Matrix Computation and its applications

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Lecture - 20
Four subspaces associated with a given matrix

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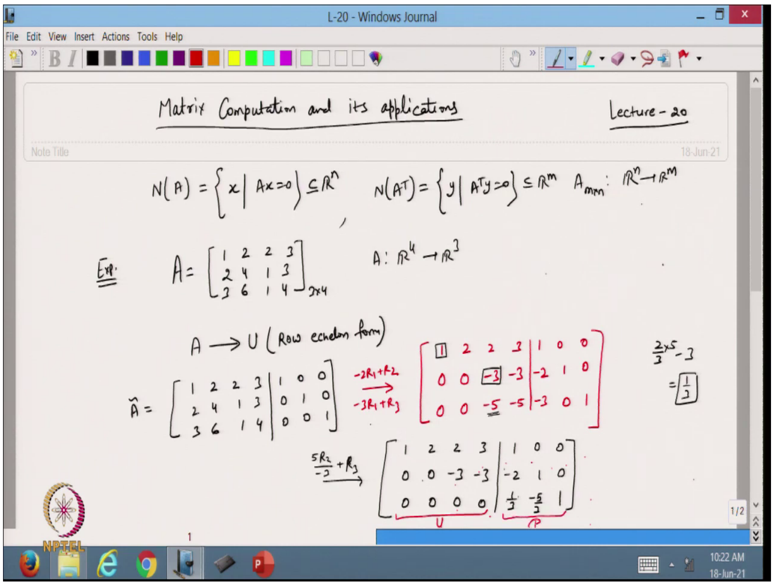

Hello viewers, welcome back to the course on Matrix Computation and its Application. So, in the previous lecture we have started with the null space. So, in this lecture we will continue with that one.

So, in the previous lecture, we have introduced the null space of the matrix A and we know that it is a set of all solutions of the homogeneous system. And, then also we have defined the $N(A^T)$ =null space of A transpose and that is also the y such that A transpose y is equal to 0.

And, we found that if we have a matrix A, that is m x n; so, it has a linear transformation or linear function from Rn to Rm. So, in this case I know that this is a subspace of Rn and this is a space of Rm so, that we have discussed.

Now, so, I will take one example and then we will go further. So, I just take one example I have a matrix A= and suppose this matrix I am taking 1 2 2 3 and it is 2 4 1 3 and 3 6 1 4.

So, this is the matrix I have taken and, now this is 3 cross 4 matrix. So, I can say that this is a map from R 4 to R 3, now, in this case just to start with this one. Now, so, what I want to do? I want to first I want to transform the matrix A into the upper triangular matrix, echelon form not the upper triangular it is because it is a rectangular matrix. So, I want to convert this into the row echelon form.

Now, what I want to do is that, I want to find out the echelon form and then with the help of the echelon form I should be able to find all the 4 subspaces associated with the matrix A.

So, for this one first we do. So, what I will do is that I will write the, the matrix A here 1 2 2 3, 2 4 1 3, 3 6 1 4 and here I am taking the unit matrix. So, I have taken this matrix now, I call this matrix as A bar. Now, so, I want to convert this into the row echelon form. So, first of all I will try to make this element 0 and this element 0. So, I will apply this row operation. So, I will take minus 2 times. So, I am taking here minus 2 R 1 plus R 2 and minus 3 R 1 minus 3 R 1 plus R 3.

So, from here I will get 1 2 2 3 and this is 1 0 0, now it becomes 0 here so, it become 0 here, this also become 0, because I am multiplying the minus 2, so, minus 4 and minus 2 plus 1. So, minus 2 plus 1 is minus 3 and minus 6 plus 3 minus 3. Similarly, I am doing the calculation here also. So, I am multiplying by minus 2 and add here, so, it will be minus 2 1 0.

Now, after this here I am getting 0, because I am multiplying my minus 3 and adding here minus 6. So, this will be also 0, then I am multiplying by minus 3. So, it will be minus 6 plus 1 it will be minus 5 and this is minus 3 and minus 9 plus 4 is minus 5. And, here I am getting minus 3. So, I will get minus 3 here and then 0 and then 1.

Now, so, this is the way I am getting. So, now, I want to eliminate this element. So, this element we have to eliminate. So, I will apply this now in the next one. So, and this is my pivot element. So, this is my pivot and this is also my pivot. So, I will apply now.

So, what I am going to do is I will divide R 2 by minus 3 and multiply by 5 and add to R 3. So, this so, I will get 1 2 2 3, this is 1 0 0 and it is 0 0 minus 3, minus 3. So, minus 2 1 0 and then here it is 0 0 0, this also becomes 0, because it will be divided by minus 3. So, it will be 1 and multiplied by 5. So, 5 plus 5 minus 5 is 0 that is same here.

So, in this case now it is minus 2 divided by minus 3. So, 2 by 3, it will become 2 by 3 and then multiply by 5 and then add to this one so, it will be minus 3. So, it is 1 by 3. So, I am writing here 1 by 3 and this is I am getting from 10 by 3. So, now, I am taking the 1 so, it will be minus 5 by 3 and this is 1.

So, now you can see from here that this is the echelon form. So, now, I can write that, this is my echelon form. So, it is my U and this is the matrix corresponding to that is P.

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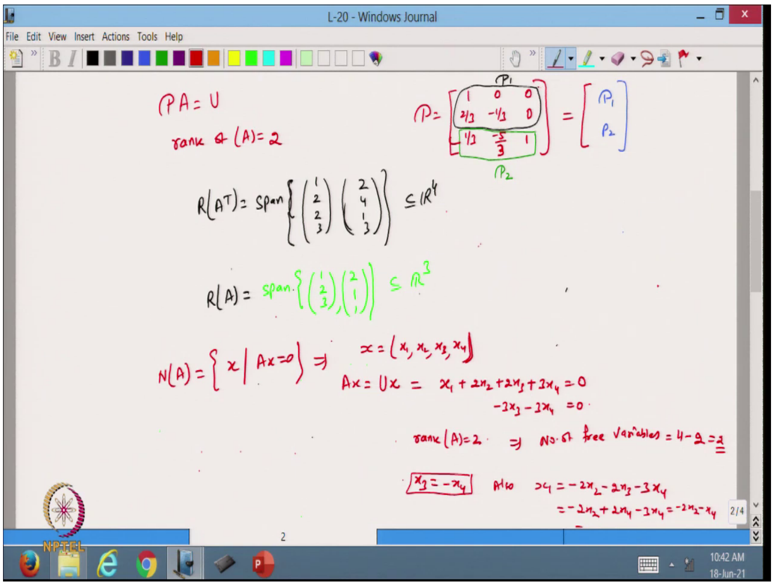

So, in this case we know that, the matrix P A that becomes U. Now, in this in the matrix also we know that, the rank of this matrix is 2 is coming 2. And, P is basically a matrix, which is using that is the transformation matrix, which is used for pre multiplication of A that becomes the U. So, it is 1 0 0, it is 2 by 3 minus 1 by 3 0 and 1 by 3 minus 5 by 3 and this is 1. So, this I am getting.

Now, the rank of matrix A is 2. So, I can write this as I can write this matrix corresponding to the non zero rows, I call it P 1 and this one, I can write as P 2. So, it becomes this matrix becomes P 1 or maybe I can use some other color. So, I can write this as P 1 P 2.

Now, from here it is very clearly we can see that, that the range space of the matrix A T. So, that will be spanned by the non zero rows corresponding to so, these are the non zero rows. So, these are the non zero rows corresponding to the given matrix, because it is a the row echelon form. So, in this case we can see that this is the non zero rows first two.

So, I will take the corresponding rows. So, it will be 1 2 2 3. So, span by 1 2 2 3 and then 2 4 1 3, 2 4 1 3. So, it is a basically span by this one. And, of course, it is we have seen that this is a transformation from R 4 to R 3 and it contain the 4 component. So, definitely it is a subspace of R 4. So, that we have taken and from here and I also know I know that the rank is 2, then the range space of A.

So, the range space of A, I can write directly from here. Now, if you see from here then this is my basic column and this is my basic column, basically not writing here maybe I will write here. So, this is my pivot and this is my pivot. So, this is a basic column and this is a basic column. So, from here I can say that, the range space of A is a span corresponding to the basic column and that will be 1 2 3 and another one will be 2 1 1 and these are linearly independent.

So, and that is equal to the rank of the matrix and definitely this belongs to R 3. So, now, we are able to find the range space of A and range space of A transpose. Now, we want to find out the other subspaces. So, let us write what about the null space of A? So, the null space of A is coming from the again from the non zero rows. So, this is my echelon form and using this echelon form. So, I can write my corresponding. So, that is equal to the x such that A x equal to 0.

So, from here you can write that the corresponding system of equation becomes 1 2 2. So, it will be I can write here that so, x I am taking from R 4. So, suppose my x is so, let x is x 1, x 2, x 3, x 4. So, I am writing my A x and A is now it is the echelon form. So, basically I am writing A as reduced to in the form of U x. And, U x is just I am writing. So, it will be x 1 plus 2 x 2 plus 2 x 3 plus 3 x 4 equal to 0 and from the last one, it is minus 3 minus 3.

So, I can write here minus 3 x 3, minus 3 x 4 equal to 0. So, I get the only these two values. Now, for this one I need to find the because this is going to have a infinite mining solution. So, x 2 and x 4, now the rank is 2. So, rank of matrix A is 2. So, from here I can know that number of free variables will be 4 minus the rank that is 2. So, it will be 2 free variable.

Now, in the 2 free variables here, if you see then we have to find out 2 variable free. So, we take the free variable corresponding to the non basic columns. So, here we have so, I can write from the first equation and from the second one, now my free variable basic non basic columns are the second and the fourth.

So, what I am going to write? I am going to write my x 3 is equal to minus x 4 from this one I can find out. And, this one I am finding also from the first equation, I can write my x 1 is equal to minus 2 x 2, minus 2 x 3 and minus 3 x 4. Now, in this case I can write minus 2 x 2 here. So, x 3 instead of x 3, I can put the minus x 4. So, it will be 2 x 4, minus 3 x 4.

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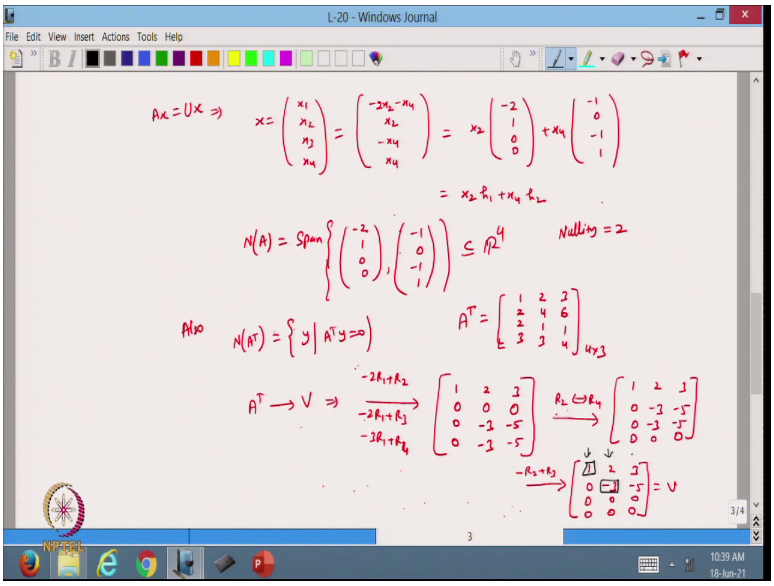

And from here so that, I can write as minus 2 x 2, minus x 4 ok. So, from here now I got this one. So, from here I can write my U x from here my x become x 1, x 2, x 3, x 4, that is equal to so, instead of x 1 I am I will write minus 2 x 2 minus x 4.

So, it will be minus 2 x 2 minus x 4 and x 2 is x 2, x 3 is minus x 4. So, x 3 is minus x 4 and x 4 is x 4. So, you can see that, we are writing each and every variable in terms of x 2 and x 4. And, from here I can write that my x 2 can be written as minus 2 1 0 0 plus x 4, it is minus 1 0, minus 1 1 ok.

So, we are able to find the x 1 in terms of x 2 x 4 and x 3 in terms x 4. So, from here now I can write my so, this I can write as a x 2 is a the non basic variable corresponding to h 1. So, this is my h 1 plus x 4 it is my h 2. So, this is the vector we are taking we represent by x 1, h 1 and h 2.

So, now from here I can say that the null space of A is span by minus 2 1 0 0 and minus 1 0 minus 1 1. So, this is my null space. And, you can see from here that this is from R 4. And, these are linearly independent so, I can also write that the nullity is 2. Now, I want to find what about the null space of A transpose? So, in the case of null space of A transpose the only thing is that it contains all the y's such that A transpose y is equal to 0.

So, in this case what we need to do is that, we want to find the A transpose, that is basically I get the matrix 1 2 2 3 and then 2 4 1 3 and then 3 6 1 4. So, this is my matrix of course, it will be 4 cross 3. And, now I have to convert this matrix into the again into the echelon form and then we have to find out the solution for this one. So, that is we have to do it again.

So, let us do this one for this example, now my A so, I will I want to transfer this matrix into the echelon form. So, what I am going to do is so, this matrix converting to some echelon form V. So, by this one I am writing. So, this is my pivot element.

So, in this case I applying this formula, I will apply this elementary transformation minus 2 R 1 plus R 2 minus 2 R 1 plus R 3, and minus 3 R 1 plus R 4 and if I apply this one I will get 1 2 3, 0 0 0.

So, I am multiplying by minus 2 and adding here. So, it is minus 2 plus 2 0, minus 2 plus 4 it is 0, and minus 6 plus 6 it is 0, again I am doing this one. So, it will be 0 and minus 2. So, it will be minus 3 and this will be minus 3 into minus 2 minus 6 plus 1. So, it will be minus 5. And, again I am multiplying by minus 3 so, minus 3 plus minus 6 plus 3 minus 3 and minus 9 plus 4 minus 5.

Now, I will swap my R 2 with R 4, because the 0 element is there already. In this case, so, it is 1 2 3 0 minus 3 minus 5 0 minus 3 minus 5 0 0 0. And, from here I will again apply the operation to make this element 0. So, what I am going to do is that, I can write minus R 2 plus R 3. So, it will be 1 2 3 and will be 0 minus 3 minus 5 0 0 0 0 0 0 ok.

So, from here you can see that this is my echelon form and this elements are the pivot elements, corresponding pivot elements. Now, I need to find that the null space of the corresponding matrix. So, for in this case I have 1 2 3.

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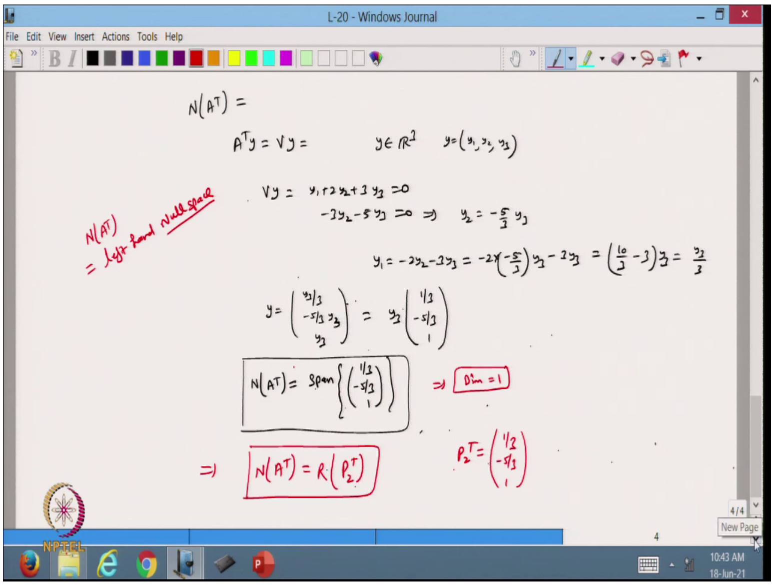

So, now, null space of A transform. So, that we are going to find. So, it will contain the y. So, I will write A transform y, that is basically v y. So, from here I can write the matrix v and y is belongs to R 3 ok. So, my y suppose I take y 1, y 2, y 3. So, here I can write v y becomes y 1 plus 2 y 2 plus 3 y 3 equal to 0, and minus 3 y 2, minus 5 y 3, equal to 0, because I am multiplying the y. So, y 1 plus 2, y 2 plus 3, y 3 and minus 3 y 2 minus 5 y 3.

Now, in this case also this is my basic columns and this is my non basic column the last one. So, I can write everything in terms of y 3. So, from here you can see that my y 2 can be written as minus 5 by 3 y 3. And, if I put here then from the first equation my y 1 can be written as minus 2 y 2 minus 3 y 3. And, y 2 is written here so, it is minus 2 into minus 5 by 3, y 3 minus 3 y 3.

And, this is 10 by 3 minus 3 so; it is y 3 by 3. Now, from here I can get that from here my y basically become. So, y 1 is becoming y 3 by 3; y 2 is minus 5 by 3, y 2, y 3 and y 3 is y 3. So, it can be written as y 3, 1 by 3 minus 5 by 3 and 1. So, I can write from here that the null space of a transpose is span by the vector 1 by 3 minus 5 by 3 1.

And, because it belongs to the R, so, that is the null space of this one and we also know that the null space of A transform is also called left hand null space, and this is equal to this one. So, it is dimension is 1. Now, from here you can see that, it is spanned by the element 1 by 3 minus 5 by 3 1. And, for this one we have taken all the transformation, first we have taken the transpose and then we convert that into the echelon form and then we are able to find.

But, if you see from the previous one and now you can see from here that the P 2. So, P was the non singular matrix such that P A becomes U. And, in that case we came to know, that the first two rows are the non zero rows in the echelon form, but the last row was the 0 rows. So, it was the 0 row. So, this was my 0 row and this was my P 2 corresponding to the 0 row.

Now, from here if you see the P 2 is 1 by 3 minus 5 by 3 and 1. So, from here we are going to introduce one thing that, we can also write that the null space of a transpose can be written as the range space of P 2 transpose. What is the P 2 transpose? Because, if you see from here P 2 is a row vector, so, I just take the column make it column vector, so, I will take the transpose.

So, it means my P 2 transpose is basically a column vector, which has the element this one and you know from here that it is spanning this one. So, if I some wants to find out what is the range set of P 2 T, then we know that, the range space of P 2 T.

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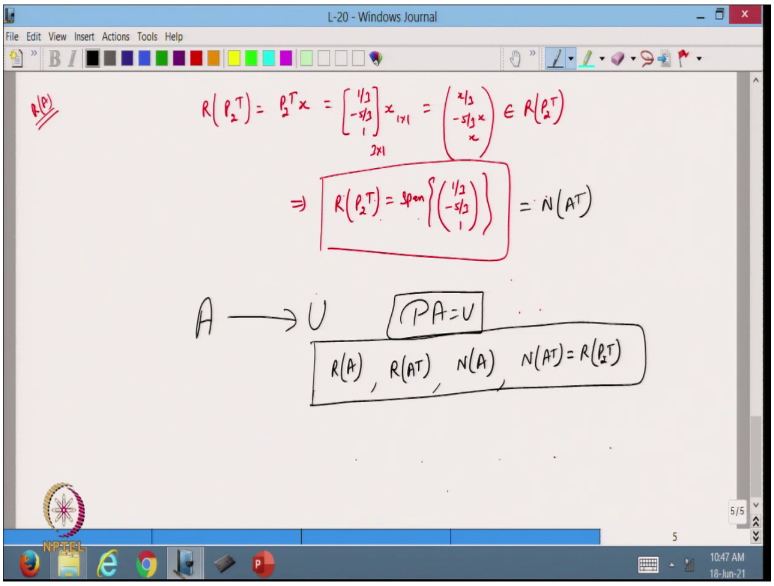

It is a column vector, so, you can take a it is a matrix with 3 rows and 1 column same as we used to find the range space of A or range space of other one, I can find the range space of this one.

So, that is equal to set of all P 2 T x. So, now, from here this is my 1 by 3 minus 5 by 3 1, 1 by 3 minus 5 by 3 1. So, this is my P 2 and it is 3 cross 1. So, I will take my x that is 1 cross 1. And, from here I will get it become x by 3, minus 5 by 3, into x and x, so, I got this vector. And, definitely this will belongs to the range space of P 2 T.

And, from here and this is just one vector, so, it is linearly independent. So, from here I can say that the range space of is span by the element the vector 1 by 3, minus 5 by 3, and 1, because x is just a real number. And, you know from here that this is coming to the null space of A transpose. It means that, there is no need to write the transpose of the given matrix and then converting this into echelon form and then writing the null space of A transform, A transpose.

So, it can be found directly by the span of the matrix P 2, that is coming from the this matrix. So, P 2 is basically we convert the matrix P into P 1 and P 2 form and then we found the P 2. So, based on this one now based on this one from the given matrix A, I have converted this into U, by P A is equal to U, then we are able to write the range space of A, by the basic columns corresponding to the columns of the in the echelon form.

We are able to write range space of A transpose, that is the left hand space range space, this is coming with the non zero rows corresponding to the echelon form. We are able to write N A directly, because it is just putting the non zero rows into x equal to 0 and finding this one. And, now we are able to find N A transpose. And, N A transpose is basically the range space of P 2 T.

So, with the one matrix transformation I am able to write, I am able to write the complete force of spaces associated with the given matrix. So, this is what we can do from this one. So, I think now we stop here.

So, today we have discussed one example and showed that using that example or the matrix A. We are able to write all the basis for the or all the basis for the corresponding subspaces associated with the given matrix. So, in the next lecture we will continue with this one. So, thanks for watching.

Thanks very much.