Matrix Computation and its applications
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Lecture - 21
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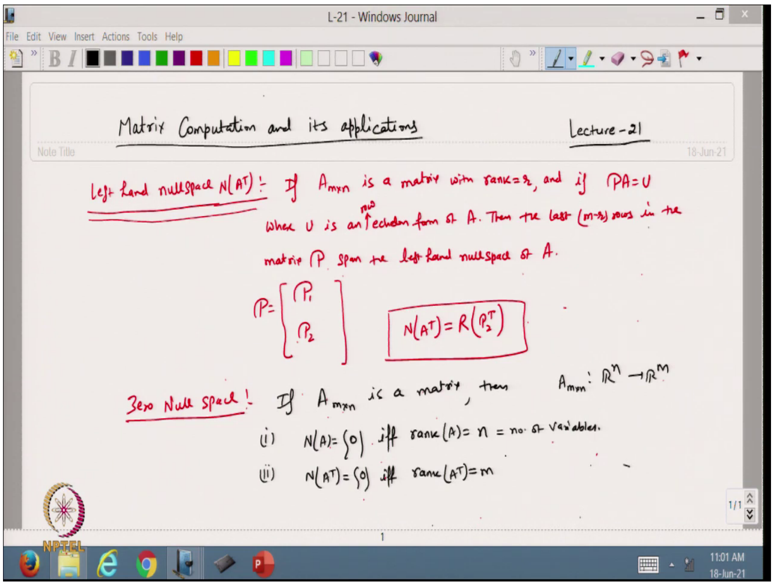

Hello viewers, so welcome back to the course on matrix computation and its application. So, in the previous lecture we have started with one relation: how we can find out the left-hand null space with the help of the matrix P, so we will continue with that one. So, today we are going to write one definition about left hand null space. So, left hand null space means we are talking about N transformation.

So, it says that if I have a matrix A $m \times n$ with rank is equal to r and r, and if PA =U, where U is an echelon form of A. So, we are talking about echelon form means row echelon form. Then the last m minus r rows in the matrix P span the left hand null space of matrix A and.

So, now here we are writing P is written as P_1 and P_2. So, from here we have that the null space of A^T is equal to range of the matrix P_2^T. So, this is my matrix P that is defined here. And the last m minus row so from here you can write that if the rank is row r. So, it is this mat is m × r and this is the remaining one m minus r × r ok. So, this is the dimension of the matrix. Maybe I can just write specifically.

So, P_1, so the dimension of will be m × r ; P_2 dimension will be m minus r × r or maybe it can be anything. So, I can write just this one; that is it. So, these things we have also discussed in the previous example. So, this is some observation about the left hand null space.

Now the next thing we want to discuss is zero null space. So, in the zero null space we say that if A is a m × n matrix then first one is null space of A will be 0, only the 0 element, if and only if the rank of A will be n, because this is the matrix A m × n matrix is from R^n to R^m. So, if the rank of matrix A is equal to n so that is equal to the number of variables. Then the null space will be only containing the only 0 element. And why is it containing 0 elements? Because the 0 is always the solution of the homogeneous system A x = 0.

Now, the second one is null space of left null space will contain only 0 element if and only if the rank of A^T is equal to m.

So, that is the observation we are going to have and this one we know.

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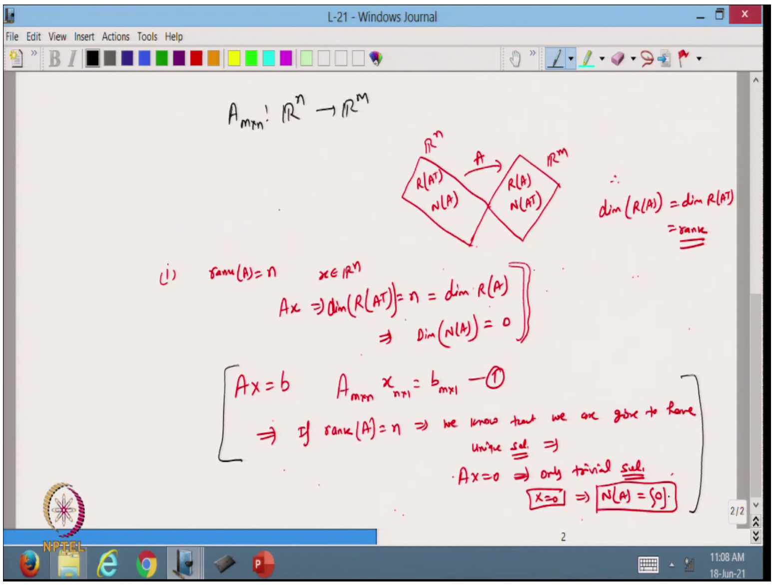

So, that we have seen that the matrix A m × n is moving from R^n to R^m and also we have the subspaces that are four subspaces we are going to have. So, this is my Rn and this is my Rm and we have a transformation that is A, and we know the range space of A transformation and N of A and this is R(A) and this is $N(A^T)$.

And we also know that the dimension of the R range space of A is the same as dimension of A^T and that is equal to rank. So, in this case we know this one. Now, as it is given that rank, suppose I will take the rank of this one. Now I just want to give the flavor of the first one.

So, the first one says if and only if. So, let I say that the rank of matrix A space rank of matrix A is equal to n. Now it is coming from here now R^n. So, x belongs to R^n, and now suppose I have Ax and rank of n is this one. So, from here you will see that the range space of this has the dimension that is equal to rank and that is n. Now from here now these subspaces are coming from the Rn, but we found that the dimension of $R(A^T)$ is n and that is also the same as dimension of R (A) this one.

So, from here you will know that the dimension of null space will be 0, because it is already coming from here and it is completely n. So, the dimension of the other space subspace in this is coming 0. Actually, these things we can just do with the help of the system A x = 0 or maybe Ax = b. Now my A is basically $m \times n$ my x is coming $n \times 1$ and b is coming $m \times 1$.

Now, from here if rank of matrix A that is coming n, then we know that we are going to have a unique solution. So, in this case if I take this 1, we are going to have a unique solution. And we are going to have a unique solution then I am going to have my Ax = 0 is going to have only a trivial solution.

And here a trivial solution means x will be 0. So, that shows that the null space of A will contain only 0 element . So, this also can be taken in this way. And these things I have just told you because later we are going to show one theorem that is about the rank and nullity theorem. So, in that theorem we will find some relation related to this one. But now as for now whatever we have written here that if the null space is 0 if and only if the rank of the matrix is equal to n the number of variables.

So, just keep in mind and later on will prove this one based on some other theory. So, these things you can familiar with. So, that is why I have taken this thing here, that this thing is what we have already done. So, from there we can make the conclusion that if the rank of the matrix is equal to the number of variables then the null space; corresponding null space is going to have a only 0 element.

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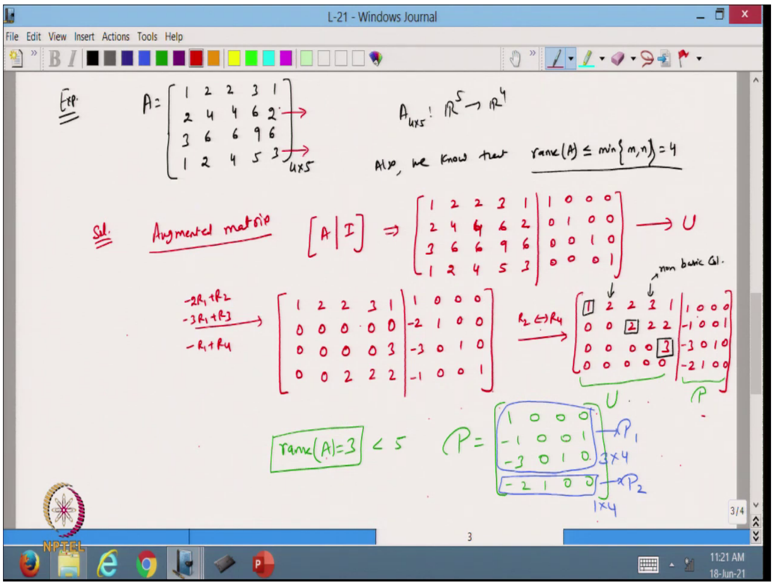

So, now based on this one I am going to take a quick example from for a given matrix A. So, this is the example I am going to take {1, 2, 2, 3, 1}, and this is {2, 4, 4, 6, 2},{ 3, 6, 6 ,9, 6 },{1, 2, 4, 5, 3}. So, 4 × 5. This is a 4 × 5 here.

Now, in this case it is a is 4 × 5, it means this is moving from R^5 to R^4. And also we know that the rank of matrix A is always less than equal to minimum of m n and that is in this case it is 4. So, the rank of the matrix is going to be 4 less than equal to 4.

Now, in this case suppose the rank of the matrix becomes 4, it means all the 4 rows will be effective to make the range space, and then we can also see that the null space will be, corresponding null space will contain only zero element. So, these things we can find out.

Now this is my matrix. So, I want for you, using this matrix I want to find all the four subspaces, I want to check whether this theorem or this observation is satisfying there and also I am going to use this one. Now, so for this one I take the augmented matrix. So, this is my augmented matrix A, identity matrix I.

So, from here this can be written as {1, 2, 2, 3, 1}, {2, 4, 4, 6, 2},{ 3, 6, 6 ,9, 6 },{1, 2, 4, 5, 3}. And now it has a four, so I will write the corresponding identity matrix of order four. So, here I am writing the identity matrix order 4 . So, that we can choose from here.

Now, I will substitute, I will transform this to the echelon form to U, so I will apply the formula. So, now, from here I will write $-2R_1+R_2$, $-3R_1+R_3$ and $-R_1+R_4$. So, directly I can write, so it is {1, 2, 2, 3, 1}, { 1, 0, 0, 0}. Now it will be 0 0 0, so - 2, so that will be also 0 and this is also 0 this is 0 and this is also 0.

So, this is coming 0 here. Now from here it is 0, because - 3 I am multiplying. So, 0 - 6 0. So, that is also 0 and -3 -9 0. So, -3 + 6 it will be 3 and the corresponding 0. So, I am writing here mine -1 +0 -2. This is also 0 - 2, so it will be 2 and 2. And from here this is my 1 0 0, so I am just multiplying here with - 2 and adding here.

So, it will be - 2 1 0 0 -3. So, it will be -3 0 1 0 and this is minus 1 and adding into R_4. So, it is -1 0 0 1. Now from here I just swap the R_2 with R_4. So, it become 1 2 2 3 1 1 0 0 0 0 0 2 2 2 and that is coming minus 1 0 0 1 it is coming 0 0 0 0 3, it is - 3 0 1 0 and the last I will get -2 1 0 0.

So, this is what I am getting now. So, from here you can check that this is my echelon form. So, that is my row echelon form and this is I can say from here that this is my P. Now based on this one, so my P if you see from here. So, the rank of the matrix A in this case is coming 3 and the number of variables. So, if you see from here the n is 5. So, my rank is three. So, it is less than 5. So, we can check from here that in this case the null space is not going to have only 0 element, it is going to have more than one. So, its dimension we have to find out. So, this is my rank of the matrix 3.

Now if you see from here P. So, P is coming basically {1, 0, 0, 0},{ - 1, 0, 0 ,1},{ - 3, 0, 1, 0} and {- 2, 1, 0, 0}. So, from here you can check that the rank is 3. So, these first three rows make it P_1. So, dimension if you see, its dimension will be 3×4 and this last row corresponds to the 0 row.

So, because this is going to be the corresponding to the 0 row. So, this will be treated as P_2 and its dimension is 1×4 and the rank was 3. So, that is why we generally do not write the dimensions here, because everything depends. So, from here I get the value of P. Now from here, so very quickly I can write all the subspaces.

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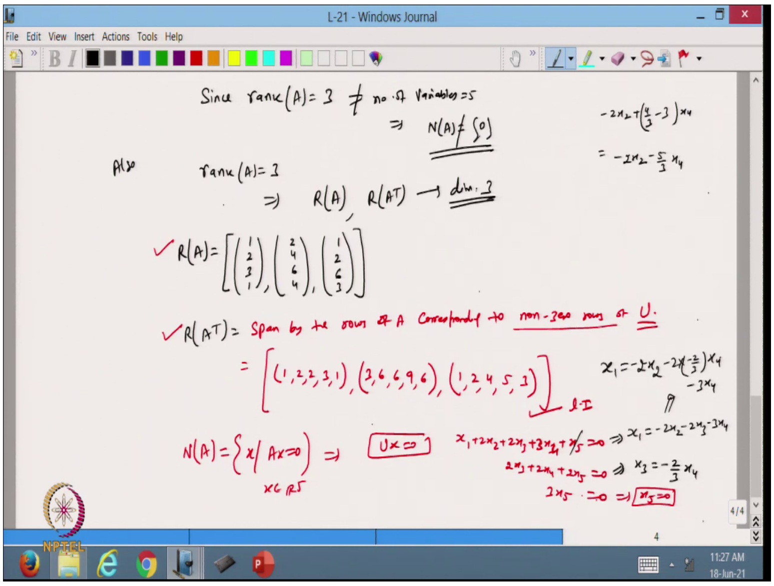

So, from here one thing is true that since the rank of the matrix A is 3 that is not equal to the number of variables that is the number of variables that is 5. So, from here that null space of A is not going to have only 0 element. So, this is what we have taken from here. Also rank since the rank of A is 3 which implies that the range space of A and A transform is going to have dimension 3. So, this one we can write down directly.

Now from here very quickly I can write the range space of A is a span of the vectors. So, I will take this matrix. So, in this case this is my pivot element, this is my pivot and this is my pivot. So, these are the three pivot elements and these are known basic columns, it means x 2 and x 4. So, from here you can write down that first, third and fifth. So, the range space of A is made up of the basic columns.

So, the basic column is basically {1, 2, 3, 1}, {2, 4, 6, 4}, because I am taking the first, third and fifth. The first third and fifth are {1, 2, 6, 3}. So, this is a span from this one. And you can check that these are also L I linearly independent. From here you can check, it contains only 1 0 0, another one contains 2 2 0 0 and the third one contains 1 2 3 0.

So, these are linearly independent. I am writing the corresponding columns here. So, it is {1, 2, 3, 1}, {2, 4, 6, 2}, {2, 4, 6, 4} and {1, 2, 6, 9}, {1, 2, 6, 3}. So, this one we have written here. And from here, so this is ok. Now I can write the range space A^T. Now from here you can check that we have also taken the swap of the matrix and then I got these three rows from the first three rows we have taken from here.

So, in this case you can check that the row has been swept with this row, because I have taken R_2 to R_4. So, R_2 to R_4 we have taken and this became the 0. So, from here you can check that the range space A transform is spanned by the rows of A corresponding to non-zero rows of U. So, corresponding to non-zero rows of U.

So, now from here you can. So, this is basically, this row I am talking about, so from here I just have to leave my second row. So, from here you can check that this is spanned by the element {1, 2, 2, 3, 1}. I am just writing in the form of columns or maybe you can also write as a row vector. So, span by {1, 2, 2, 3, 1} and second one is I am leaving this one. So, it will be {3, 6, 6, 9, 6} and {1, 2, 4, 5, 3}.

So, these are the corresponding vectors of the rows I have taken, because this is the 0 row, but it was coming in second place. So, the second place row is this one. So, we have taken this one. So, now, from here, this is a linearly independent vector, you can check that this is a linearly independent vector I have taken, and from here I know that the dimension is three. So, it is equal to the rank.

So, now from here I am able to find R(A) and $R(A^T)$. Now from here quickly I can find my null space and A. So, N (A) is basically all the x such that A x = 0, and this one we can find from U x = 0. So, in this example that is why we are taking this one because we are swapping the row also. Now I am going to take the corresponding non zero rows.

So, from here you will get the first row. So, we are getting from here $x_1+2x_2+2x_3+3x_4+x_5=0$, because here the x is coming from R^5, because it will transform from R^5 to R^4. And then from the second non zero row it is 2 to 2. So, I can write from here that it is $2x_3,+2x_4+2x_5 = 0$.

And then from the third row it is $3x_5$=0. So, this one we can write. Now from here that gives me that x_5 is always going to be 0. So, once I get the x_5 = 0. Now here the non basic columns are only x_2 corresponding to x_2 and x_4. So, I can write from here that my, so I am taking x_2 and x_4.

So, I can write from here, so x_5 is 0 I can write from here $x_3 = -\frac{2}{3} x_4$. And from here I can write my x_5= 0. So, this part becomes 0. Now from here I can write my x_1 =-2 x_2 and then - 2 x_3 and -3 x_4. And from here I can substitute the value of x_3 here, so that I can write my x_1= 2 x_2 - 2.

So, $x_3=\frac{-2}{3}x_4 -3x_4$ and that becoming, so it is -2 x_2 it became 4/3 because it is 4/ 3 and - 3. So, it is (4/ 3 - 3) x_4 and that gives me - 2 x_2. So, it is 4 - 9. So, it is $\frac{-5}{3}x_4$.

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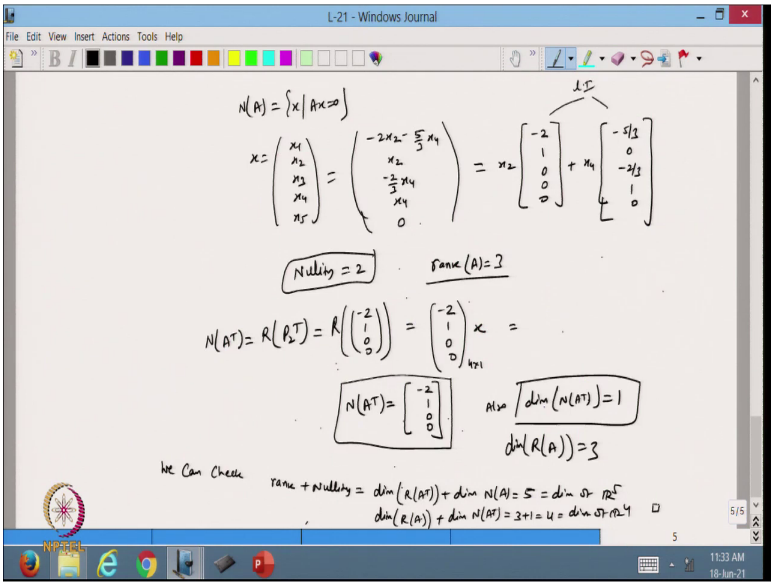

So, based on this one now I can write my null space. So, it corresponding to the x such that A x is equal to 0. So, x is basically x_1, x_2, x_3, x_4, x_5 and that is coming.

So, x_1 is what I have written. So, this is my x_1. So, it is x_3 is $\frac{-2}{3}x_4$, x_4 is x_4 and x_5 is 0. And from here I can write x_2 as {- 2, 1, 0, 0, 0 } just the linear combination and + x_4 is {$\frac{-5}{3}$,0, $\frac{-2}{3}$, 1, 0}. So, from here you can see that the null space is spanned by 2 vectors and these are linearly independent. you can check with this one. So, the nullity in this case is 2. And you can also verify from here that the dimension the rank was 3 in this case, and from here it was the r_5. So, nullity is 2, rank is 3, and from here you can see that it is 2 plus 3 is 5.

Now, this is I have written; now null space of a transpose we can range space of P_2^T. So, in this case I want to write directly from here. So, this is, so P_2 is this one { - 2, 1, 0, 0 }. So, this is basically we are writing corresponding to the 0 row. So,{ - 2, 1, 0, 0 }. So, the range space of the vector { - 2, 1, 0, 0 } and what is this? So, it will be equal to { - 2, 1, 0, 0 }. So, it is 4 × 1 into x.

So, I can write from here that the null space of A^T is spanned by {-2, 1, 0, 0}. So, this is basically we are writing directly from here. So, now, from here the null space of A^T is this one and its dimension is also dimension of N (A^T) is 1. So, the dimension is 1 here, and I know that the rank of A is 3. So, the range space of A that I know is 3. So, it is 3 and +1 =4. Now, it is coming from R^5 to R^4.

So, from here you can check that it is going from R^4 to 4. So, the range space of this 3-dimensional N(A) is 2 dimensional. So, it becomes 5 in this case and it is 3 + 1= 4. So, you can check from here that in this case I can check that the rank plus nullity. It means I am taking the dimension of range space of a transpose plus dimension of N(A) that is equal to 5 equal to the dimension of R^5, and dimension of R(A) plus dimension of N($A^{T)}$ that is also 3 + 1 =4 and that is equal to the dimension of R^4, because this map was going from R^5 to R^4. So, we will make some conclusions based on this one. So, let me stop here today.

So, in the today lecture we have discussed two very important relation or topic that how we can write down the all the four subspaces related to the matrix after converting the matrix into the row echelon form, and then also we have shown that if the rank of the given matrix is equal to the number of variables, then definitely the null space corresponding to that is going to be only zero null space or may be trivial null space. So, in the next lecture we will continue with this concept. So, thanks for watching.

Thanks very much.