Matrix Computation and its applications
Dr. Vivek Aggarwal
Prof. Mani Mehra
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture - 23
Application of diagonal dominant matrices

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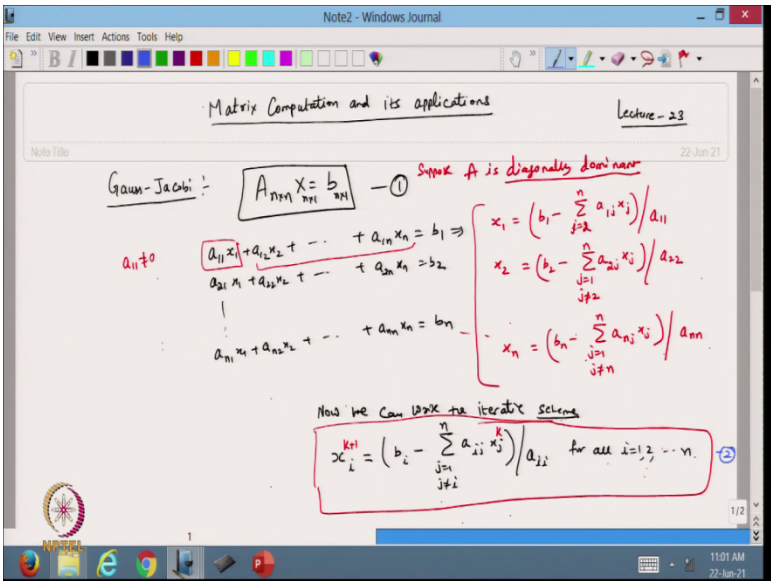

So, hello viewers, welcome back to the course on Matrix Computation and its application. So, today we are going to discuss the sufficient condition for convergence of the Gauss Jordan methods. So, let us discuss that one.

So, in the previous lecture, we have discussed the definition of diagonal dominant matrix. So, today I am going to discuss how the diagonal dominance matrix is used to show the method Gauss Jordan, sorry Gauss Jacobi basically, Gauss Jacobi method.

So, it is not the Gauss Jordan, Gauss Jacobi we are talking about. So, suppose I have a matrix A and that matrix is a n × n and we have a system of equation Ax=b. So, this is n ×1 and this is n × 1. So, this is my system and it this system is a very big system, I denote a 3 × 3 matrix or it may be a million by million matrix. So, in this case, I want to solve this system to find out the solution.

So, in the real applications, we always go with this type of matrix, so that the dimension of the matrix we can have a very large dimension. So, we apply the iterative process. So, in the Gauss Jacobi method, suppose we have this one. So, I can write this matrix as $a_{11}x_1+a_{12}x_2+\ldots\ldots..+a_{1n}x_n=b_1$. Similarly, we can write $a_{21}x_1+a_{22}x_2+\ldots\ldots..+a_{2n}x_n=b_2$; … $a_{n1}x_1+a_{n2}x_2+\ldots\ldots..+a_{nn}x_n=b_n$. So, this is my corresponding system of equations. Now, what I am going to do is that, in this equation. So, from here I am just keeping these things on the left hand side and taking the other terms on the right hand side. So, what I am going to do is that, I am going to have here $x_2=(b_1 - \sum_{j=2}^{n} a_{1j}x_j)/a_{11}$.

Now, I can write from here that I am moving from 1 to n. So, in this case I can have this j. So, basically I will take j from 1 to n and I can write that the j is not equal to i, and this is true for all i; it means suppose I am talking here. So, it is 1, 2, 1, 3. So, maybe I can write here. So, I will just write j or maybe I can write from here 1 j. So, this one I can write and then divide by a_{11}.

So, my matrix $a_{11\neq 0}$ and also. So, this is my matrix and also I am writing that, let us suppose matrix A is diagonal dominant. So, by this definition we already know what the diagonal means. So, suppose we are also considering that A is a diagonal dominant matrix and then we are going to write the Gauss Jacobi method for solving this system iteratively.

So, I can write like this one. Similarly from the next equation, I can write my x_{22}; so that can be written as $x_2=(b_2 - \sum_{j=1}^{n} a_{2j}x_j)/a_{22}$ for $2\neq j$. So, it is {2, 2, 2}. So, maybe I can write here, just for the indexing I can write from 2. So, all this I can write in the form of this equation. So, it will be b_n summation j from 1 to n; $a_{nj}x_j$ and then divided by a_{nn}, j is not equal to n. So, $x_n=(b_n - \sum_{j=i}^{n} a_{1j}x_j)/a_{nn}$ for $n\neq j$. So, whatever I am taking, this will correspond to this system. So, now, from here this becomes how we can write our system. Now, based on this system, I can make this system in the iterative form. So, from here you can write this as. So, each equation can be written now. So, I can write this equation as. So, now, we can write the iterative scheme; how? So, I am finding x that is; maybe I should write here not x_{11}.

So, I can write my x_k as b_k minus summation j from 1 to n a_{ij} x_j, $j \neq i$ divided by a_{ii}, and this is true for all i is equal to 1, 2, 3,…. n. So, this one I am writing or maybe I can write this as; because I am writing i, so I should write here i only, not k, so I will write i. So, my x_i is equal to b_i minus this part, where $j \neq i$. So, this is the corresponding system, this system I have written with the single equation. So,$x_i^{k+1} = b_i = \sum_{j=i}^{n} a_{ij} x_j^k$ for all i=1,2,….n and i$\neq j$. Now, based on this equation, I want to make it an iterative process. So, in the iterative process what I am going to do is that, suppose I am finding at k + 1 step and this value is given to me at the k step. So, this is the corresponding iterative scheme we get. So, this is my iterative scheme.

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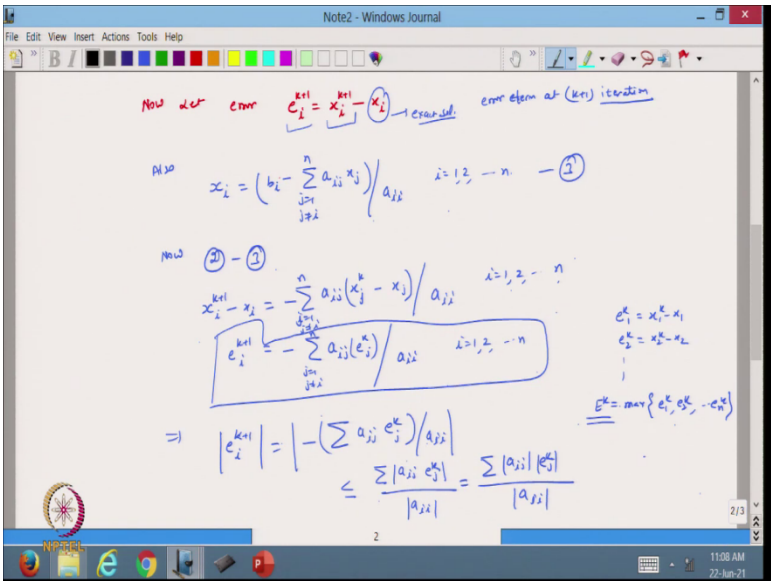

Now, this is my iterative scheme. Now, let me introduce the error terms, error e_i I am taking. So, this is suppose I am calculating at the k +1 step, that is equal to I am finding the value of x_i at k + 1 step minus this is the exact value. So, this is the error we are taking. So, basically this is the exact solution.

So, this is basically the exact solution, because in the iterative scheme we are going to improve our solution, starting with the initial solution and this is the exact solution; whatever the solution is there, if we are able to solve the system exactly, then this is supposed to be the exact solution. And this is the solution at the k +1th step and I am taking the error e_i at that time. Now, this equation I can give the name as equation number 2. Also I can write my x_i is equal to b_i minus summation j from 1 to n and $j \neq i$, a_{ij} x_j divided by a_{ii}, where i = 1, 2,…n. So, $x_i=(b_i - \sum_{j=i}^{n} a_{ij} x_j)/a_{ii}$ for $i \neq j$. So, this equation I write (3). Now, (2) -(3); I subtract this equation, 2nd equation minus the 3rd equation. So, what I will get is, I will get my x_i at k +1th step - x_i and this b_i will cancel out, because b_i and b_i are the same.

So, if you see, I will get only summation. So, I may be this minus this. So, - x_i. So, from here, I am getting - a_{ij} x_j ; k+ 1; if you see from here, - x_j. So, this one I can write, divided by a_{ii}. So, the same thing we are getting.

So, minus plus. So, I may be, I can take the sign minus, it just does not matter. So, this one I am getting. So, now, from here, you can check that this I can write as e_i^{k+1}, the error at the k + 1th step and this is true for all I= 1, 2,…. n.

So, this will be minus summation a_{ii} and everything is same; i from moving from 1 to n, j is moving from 1 to n and $j \neq i$. Similarly here j is moving from 1 to n and $j \neq i$ and this will become e_i^k, this value is divided by a_{ii}. So, this is true for all i, this one. So, I got this relation.

So, this is an error term we are getting at. So, I can write that this is the error term, error term at k + 1th iteration. Now, so you can see that I have e_1^k, that is x_1^k - x_1; x_2^k - x_2 like this one. So, what I do is that, I choose e_k, that is maximum of all e_1^k, e_2^k, …, e_n^k. So, I just take the maximum in this case.

So, now from here I can write this equation as. So, before that one, I can just get rid of this negative sign. So, I can just write from here, I can write e_i^{k+1} magnitude; this one I can write as minus summation a_{ij} e_j^k/ a_{ii} this one. And this can be written; as I know that, this can be written as summation a_{ij} e_j^k/ a_{ii} . So, I have taken this modulus value inside the summation and I got this less than equal to relation so and this is true for all i's, so that is understood.

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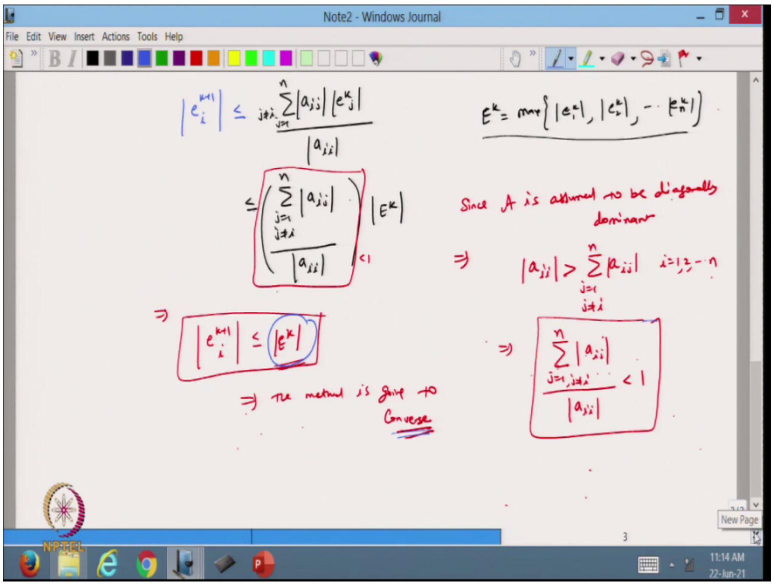

Now, from here, I am going to apply this one. So, I can write this as e_i^k modulus value is always less than. So, I am just taking summation. So, this one I can write as separate values a_{ij} e_j^k/ a_{ii} this value. So, now, what we are going to do is that, now I am going to take it. So, this is the value here. So, what I am going to do is, this is I will take the magnitude. So, I can write from here that, summation a_{ij} e_j^k divided by a_{ii} and that is true for all j =1,2,3…n and $j \neq i$.

Now, I choose my capital E_k as maximum of e_1^k, actually I have taken this magnitude, e_2^k, e_3^k, ….,e_n^k; just I have taken the magnitude, here I am just showing this one. So, we are just taking the magnitude. So, now, from here I can write this as; because it is coming in each one, so I just take the common. So, I can write from here that, this will be equal to summation j from 1 to n and $j \neq i$, a_{ij} and divided by a_{ii}, taking this common a_{jk}. So, this is e_j^k, I will just write here E_k. So, this is the maximum value I am taking.

Now, from here, now I will use the property of A. Now, since A is assumed to be diagonal dominant, which implies that a_{ii} modulus value, that is going to be greater than summation a_{ij} modulus value. And I will take j from 1 to n and $j \neq i$ and i are moving from 1, 2,….n. So, this one I am going to take. So, this is the definition of a diagonal dominant matrix. So, from here I can write. So, I just take this element here. So, I can write from here that the $\sum_{j}^{n} a_{ij}$ modulus value divided by a_{ii} and $j \neq i$. So, it will be less than 1 for all i. So, this is less than 1. So, now, if it is less than 1; so if you see from here, this term is the same as this one. Now, from here, I can write that e_i^{k+1} will be less than. So, this terminology is here and this is less than 1. So, I can write that this is less than 1. So, from here I can write that this is equal to E_k modulus value. Now, this is the maximum error at the kth step, at the error at the k + 1 step h in each of the x_i is less than this one.

So, which implies that the method is going to converge. Why is it going to converge? Because the error introduced at the kth iteration and the error introduced at the k + 1th iteration. So, the error at the k + 1th iteration is becoming lesser than the maximum error at the kth iteration; it means that as the iteration will grow, the error is going toward 0. And so, the error is going toward 0, it means that the matrix is going to converge.

So, that is the way we have a sufficient condition that the given matrix is dominant, then the Gauss Jacobi method is going to converge, the same way we can have the method Gauss Seidel. So, this is one of the applications of the diagonal dominant matrix. Another important application we are going to do is that all diagonally dominant matrices are nonsingular.

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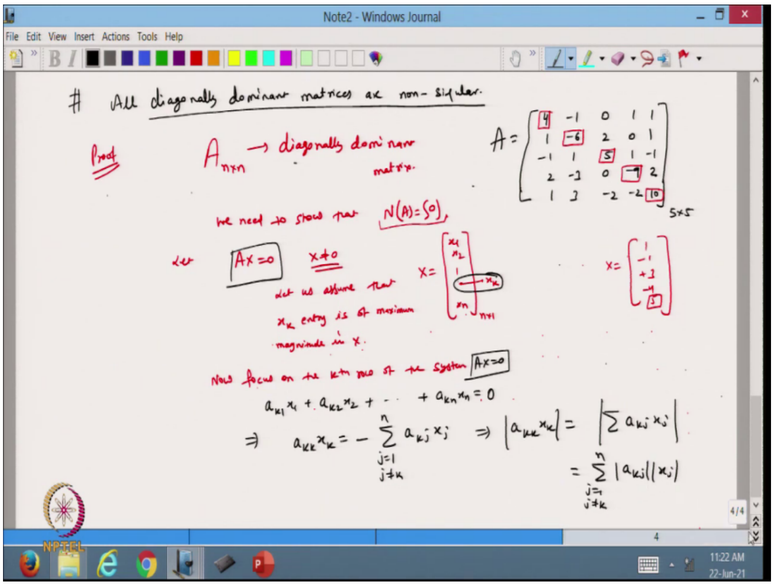

So, this is one of the another application; suppose I take the matrix A and this matrix is suppose I take 5 ×5 matrix. So, like I take matrix 4 here, {1,- 1, 0, 1, 1} and suppose I take {1, -6, 2, 0, 1}, {-1, 1, 5, 1, -1}; then {2, -3, 0, - 9, 2} and the last is {1, 3, -2, -2,10}. So, this is a 5 × 5 matrix.

And if somebody ask me that, whether this matrix show that, whether this matrix is singular or non-singular; then we have either we have to convert this matrix into the suppose rho echelon form and then from there I can say that, if the rank of this matrix is equal to 5, then this matrix will be non-singular.

But it is just a 5 ×5 matrix, maybe it can be a 10× 10 or 20 ×20; but now from here, if you check that this matrix is diagonal dominant, because this $4 > |-1| + |1| + |1|$, 6 is greater than 1 +2 +1=4, 5 is greater than $|-1| + |1| + |1| + |-1|$ =4 and here it is 9 is greater than the $|2| + |-3| + |2|$ =7 and 10 is greater than the $|1| + |3| + |-2| + |-2|$ =9.

So, this matrix is a diagonal dominant matrix and from there I can say that this is a non-singular matrix. So, that is the benefit of checking the diagonal dominance; because once we get that the matrix is diagonal dominance, then easily we can say that this is a non-singular matrix. So, let us prove this one. So, how are we going to prove? So, suppose we have a matrix A that is diagonal dominant.

So, suppose I take $n \times n$ matrix. So, this is a diagonal dominant matrix. So, this is what we have assumed. Now, we need to show that, if the null space of A, it just contain the 0 element; then we are done, because if we have seen that if the null space of A matrix contains with a 0 element and the matrix is a square matrix, then the number of columns are linearly independent or the number of rows are linearly independent. And then it means that, the rank of the matrix will be equal to the number of variables, that is n.

So, now we are going to show these things. So, let we have $Ax = 0$, such that $x \neq 0$. So, I am taking the system $Ax = 0$ homogeneous system, such that assuming that $x \neq 0$. Now, suppose $x \neq 0$. So, let us assume that. So, because x is there, so x is basically this one; $x_1, x_2, \ldots x_n$, it is a $n \times 1$ system. So, let us assume that, I just take here somewhere x_k.

So, let us assume that x_k, x_k entry is of maximum magnitude in x. So, let us assume that; because x is going to be a vector, so suppose my x_1 takes just as {1, - 1, -3, 4, 5}, suppose I take this one - 4 or. So, in this case, I will assume this value. So, let that be my x_k. Now, focus on the kth row of the system; because we are getting that the x_k is of maximum magnitude, so I am talking about only the row corresponding to the kth. So, we are talking about the kth row only.

So, let us take this one. So, now, I am talking about only the kth row. So, it will be equal to $a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n = 0$. So, I am writing the system a x = 0. And in that system, I am taking only the kth row. So, this is my system, a homogeneous system I am talking about and from here I get this value. Now, I am going to write this system $a_{kk}\, x_k$ and take all the terms on the other side.

So, that can be written as minus summation $a_{kj}\, x_j$, where j is moving from 1 to n and $j \neq k$, this is what I am going to have. Now, from here, I just take the magnitude of the modulus value. So, I just take modulus this value. So, basically this will removed and this one I can write as $a_{kj}\, x_j$ and j is 1 to n, $j \neq k$. Now, this is coming with all values. So, from here I just take and I can write from here that my x.

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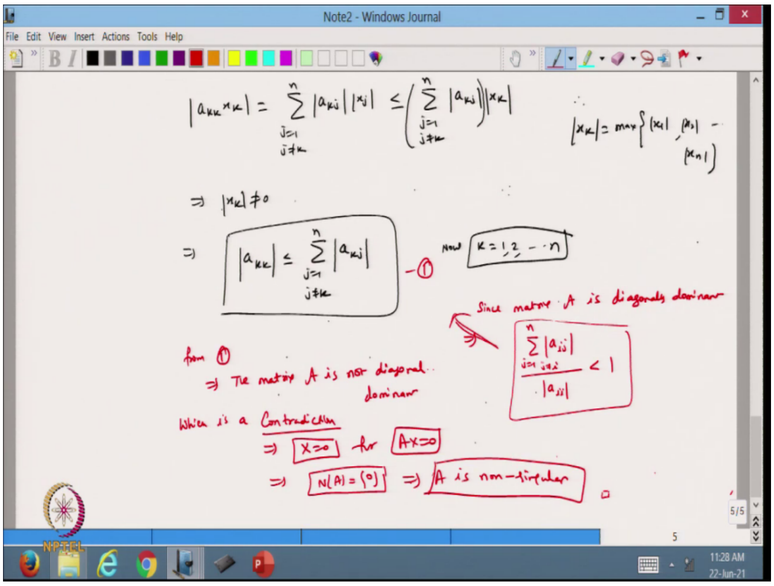

So, I can write from here that the a_{kk} x_k $=a_{kj}$ x_j and j is moving from 1 to n, j$\neq$ k. This can be written as summation a_{kj} and this one I just take; because we have chosen that my x_k is of magnitude larger than any of the components, it means I can write here x_k, because x_k magnitude is maximum of x_1, x_2,…. x_n that we have taken.

So, I have just taken this one and just taken it outside; because otherwise it is coming with each member, but I have taken the x_k in all and taken the common from there. So, I have just written like this one. Now, from here now x_k $\neq$0 of course, because it is a maximum value we have taken; so from here I can write that my a_{kk} is less than equal to summation a_{kj}, j from 1 to n and j $\neq$ k and this is true for all. So, here I am taking the k, so this k can happen for any value. Now, k = 1, 2,…. n; it can happen everywhere, anywhere.

Now, from here, if you see from here; then matrix A is diagonal dominant, which implies as we have shown in this case also. So, which implies that, the summation a_{ij} divided by a_{ii} <1 and j $\neq$ i and this is true for all. So, this is what I have just written this value, but here it is this one. So, from here I can just give the number here, so maybe I can give it 1.

So, from 1, we, which implies that the matrix A is not diagonal dominant; because if it is diagonal dominant, then it has to satisfy this condition, because I can divide by this one and this just it will be less than equal to 1, because I can take this side, but here it should be only strictly that less than 1. So, from here I can say that, based on this condition I can say that, 1 is not diagonal dominant, which is a contradiction.

So, which implies that $Ax = 0$. And so, which is a contradiction means; which implies that the $x = 0$ for $A x = 0$, because in we have started with $A x = 0$ and we assume that $x \neq 0$. But from here we got the contradiction; means our assumption was wrong, we have to choose x should be equal to 0. So, which implies that the null space of A should contain only 0 element and from here I can say that the matrix A is non-singular. So, the given matrix will be non-singular. So, that is the proof of this thing.

So, we will stop here today. So, in today's lecture, we have discussed the property that is diagonal dominant matrix and based on that we have shown that, how the iterative method, that is Gauss Jacobi has the convergence and how a diagonal dominant matrix is a non-singular matrix. So, we will continue with this one in the next lecture. So, thanks for watching.

Thanks very much.