Matrix Computation and its applications
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Lecture - 24
Application of zero null space: interpolating polynomial and wronskian matrix

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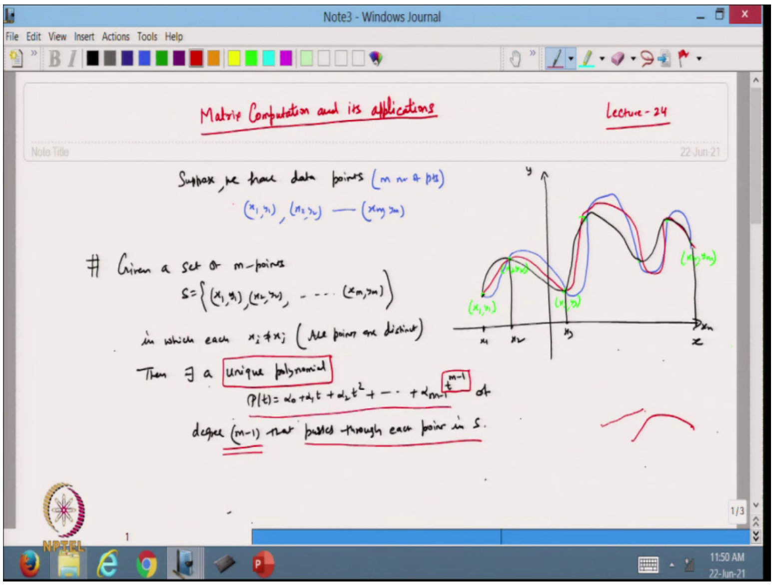

Hello viewers, welcome back to the course on Matrix Computation and its application. So, today's lecture we are all going to discuss a few more applications of the statement, that the null space if the null space of the matrix A is only the 0 null space, then the corresponding matrix, if it is a square matrix is a nonsingular matrix. So, we are going to take the application based on this one. So, let us do that.

So, today we are going to discuss one more term and that is very useful these days. So, now, suppose we have data, suppose we have data points. Let me just take the x and y axis. And, suppose I have some data points that are given to me. So, these data points can be anywhere. And, suppose this data I can write as $\{ (x_1, y_1), (x_2 y_2), \ldots (x_n, y_n)\}$. So, in this case suppose I have m number of elements. So, suppose we have the data points, that is $\{ (x_1, y_1), (x_2 y_2), \ldots (x_m, y_m)\}$, so, m number of points are there number of points. So, these are the data points given to us. And, now suppose I just want to interpolate these points with the polynomial.

So, now, what I do is that I just take a polynomial passing through this point. So, suppose this is the polynomial and I am going from here, then reaching here this one, suppose this is the point here. Suppose this is not the point, it is just maybe this point is there.

So, let us suppose I have this number of points and I take a polynomial that is passing from these points. But, some student says that, I can draw another polynomial not like this one, maybe I can have a polynomial like this one going from here, then this one passing through all points and then this one. Maybe some other students can plot another polynomial like this one, this polynomial.

So, by this way we can have a large number of polynomials, we can show which are passing through these points. So, based on this one I just give you one statement that gives a set of m points. So, I take the set S as points are given to me (x_1, y_1), which means this is my corresponding x_1 and this is the point corresponding to, say, that is y_1. So, $\{(x_1, y_1), (x_2, y_2), \ldots (x_m, y_m)\}$. So, m number of points are there given to me in which each $x_i \neq x_j$. Means all points are distinct, this is my x_1, this is corresponding to x_2, this may be x_3 and this is the last one is x_n. So, all my x_i's are different. We know that one. It means we are taking the point at this point, and this point, and this point only once never two times. So, that is we are taking.

Then, so, we give the statement then there exist a polynomial or may be I can say there exist a unique polynomial, that I write as a $P(t)=\alpha_0+\alpha_1 t+\alpha_2 t^2+\ldots+\alpha_{m-1} t^{m-1}$. So, its degree is m-1 that passes through each point in S.

So, it shows you that the first thing is that we are going to have a unique polynomial. And, its degree will always be m -1, because we know that we suppose we have 2 points, then we can have a line passing for this one. So, it is of degree 1, when we have 3 points suppose I have 1, 2, 3. So, we can have a quadratic polynomial from this one.

So, we have 3 points, but it is going to be only quadratic. So, we have m points and we are going to have a unique polynomial of degree m-1. So, from here one thing is clear, that we have drawn 3 different types of polynomials, but by this theorem we are going to show that only 1 polynomial, that is a unique polynomial is going to pass through all this point. So, this one we are going to use.

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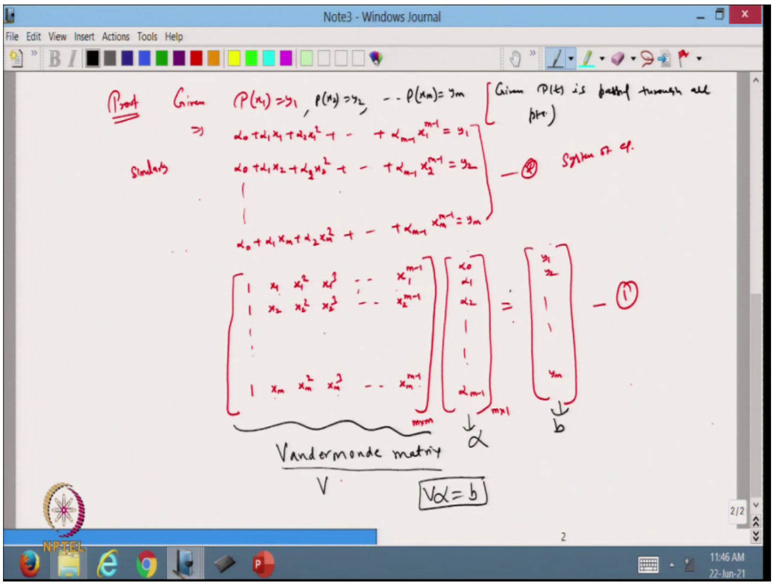

Now, so, the proof of this one, so, in this case we are going to have a unique polynomial. So, given that polynomial at $P(x_1)=y_1$, because it is passing through this point. Which implies that $\alpha_0+\alpha_1 x_1+\alpha_2 x_1^2+\ldots+\alpha_{m-1} x_1^{m-1} = y_1$. Similarly, because it is going to pass through all points, so, given that given P(t) is passing through all points. It means that

$P(x_1)=y_1$, $P(x_2)=y_2$,$P(x_3)=y_3$,..., $P(x_{m-1})=y_{m-1}$,$P(x_m)=y_m$. So, from here I can write as

$\alpha_0+\alpha_1 x_2+\alpha_2 x_2^2+\ldots+\alpha_{m-1} x_2^{m-1} = y_2,\ldots,$ $\alpha_0+\alpha_1 x_m+\alpha_2 x_m^2+\ldots+\alpha_{m-1} x_m^{m-1} = y_m$. So, this is the way we can write the corresponding matrix. $\left[\vdots\ 1\ x_1\ x_1^2 \ldots x_1^{m-1}\ 1\ x_2\ x_2^2 \ldots x_2^{m-1}\ \ 1\ x_m\ x_m^2 \ldots x_m^{m-1}\ \vdots\right]\left[\alpha_0\ \alpha_1\ \alpha_2\ \vdots\ \alpha_{m-1}\right]=\left[y_1\ y_2\ y_3\ \vdots\ y_m\right]$. So, now if you see from here, this corresponding matrix, it has some special name and that is it is called Vandermonde matrix.

So, and represented by V, so, it is just the matrix $1\ x_1\ x_1^2 \ldots x_1^{m-1}$ like this. So, it is called the Vandermonde matrix. And, suppose this α_i just write as maybe some α and this is what I can write as b. So, I get the system V α=b. Now, I am going to show that this is a unique one. So, this polynomial is unique. So, this polynomial is unique means the solution of this system should be unique.

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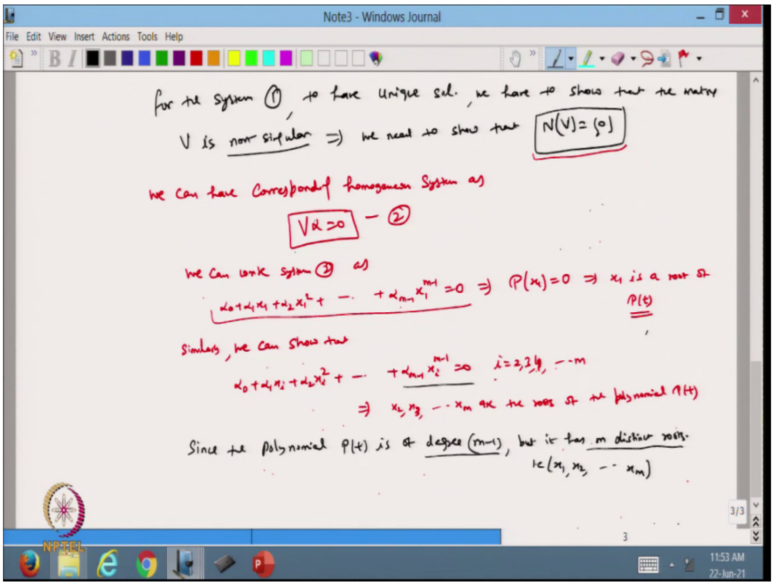

Now, for system 1, to have a unique solution, we have to show that the matrix V is non-singular, because if it is a non singular only then we are going to have a unique solution that we already know from the matrix theory. So, which implies that we need to show that the null space of the matrix V is just a containing the 0 element. So, this is equivalent to saying that the null space of N(V) = 0, that we have already seen in the previous results.

So, now we are going to show these things. Now, what are you going to have? So, I want to show this thing now. So, let us write. So, we can have corresponding homogeneous system as $V\alpha = 0$, where V is this matrix and α is this value. Now from here we have so, $V\alpha = 0$. So, from here we can write from here. And, α is made up of this $\alpha_0, \alpha_1, \ldots \alpha_m$. So, from here I can write, we have or maybe we can write. So, I will just write the first one. So, first one will be $\alpha_0+\alpha_1 x_1+\alpha_2 x_1^2+\ldots+\alpha_{m-1} x_1^{m-1}=0$. So, this is the first equation. Now, which implies if you see from here, that I am putting the x_1 in the given polynomial and getting the value equal to 0, which implies that $P(x_1)=0$, which implies that x_1 is a root of the polynomial P(t). So, this is the polynomial I have P(t). So, that the polynomial we have defined this one and it is going to show that x_1 is a root of this P(t) by this way.

So, similarly we can show that $\alpha_0+\alpha_1 x_i+\alpha_2 x_i^2+\ldots+\alpha_{m-1} x_i^{m-1} = 0$ for i=2,3,….m, which implies that x_2, x_3 all x_m are the roots of the polynomial P(t) that we are defined.

So, from here one thing is true that since the polynomial P(t) is of degree m-1, we know that the polynomial is of degree m-1. But, it is distinct, it has so, you can see from it that it has m distinct roots, it is of degree m-1, but it is having m distinct root that is $x_1, x_2, \ldots, x_m$, that we have shown from here, this one.

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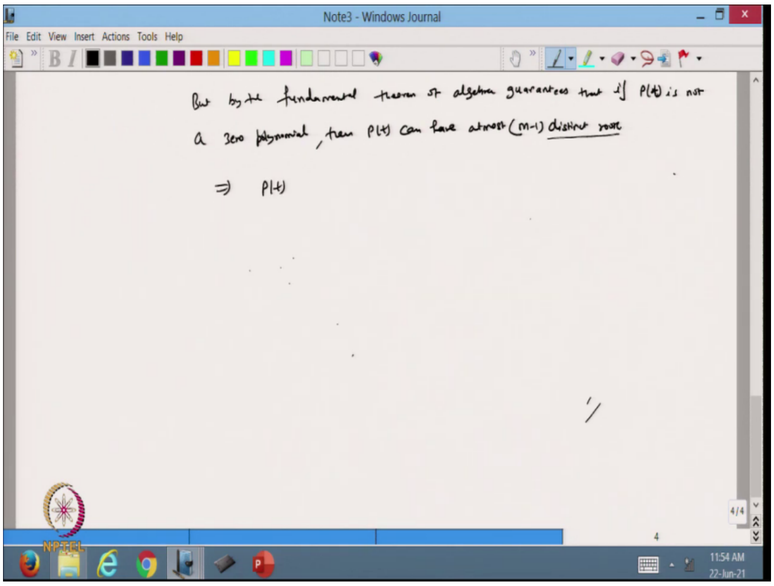

But, that we know that by the fundamental theorem of algebra, that guarantees that if P(t) is not a zero polynomial, then P(t) can have at most m-1 distinct roots because if it is of degree m-1 then it can have a maximum m- 1 distinct root. So, which implies that my P(t)=0, but that is not the 0 polynomial we have taken this one passing through these points.

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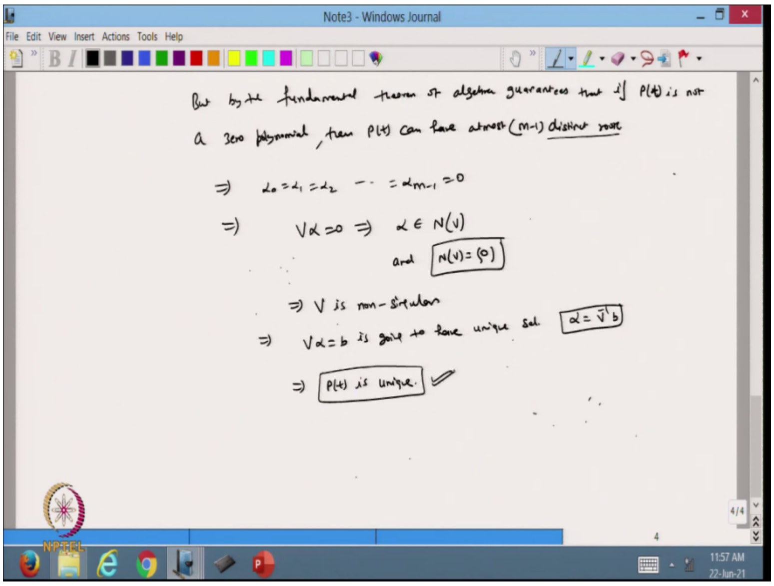

So, which implies that, that this is the only possibility, that $\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_m = 0$. So, it shows that this polynomial should be equal to 0, itself only. And, which implies. So, if all these α_i's = 0, then from the system 1 from the system 2, I can say that, which implies that $V\alpha = 0$ gives you that α belongs to the null space of V.

Because, it is going to 0 and the null space of V contains only 0 element. Because definitely α will be definitely in the null space of V and the values I am getting $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m$, all these values I am getting 0. So, null space will contain only this element, only 0 element.

So, that implies that the corresponding matrix V is non singular and if it is a non singular matrix. So, which implies that V alpha is equal to b is going to have a unique solution and that solution I can write as V inverse b. So, if it is going to have a unique solution it means the alpha i's we are going to get in this case are unique.

So, from here I can say that my polynomial P(t) is unique. So, whatever the points you are taking you can draw different different polynomials, but from by this theorem we are going, we have shown that only a unique polynomials is going to be that passing through the all the points. So, that is the proof of this one. So, this polynomial will be unique.

And, this polynomial you can find in a different way, but this is going to be unique if it is passing through all the points. So, that is one of the applications of this property of null space containing, null space is going to be only 0 space. So, this is one of the applications we have taken.

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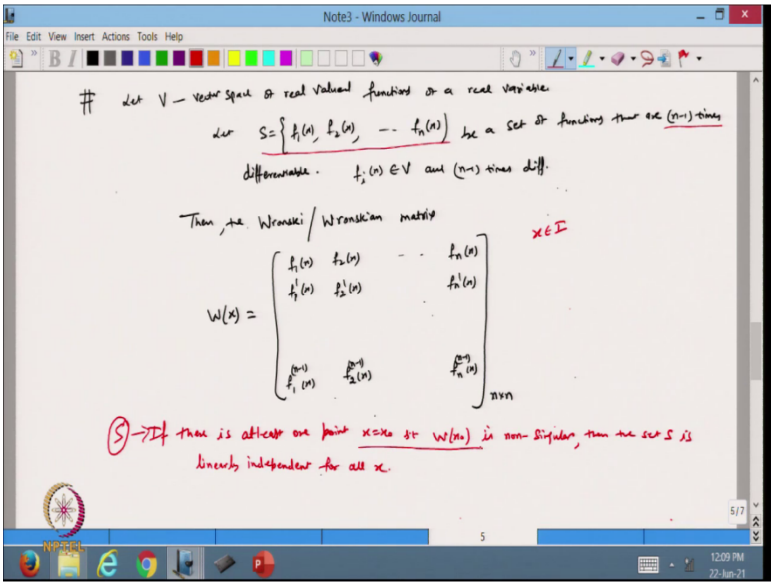

Another application I want to apply for, I want to show you. Now, this is the application we are going to discuss. Suppose I have a vector space V is a vector space of real valued functions of a real variable. And, let I take a set S, which contains a number of functions coming from this one.

So, be a set of functions that are n-1 times differentiable. It means that, each $f(x_i)$ belongs to the vector space V and n times and n -1 times differentiable. So, that is the meaning of this one. So, and the set S is there. Now, so, then the there is a word term Wronski or we also some books write like a Wronskian, then the Wronskian matrix or Wronski matrix. So, that this is the matrix we are going to write $f_1(x), f_2(x),\ldots f_n(x)$. So, I am going to write this matrix with the function. Second row is just the derivative taking the derivative of each function. And, similarly I can go and I can take n-1 time derivative, I am writing in the same column. So, I get this matrix, it is n $\times$n matrix and we call it the Wronski matrix. So, this matrix I have written.

Now, so, now, the statement this is what we are now if there is at least one point that is $x=x_0$. So, x is below basically x belongs to some interval in the given domain. So, this is a standard. If there is at least one point $x = x_0$, such that the value of the Wronskian at x_0. So, if it is non singular then the set S is linearly independent for all x. So, this is my statement, S means statement.

What I'm going to do is, I take a vector space and suppose that the factor space is made up of functions and suppose I take a set S and somebody asks me . Tell me whether this set is linearly independent or dependent, then how we are going to check whether this set is linearly independent or dependent? So, this is what we are going to do. So, and in this case we are finding that these functions are n-1 time differentiable.

Then, we have written the Wronski matrix and then we say that, if I choose any point only one point x equal to x_0 and I found that, at this point this Wronskian matrix is non-singular. Then just I can say from there that this set S is going to be linearly independent. So, that is the powerful way, that only one point if it is satisfied it is satisfying for all. So, this one we are going to prove. So, let us do this one.

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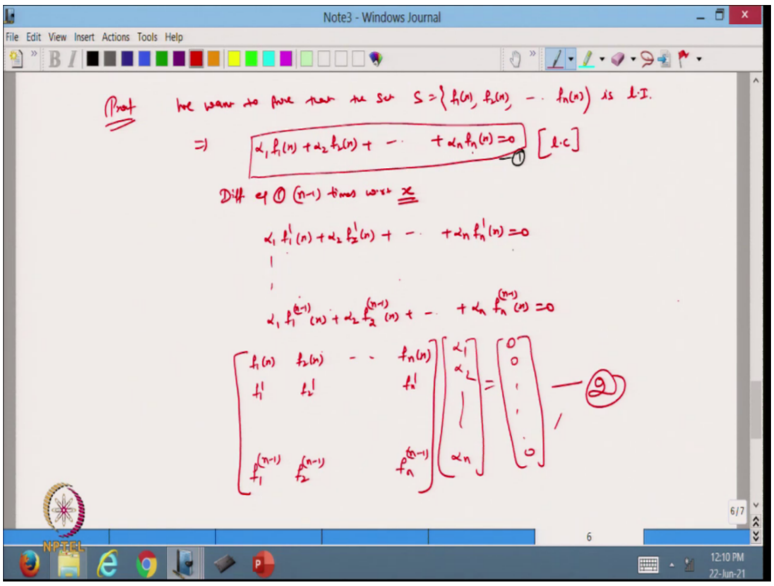

Proof: We basically want to prove that the set S of the function $f_1(x)$, $f_2(x)$, $\ldots f_n(x)$ is linearly independent. So, this one we want to show. So, suppose I have a function and I want to show the linear method. So, which implies that, I will take some scalars may be I can write $\alpha_1 f_1(x)+\alpha_2 f_2(x)+\alpha_3 f_3(x)+\ldots+\alpha_n f_n(x)=0$. So, this is my linear combination and I am putting it equal to 0. So, this one we have taken.

Now, after doing this one, I can write this as equation number (1). Now, how to prove this one? So, I will try to differentiate equation (1), n -1 times with respect to x. Because, I assume that my functions are differentiable and it can have n-1 times differentiable function even. So, I can write here $\alpha_1 f_1^1(x)+\alpha_2 f_2^1(x)+\alpha_3 f_3^1(x)+\ldots+\alpha_n f_n^1(x)=0, \ldots.$

$$\alpha_1 f_1^{n-1}(x)+\alpha_2 f_2^{n-1}(x)+\alpha_3 f_3^{n-1}(x)+\ldots+\alpha_n f_n^{n-1}(x)=0.$$

So, from here I can write this system as

$$[f_1(x) \quad f_2(x) \quad f_3(x), \ldots \quad f_n(x)\, f_1^1(x) \quad f_1^1(x) \quad f_2^1(x) \ \ldots \ f_n^1(x) \ \vdots \ f_1^{n-1}(x) \quad f_2^{n-1}(x) \quad f_3^{n-1}(x) \ \ldots \ f_n$$

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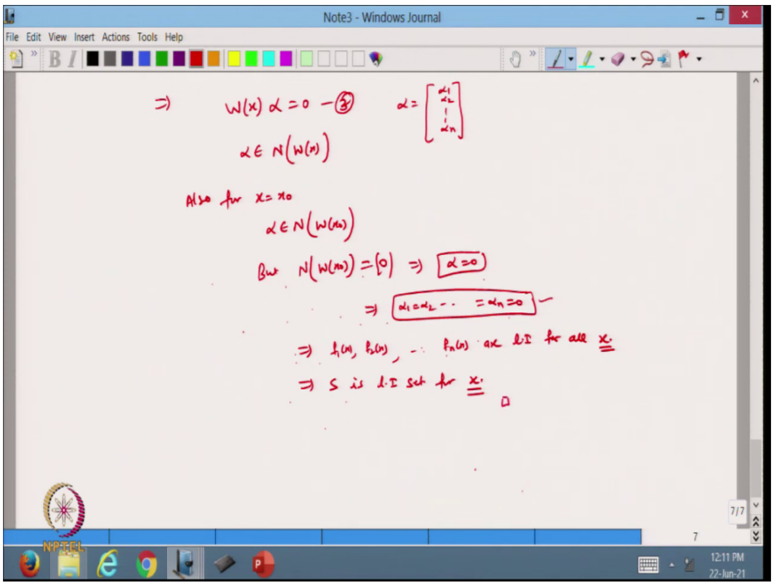

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And, now from here this can be written as you can write from here that this becomes a Wronskian matrix. So, because this is Wronskian matrix and this is α. So, I can write as W(x) $\alpha = 0$-----(3). So, this is a system I can write as equation (3), where α is basically $\alpha_1, \alpha_2, \ldots \alpha_n$.

Now, it is given to me now. So, from here it is a homogeneous system. So, homogeneous system is basically from here I can write, that α belongs to the null space of the Wronski matrix. Also now it is given to me that for $x = x_0$, α will belong to the null space of x_0, because it is true for all x.

So, if I choose just $x = x_0$ then definitely α will be there, but null space is just the 0 element it is written from here, that is if there is at least one point where this matrix is non singular. So, it means if it is a nonzero, it means that the null space of this is going to have only one element. So, which implies that my α should be 0, and which implies that my $\alpha_1, \alpha_2, \ldots \alpha_n$ all should be 0.

And, if it is 0, then it shows that the functions $f_1(x)$, $f_2(x)$, $\ldots f_n(x)$ are linearly independent for all x. Because, here we are representing with just one point and then shows it shows that the α_1, α_2, … α_i's all are 0 and it this is the linear combination I have taken and from this linear combination we found that $\alpha_1, \alpha_2, \ldots \alpha_n$ are 0 and it is true for all x in this case now.

So, from here I can say that this is linearly independent for all x. So, which shows that S is a linearly independent set and that is true for all x. So, this way we can prove the set of functions whether they are linearly independent or dependent. So, after this we will stop here.

So, today we have discussed two important applications of the statement about the null space, 0 null space corresponding to the given matrix. And, we have shown that the interpreting polynomial for the given data is always unique and then we have shown how we can check the given set of functions to be either linearly independent and dependent. So, I hope you have enjoyed this lecture, thanks for watching.

Thanks very much.