will be less than or maybe I should take it should be it can be less than n because it is if it is not the minimal it means α is the minimal and a if α is the minimal then it definitely will contain the vector less than n numbers and which are also linearly independent and spans U.

Now, α contains a number of linear independent vectors. So, from here what you are getting that β is also one of the basis for U and α is also one of the basis for U and which having the less than number of less which has the vector number of vectors less than n. So, which implies that these things are not possible.

Because if I say. So, this can be seen that if I say that the dimension of U is n, then its basis definitely should contain the n number of elements, but here if I take alpha as a minimal spanning set less than beta, then it contains the vector less than n. So, which is not possible. So, from here we can say that this is the minimal spanning set. So, this way we can prove these things. So, which implies that beta is a minimal spanning set for U.

Now, the same way we can discuss the 3rd one. So, the 3rd one says that beta is a maximal linearly independent subset of U. So, since it is the maximal linearly independent subset it can be, we cannot have a subset of U which contains the linearly independent vector having more numbers than this set beta.

So, let us prove this one. So, let us do this one. So, we go by the contradiction let β is not a maximal linearly independent subset of U. So, it denotes the maximum linearly independent set. It means that suppose I have a set S which contain the elements maybe I can take the elements as $\{v_1, v_2, \ldots, v_n\}$, v_{n+1} or maybe v_k. So, it contains the k number of vectors and s is linearly independent it means the vectors are linearly independent. So, it means that I am considering that beta is not the maximal linearly independent subset of U.

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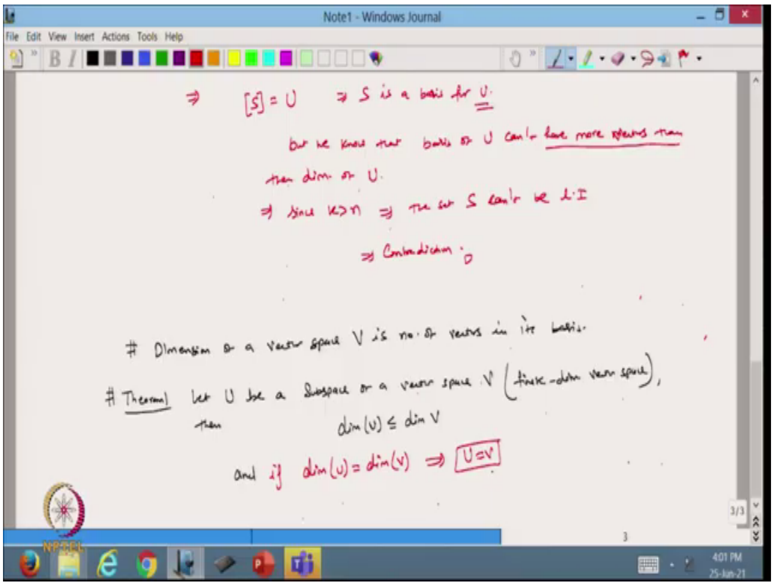

So, if it is not the maximal linearly independent of U, I take the other set S which contains the linearly independent vector which is in number k that is more than the n. So, here I am taking that the k is more than n. Now we already know. So, if it is the maximal which implies. So, if it is not the maximal this is the maximal. Now from here I can say that from here I can say that the set S is linearly independent and it also spans U.

So, which implies that S is a basis for U. Now, if it is a basis for U, but we already know the dimension of U, but we know that the basis of U cannot have more elements, more vectors cannot have more vectors than the dimension of U. Because if U is a n dimensional then its basis cannot have more vectors than any numbers. So, which implies that, but k is greater than n.

So, from here I can say that since k is greater than n which implies that the set the set S cannot be linearly independent because it is a if it is the basis containing n elements we know that the more elements, then n will be linearly dependent. So, which is a contradiction. So, which implies that the given set is the maximal set. So, that shows that beta is the maximal linearly independent subset of the vector subspace U.

Now, after that we also know one of the other definitions, we know that the dimension of a vector space V is the number of vectors in its basis. So, that we already know. So, based on this one, I want to discuss one theorem. Let U be a subspace let U be a subspace of a vector space V.

So, we are talking about finite dimensional vector space. So, let U be the subspace of the vector space V, then dimension of U will always be less than dimension of V because U is a subspace of V and if dimension of U is equal to dimension of V. So, that implies that U will be equal to V. So, this is another important theorem.

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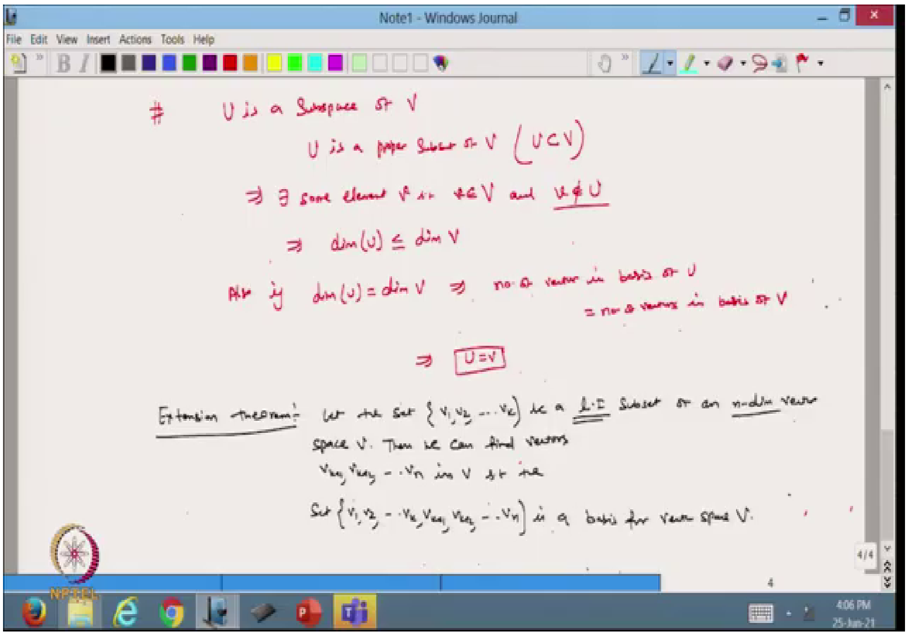

Now, this one we can see very clearly since U is a subspace of V. So, I can say that U is a proper subset of V that is U is proper subset of V. I am considering that it is not equal to V which implies that there exist some elements V such that such some elements V, such that V belongs to capital v belongs to the vector space and does not belongs to the given subspace U.

It means, dimension of U has to be less than dimension of V because I get a 1 vector which is not belongings to the given subspace it means that the basis of U cannot span the entire V because V is a one vector I found which is containing in the given subspace U and which is not in the V.

So, that implies the dimension of U is equal less than equal to dimension of V. Also if

Dim(U)=Dim(V) which implies that the number of vectors in basis of u is equal to number of vectors in basis of V and we know that if v is a n dimensional vector space then all the basis will contain n number of vectors. So, which implies that this is possible only when U is equal to V. So, from here I can say that the dimensions are the same then the set U and v will be the same. So, this is another important terminology.

Now, there is another theorem we want to discuss that is called extension theorem. So, in the extension theorem basically what we are going let the set I take a set which contain elements $\{v_{1,} v_{2,} \ldots, v_k\}$So, let the set it contains k elements be a linearly independent subset of an n dimensional vector space v, then we can find; we can find vectors that is $\{v_{k+1,} v_{k+2,} \ldots, v_n\}$ in V such that the set.

Now, I take the set $\{v_{1,} v_{2,} \ldots, v_k, v_{k+1,} v_{k+2,} \ldots, v_n\}$ is a basis for vector space V. It means, if I have a vector space n dimensional vector space and in that vector space if I get a set of vectors k number of vectors which definitely $k < n$, then we can extend these vectors because these are linearly independent.

So, I can extend these vectors to the n numbers such that it becomes the basis for the vector space V. So, in that case it will be linearly independent as well as it will span the whole v. So, this is the extension theorem. So, this theorem will be used in future for the extension of the basis.

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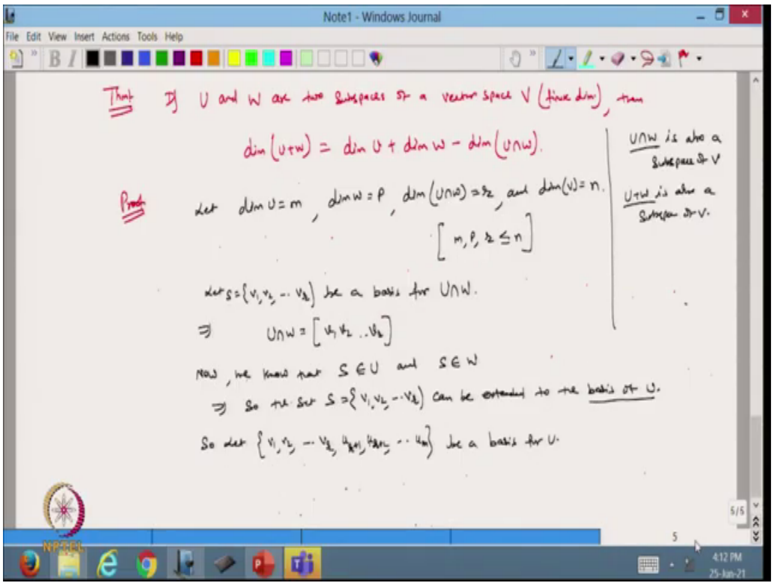

Now, I want to discuss a very important theorem that if U and W are two subspaces of a of a vector space V. So, that is finite dimensional, then dim (U+W)=dim(U)+dim(W)-Dim(U∩ W). So, in this way we can find out the directly the dim (U+W) because we already know that U ∩W is also a subspace of V that we already know and we also know that U + W is also a subspace of V.

So, those are two things we already know. So, this condition will be held. So, let us do the proof for this one. So, let I take that let first I assume that the let dim(U) = m, dim(W)=p, dim (U ∩W)=r, dim(V) = n.

Now, of course, m, p, r definitely will be less than equal to n because it is a subspace. So, definitely the dimension will be less than equal to n. Now I will take that let I will take a set { $v_1, v_2, \ldots, v_r$ } be a basis for the subspace U ∩ W which implies that U ∩ W is a span by this one and these are linearly independent.

Now, I just call this set. Maybe I will call S now we know that S belongs to U and S also belongs to W because these vectors will definitely lie in the U also and W also because they belong to the intersection now from here. So, the set S. So, this is the set which contains the r number of vectors can be extended to the basis of U because this set also belongs to U, but I can extend with the help of extension theorem I can make the basis of U. So, let I take the set

$\{v_{1,} v_{2,} \ldots, v_{k}, v_{k+1,} v_{k+2,} \ldots, v_{m}\}$. So, I will take or I should instead of v_1 will take them as a u. So, let us take u and u_m. So, this is a basis for the U.

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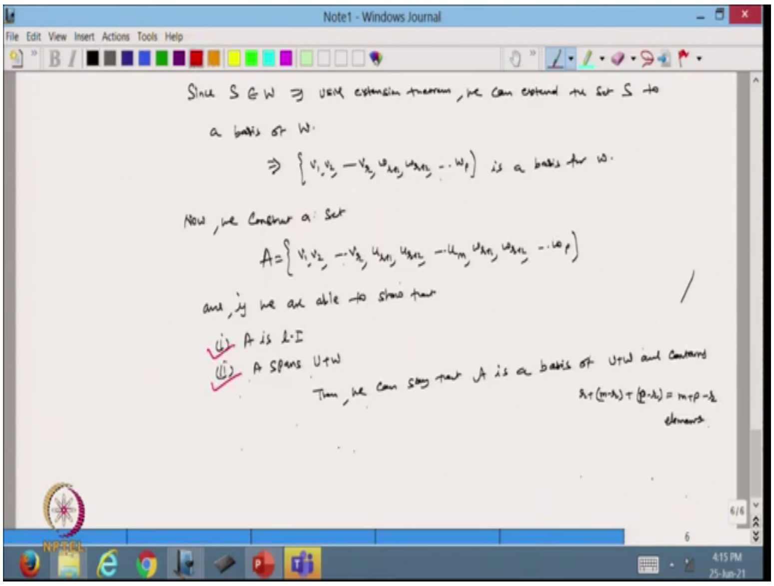

Now, since S also belongs to W so, which implies using extension theorem. So, using extension theorem we can extend the set S to a basis of vector space subspace W. So, which implies I can have a vector the set of vector $\{v_{1,} v_{2,} \ldots, v_{r}\}$ and then I can write $\{w_{1,} w_{2,} \ldots, w_{r}\}$. So, it goes up to p. So, it is p.

So, now is a basis for w. Now, we construct a set. So, I construct a set suppose I write A and that set contain the element $\{v_{1,} v_{2,} \ldots, v_{k}, v_{k+1,} v_{k+2,} \ldots, v_{m}\}$. and $\{w_{r+1,} w_{r+2,} \ldots, w_{m}\}$. So, we construct the set containing all the linearly independent vectors.

Such that. So, let me construct this and if we are able to show that the first thing is that A is linear independent and the second one is A spans U + W, then we can claim that A is a basis of U + W and contains. So, it contains r + m - r + p - r. So, that becomes basically m + p - r elements. So, these things will be there. So, now we need to show that A is linearly independent and A spans the whole U+W Now, so, let us prove the first one.

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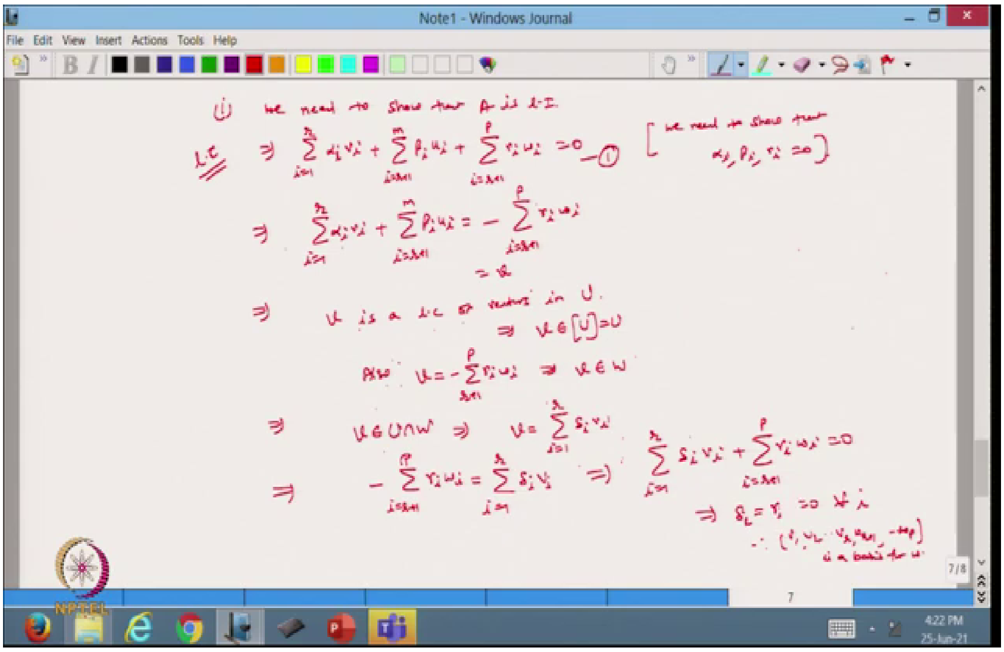

We need to show that A is linearly independent. So, this one we need to show. So, from here I will write I will take the linear combination of the elements this which implies that suppose I take $\sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i + \sum_{i=r+1}^{p} \gamma_i w_i = 0$.....(1). suppose I take the linear combination for this one. So, this is the linear combination. So, we need to show that α_i's, βi's and γi's all are 0. So, that we need to show. Now from here I can write this as $\sum_{i=1}^{r} \alpha_i v_i +$

$$\sum_{i=r+1}^{m} \beta_i u_i = - \sum_{i=r+1}^{p} \gamma_i w_i = v$$

And let this be equal to v. I just take. It means I am taking this as a vector v. Now from here I can say that v is a is can be written as a is a linear combination of vectors in U because v_i's and U_i's are the basis for the subspace U and taking this linear combination I am getting the value v.

So, which implies that v belongs to U the spanning of U basically it is U also v is $- \sum_{i=r+1}^{p} \gamma_i w_i$ which implies that v can also be written as a linear combination of w_i's. So, it means that v also belongs to w. Now from here if v belongs to W, then which implies that v also belongs to U∩W. So, if belongs to U∩W implies that v can be written as some linear combination $\sum_{i=1}^{r} \delta_i v_i$ because these are the basis v_i are the basis for U∩W.

So, now from here I can write that. So, v is already this one. So, from here I can write that $-\sum_{i=r+1}^{p} \gamma_i w_i$ is equal to $\sum_{i=1}^{r} \delta_i v_i$ and from here I can write that $\sum_{i=r+1}^{p} \gamma_i w_i + \sum_{i=1}^{r} \delta_i v_i = 0$.

So, now it is a v_i and w_i already we know that they are the basis for w. So, from here it is very easy to check that it means that δ_i and w_i all are 0 for all i because the set $\{v_1, v_2, \ldots, v_r\}$ and $\{w_{r+1}, w_{r+2}, \ldots, w_p\}$ is a basis for w. So, from here all the δ is and will become 0. So, if δ_i and will become 0, then from here. So, I just can give the name of this equation. So, let us write one.

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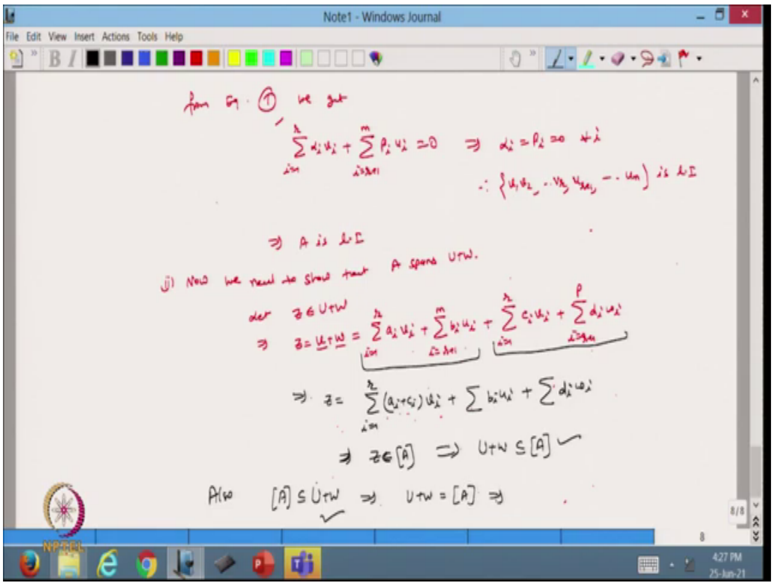

Then from equation 1 we get. So, now, γ_i it became 0. So, γ it became 0. So, from here I will get that $\sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i = 0$, which implies that all α_i's and β_i's are 0 for all i because this set $\{v_1, v_2, \ldots, v_r\}$, $\{u_{r+1}, u_{r+2}, \ldots, m\}$ is linearly independent that we already know because it is a basis.

So, from here we can say that which implies that the set A is linearly independent. So, this is we are able to show that A is linearly independent. Now we need to show that it spans U+W. So, it is very easy to show now. So, the 2nd one we need to show. So, now, we need to show that A spans U+W.

Now, let I take element z belongs to U+W , which implies I can write z as some element u+w and which again because if it is in the u, I can write this as a linear combination

$\sum_{i=1}^{r} a_i v_i + \sum_{i=r+1}^{m} b_i u_i$ because this is the basis for this one plus.

Similarly, I can write $\sum_{i=1}^{r} c_i v_i + \sum_{i=r+1}^{p} d_i w_i$ because this element w belongs to w and this u belongs to u. So, u can be written as a linear combination of this one and w can be written as a linear combination of this one.

So, after writing this one I can say that which implies because this can be combined. So, I can write that $z = \sum_{i=1}^{r} (a_i + c_i) v_i + \sum_{i=r+1}^{m} b_i u_i + \sum_{i=r+1}^{p} d_i w_i$ and which is if you see that is the linear combination of the vectors of A.

So, which implies that z belongs to the span of A. So, from here we can see that z belongs to the span of A. It means that I started with a z from u. So, from here I can write that U + W is contained in span A. Also because A definitely contains the elements the vectors and all vectors are coming from U + W.

So, also I can say that a span is subset of U + W because I can take the element A from the span of a and that can be shown with this that that belongs to U + W. So, from here I can using this and this condition, I can say that U + W is equal to the span of A. So, it means from here I can say that A is a basis for U + W.

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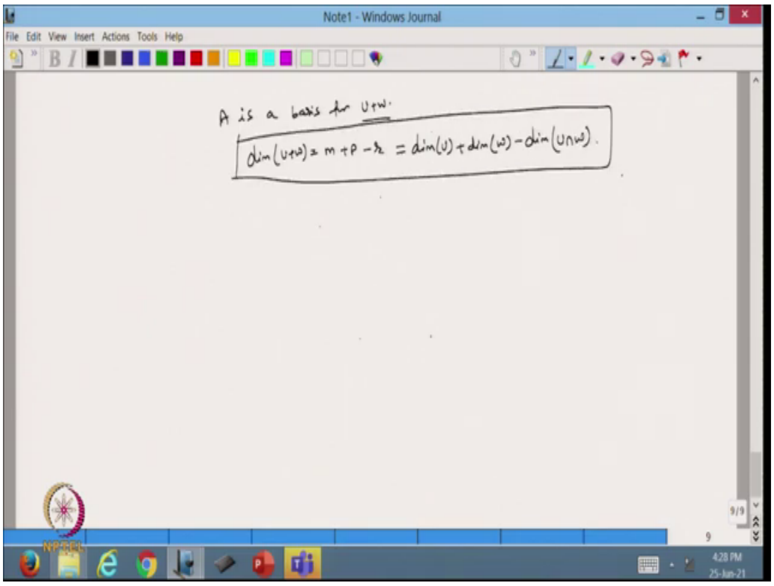

Now, if it is a basis of U + W then definitely I can check that the dimension of U + W is. So, we have shown this one that it contains m + p - r elements is m + p - r. So, that is equal to dim(U+W)=dim(U)+dim(W)-dim(U∩ W). So, that is the proof of the given theorem.

So, now we will stop here. So, today we have discussed very important theorem that the

dim(U+W)=dim(U)+dim(W)-dim(U∩ W). So, I hope you have enjoyed this one. Thanks for watching.

Thanks very much.