Matrix Computation and its applications

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Lecture - 22
Linear Independence of the rows and columns of a matrix

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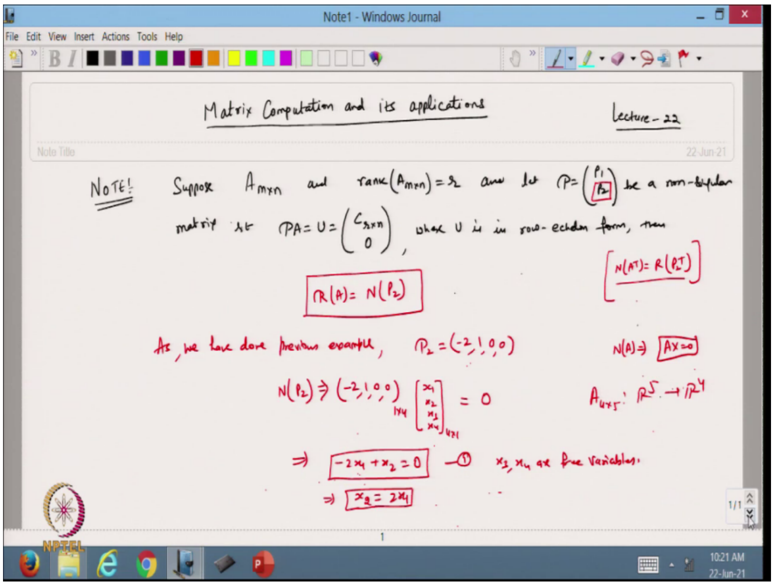

Hello viewers, welcome back to the course on Matrix Computation and its Application. So, in the previous lecture, we have discussed one example and show the dimensions of range space and null space. So, today we are going to continue with that one. So, let us do this one.

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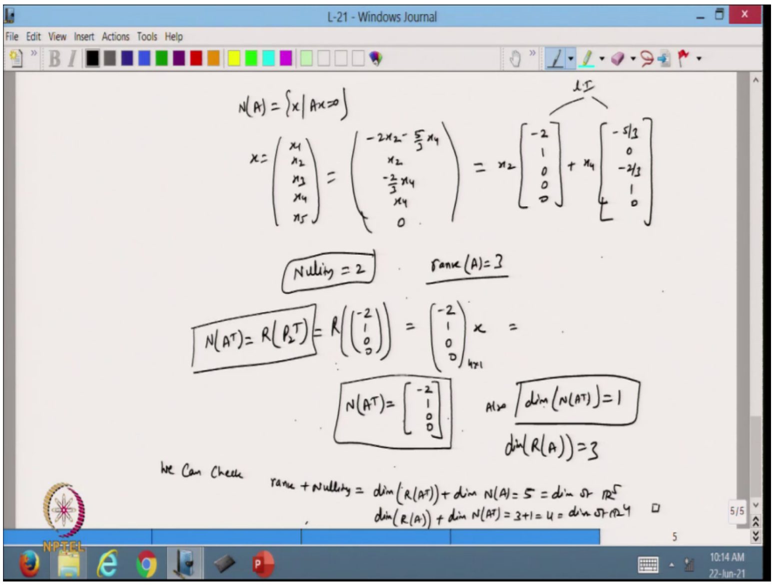

So, today I am going to discuss other properties of the corresponding system. So, if you remember that in the previous lecture, we have discussed one example and we showed the range of the given matrix.

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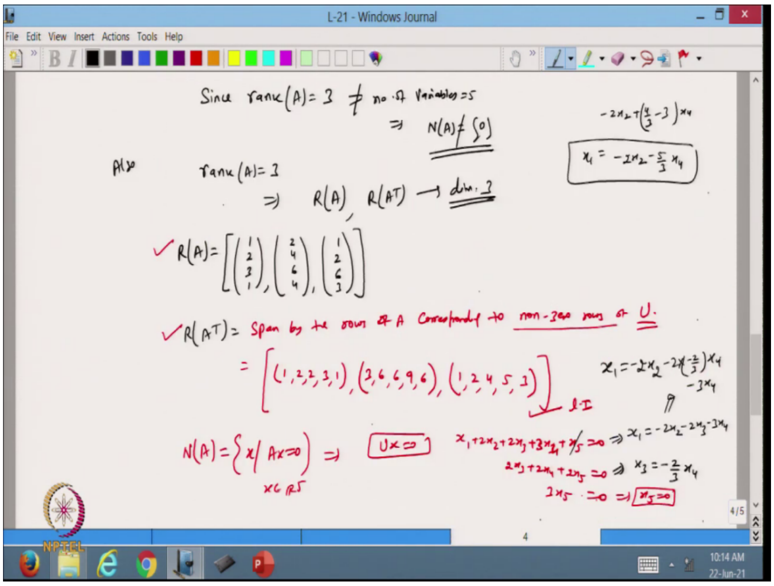

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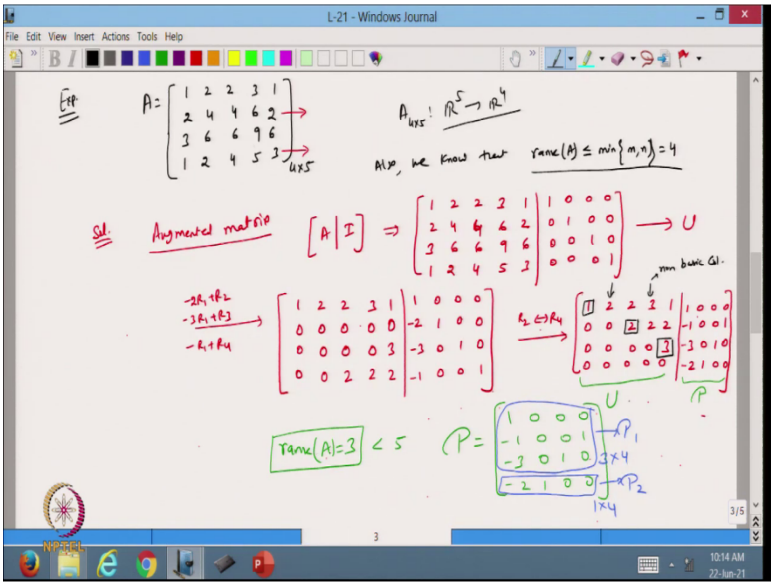

So, this was my matrix. And we have discussed the range space of a transpose. And then we have also shown that the P matrix the non non-singular matrix P. So, the row corresponding to the 0 rows in the echelon form, and then using this one we have showed this relation that range space of P_2^T is equal to the null space of A^T. So, the same thing we are going to discuss now.

So, it say that suppose we have a matrix A that is m × n, and let the rank of A = r. And let we have a matrix P that is given by P_1 and P_2 as discussed in the previous lecture be a non-singular matrix such that PA=U that is the row echelon form. So, this will be definitely some matrix C that is r × n and this will be 0 because the rank is r. So, this matrix is going to have only non-zero rows that are r in number, and n is the number of variables. So, this is there, where U is in row echelon form.

Then, so then we are going to have one relation, and this relation is that the range of A can be written as the null space of P_2 this one, so where this is my P_2. Earlier we also discussed that $N(A^T)$ that is a range space of P_2^T. So, this is what we have discussed in the previous lecture.

Now, I want to discuss this one. So, as we have done previous examples, so from there I will, so this one already we know. So, P_2, I am just going to get what the P_2 is there. So, basically if you see, then my P_2 ={ -2, 1, 0, 0}, so that is my P_2. My P_2 is basically this row { -2, 1, 0, 0}.

So, this is my row vector. Now, I want to find what will be the null space of P_2. So, basically we are going to write P_2 that is { -2, 1, 0, 0}. So, it is a row vector that is basically 1 × 4, and then I will apply. So, this will give me this into I am applying over the element 4. So, I am suppose I am taking it as x_1, x_2 , x_3, x_4 so it is basically 4 × 1.

And then it should be equal to 0 because whenever we are going to deal with N(A), it means we are going to deal with this system. So, I am going to deal with the null space of P_2 implies that P_2 this is our matrix that of dimension 1 × 4 with this where the number of variables, and that is equal to 0.

So, from here I can write this one as $-2x_1+x_2=0$. So, I am getting only one equation from here. So, this one we have done, only one equation is there. And my matrix was from R^5 to R^4. So, that also we have to keep in mind that our matrix was 4 × 5 so moving from R^5 to R^4.. Now, this is our equation. Now, from here you can see that x_3 and x_4 are basically free variables because they are multiplied by 0. So, from here I can say that this is the equation number 1, and x_3, x_4 are free variables.

So, from here on out, I can write down. So, now, from the previous matrix, we came to know that the rank of A is 3 . So, the range of A is this one. So, it is made up of this one. So, this one we are going to do now. So, from here, I can say that my x_1 or maybe my x_2 can be written as $2x_1$. So, this relation I can write x_2 is becoming equal to this one.

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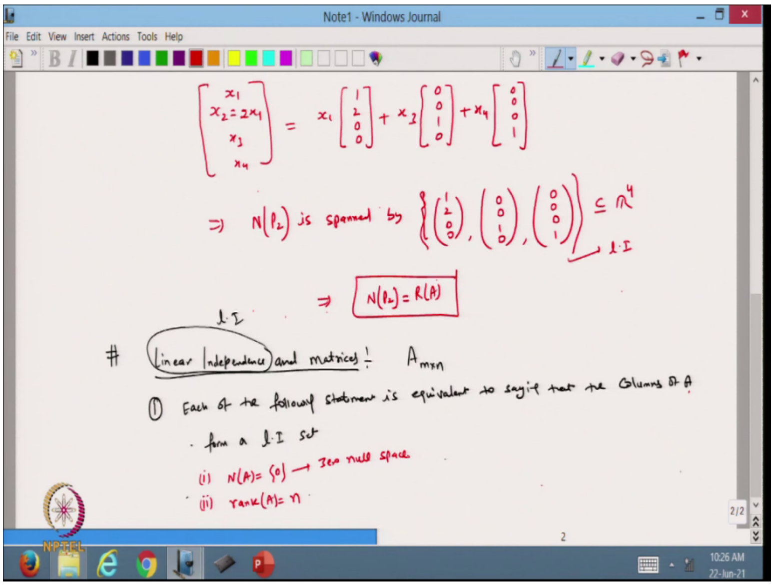

So, from here I can write that my x2 is equal to $2x_1$, and x_1 is always there, and x_3 x_4. So, this is my vector I am going to have. So, that vector can be written as because this vector is basically I need to find the values of x_1, x_2 , x_3, x_4. So, x_1, x_2 , x_3, x_4 can be written like this way.

So, from here, I can write that I can take the x_1 common. So, it will give me $x_1(1, 2, 0, 0) + x_3(0,0,1,0)+x_4(0,0,0,1)$. It means that the null space of P_2 is spanned by this set of vectors{1, 2, 0, 0}, {0, 0, 1, 0}, and {0, 0, 0,1}. And you can check that this set is linearly independent and is. So, this will be the subspace. So, it belongs to the set R^4. And this will be.

So, now, if you see from here that in this case my range space of A is spanned by the element {1, 2, 3, 1}, {2, 4, 6, 4}, and {1, 2, 6, 3}, but there are three number of vectors are there, and they are linearly independent. The same thing is coming here. Only the thing is that we have the same number of vectors, but these vectors are not the same as the we have got in that range space of A.

But so in this case we can say that the basis vectors have been changed but the dimension is the same. So, from here you can check. So, from here you can imply that the null space of P_2 is always equal to, so this can be written as the range space of A. So, this one we have done. So, this one this way we can write down these things.

So, after doing this one, now we are going to start with some other things like linear independence and matrices. So, suppose we have a matrix A that is m × n matrix. Now, first one that each of each of the following statements is equivalent to saying that the column of A forms a linearly independent, so this word can be written as lI. So, a linearly independent set if the null space of A contains only 0 element.

So, which of the following statements is equivalent to saying that the columns of A form a linearly independent set? So, because in this case we have a matrix A is m × n, so I know that it is going to have n number of columns and m number of rows. So, the columns of A forms a linearly independent set, it is equivalent to saying that the null space of A is 0, and the rank of A is going to be equal to n the number of variables. So, this is a 0 null space.

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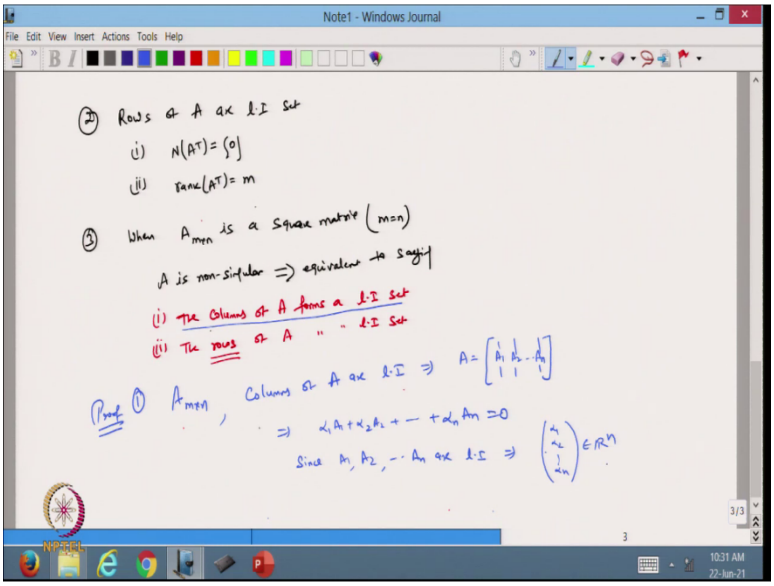

Second one. Now, I can talk about rows of A being a linearly independent set, it is equivalent to saying that null space of A^T is just the 0 element. And the second one is that the rank of A^T is equal to m; and m is the number of rows in that given matrix. So, now the third case is that what will happen when the matrix A is m× n is a square matrix. Square matrix is when m = n.

Then A is non-singular, so that is equivalent to saying, so this is equivalent to saying, the first one is that the columns of A, the columns of A form a linearly independent set. So, all the columns of A are linearly independent to each other. And the second one is that the rows of A form a linearly independent set. It means that the rows of the matrix are also linearly independent to each other. So, these things we have to keep in mind.

Now, using this one, it is very clear. So, I can discuss the proof. Now, columns of A form the linear independent set, and the rows of A form the linear independent sets. So, we can say that suppose I have the matrix A that is m ×n matrix. Now, I am saying that the columns of A are linearly independent. So, this one I am first, I am doing this one saying that the columns of A are linearly independent. That is equivalent to saying that the null space contains 0 or the rank is equal to n. So, this is the proof for the first one.

So, I am saying that the columns of A are linearly independent which implies that, so I have a matrix and that matrix is basically it has the column, so that column I can write as A_1, A_2,….. A_n, n is the number of columns. And these are linearly independent. So, this are linearly independent means that if I take the linear combination $\alpha_1 A_1 + \alpha_2 A_2 + \ldots\ldots + \alpha_n A_n = 0$. So, it is there. Then A_1, A_2,….. A_n, are l I which implies that the set $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ that belongs to R^n, belongs to so this belongs to R^n. So, it is going to be that. So, if it is a linearly independent, then I can write even that $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$, all are 0.

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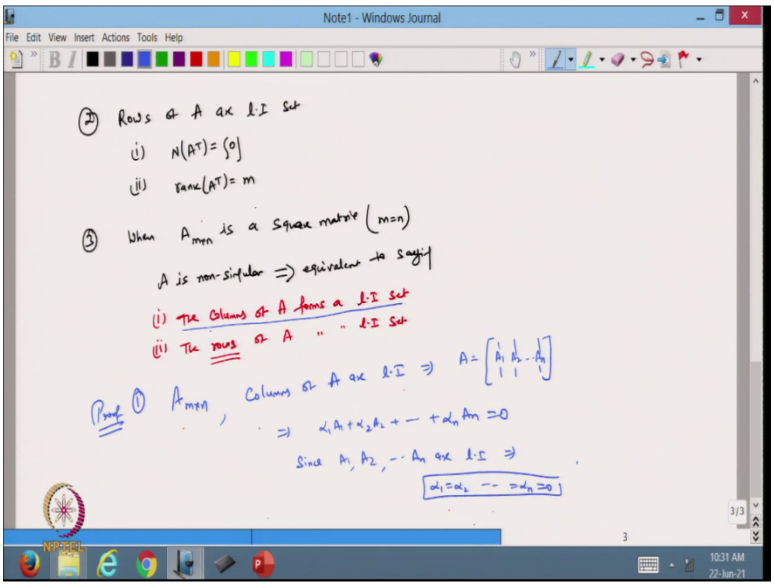

So, if all are 0, then from here.

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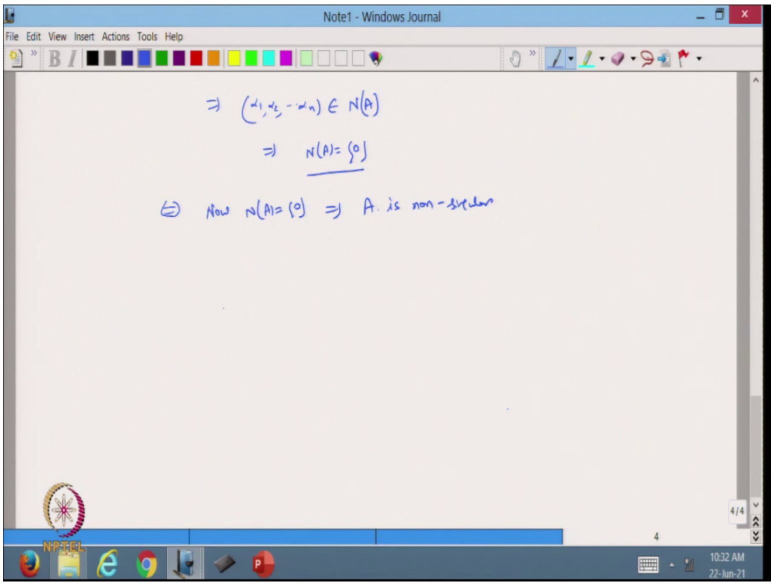

Which implies that the set $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ belongs to null space of A. But in this case, all the vectors are linearly independent and showed that this α is 0. So, from here I can say that null space of A will be only 0 element because all the α is 0. We are able to show this one that the null space will contain only 0 elements.

And now from here you can say that this is equivalent to saying that the matrix. So, now the null space of A contains only 0 element which implies that the matrix A is non-singular. If it is now these things I can say only when it is a square matrix.

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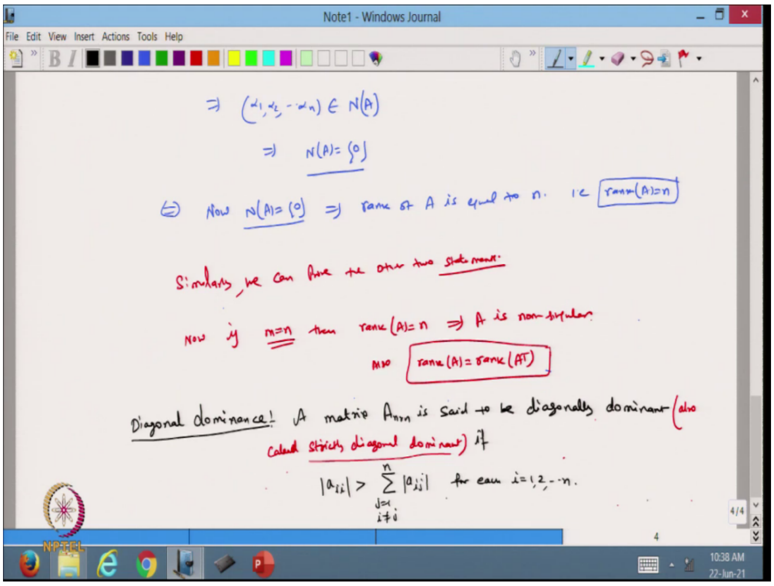

So, from here I can write that from here I can say that the rank of matrix A = n, so that I can say because null space contains only 0 element. So, I can say that the rank of the matrix will be n. So, the rank of A is n because if it is a square matrix then I know that the rank of the matrix A will be n, and then it will be non-singular. So, this is the same thing we can write down.

So, similarly we can prove the other two. So, these are the other two properties that we have or the statements, so other two statements because in that case we have to deal with only rows. So, the rows of A are linearly independent. Then the same way we can write down that the null space of A^T will be 0 and the rank will be equal to the number of rows.

And if it is a square matrix, then we can say because if m is equal to n, now if m is equal to n, then rank of A is equal to n which implies that matrix A is non-singular. Also the rank of A is equal to the rank of A^T, so that we already know because we take the matrix A to find the rank, then you just take the transpose of that matrix and then again find the rank that will be the same. So, we already know the rank of the matrix and its transpose are always the same. So, these things we are using this one.

Now, so I want to use this statements for solving some real applications that if the null space contains only zero element, then the this is true that the given columns are linearly independent or the corresponding rows are linearly independent; or if the matrix is a square matrix, then it is it will be non-singular. So, these things we are going to use for discussing some important things.

So, the first thing we are going to discuss is a very important topic and that is called diagonal dominance. So, a matrix A so that is of order $n \times n$, so I am talking about the square matrix, which is said to be diagonal dominant or in some books sometimes it is also written as also called strictly diagonal dominant, in some books it is written as a strictly diagonally dominant. xc

So, matrix A is said to be diagonal dominant if I take the elements at the main diagonal taking the modulus value. So, if $|a_{ii}| > \sum_{j=1}^{n} |a_{ij}|$; for I=1,2,….n and $i \neq j$.

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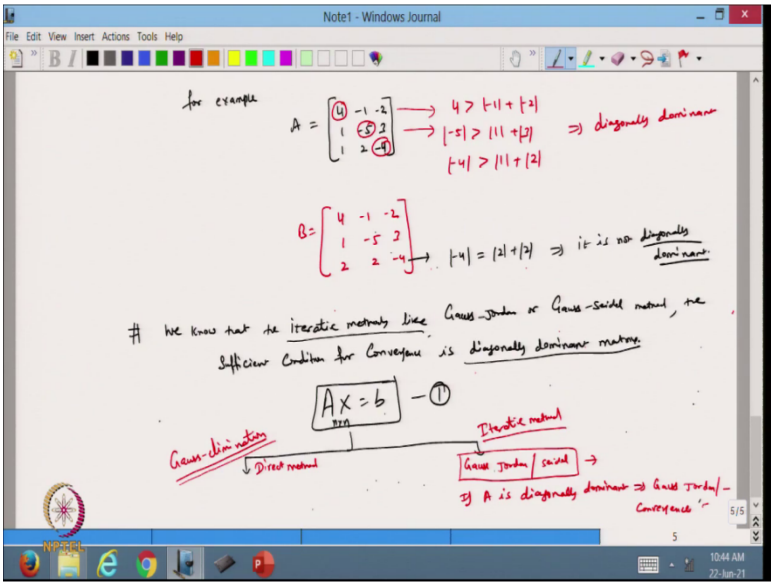

So, this is called the diagonal dominant matrix for example, suppose I take the matrix A that is I may take 3×3. So, in this case, suppose I write here {4,- 1, - 2},{1,-5,3},{1,2,-4} and this is supposed to be -4. So, in this case, if you see, then these are my diagonal elements. And from here I can say that the $|4|>|-1|+|-2|$, so that is in the first row I am talking about.

And in the second row we are writing diagonal elements. So, $|-5|>|1|+|3|$; and similarly $|-4|>|1|+|2|$,. So, everywhere it is coming. So, it is called a diagonal dominant matrix.

But suppose I take another matrix B, and if I write the $\{4,-1,-2\}$, $\{1,-5,3\}$, $\{2,2,-4\}$. So, in this case, if you see that starting from here, then in this case $|-4|=|2|+|-2|$. So, in this case, we are going to have only equality signs. So, this sign is not there. So, in this case, we say that it is not diagonally dominant. So, these types of matrices are playing a vital role in numerical analysis or for the convergence of some numerical methods or iterative methods.

So, based on this 1 I want to show, now, we know that with the iterative methods like Gauss-Jordan or Gauss-Seidel methods the sufficient condition for convergence is diagonal dominant matrix. So, with iterative methods like Gauss-Jordan or Gauss-Seidel, we know that the sufficient condition for convergence is that the matrix should be diagonal dominant. It means we have a matrix $AX = b$. So, this is a system of equations. And this is suppose $n \times n$.

So, I know that this type of system can be sold with the help of, for example, I assume we have done Gauss elimination. So, in the Gauss elimination, if you see we use the pivot pivoting, so pivoting is basically to have the largest element at the diagonal places. So, this type of thing is also related with the diagonal dominant matrix, and others are iterative methods. So, this is a direct method basically, and another is the iterative method. So, the iterative method is like Gauss-Jordan or Seidel.

So, this type of method is if we apply we are not sure that we are going to have the solution or not, but in this case we are going to get the solution if the matrix is diagonal dominantly. So, that means, that, that is called the sufficient condition that if the matrix is diagonal dominant. So, if A is diagonal dominant, if A is diagonal dominant which implies that Gauss-Jordan or Seidel are convergent. But the converse is not always true. If the methods are converging that also does not mean that the matrix should be diagonal dominant.

But if the matrix is diagonal dominant, then definitely it is going to converge. So, these types of things are very important to deal with the iterative processes for solving the system of equations. So, those things are what we are going to discuss in the next lecture. So, let me stop here.

So, in the today lecture, we have discussed about one another properties of the that is related with the matrices that if the null space of the matrix corresponding matrix is only the zero space, then we can have that the given matrix is going to have the columns are going to be linearly independent, or the rows are going to be linearly independent. And then we have also discussed another definition of diagonal dominant matrix. So, we will use this one to show some properties in the next lecture.

So, thanks for watching. Thanks very much.