**Matrix Computation and its applications**
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**Lecture - 02**
**Vector spaces**

Hello viewers. So, welcome back to the course on Matrix Computation and its application. So, in the previous lecture, we have started with the basics, now we will continue with that one and we are going to define another algebraic structure that is called field.

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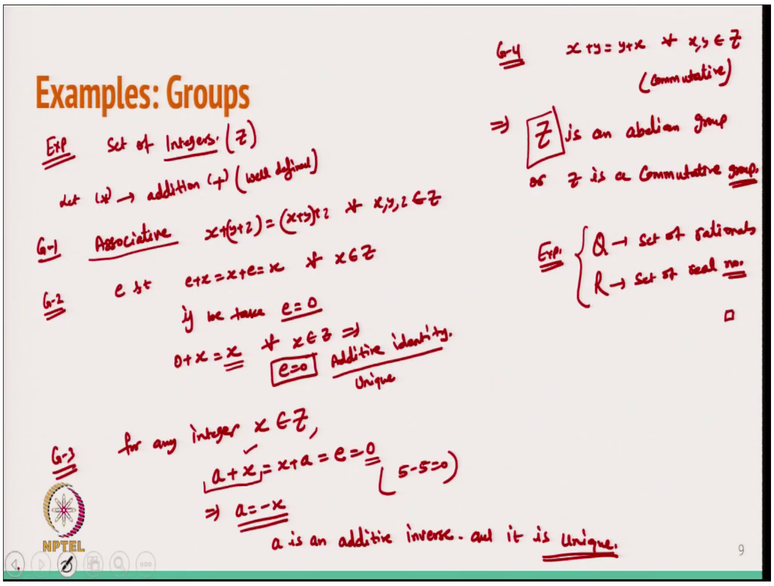


So, in the previous lecture, we have started with the groups and then, we have shown that the set of integers is an abelian group under this operation binary operation that is addition. And the same way we can define that the set of rational numbers or the real numbers, are also the group under the operation addition or even I can show that they are also abelian groups under multiplication.

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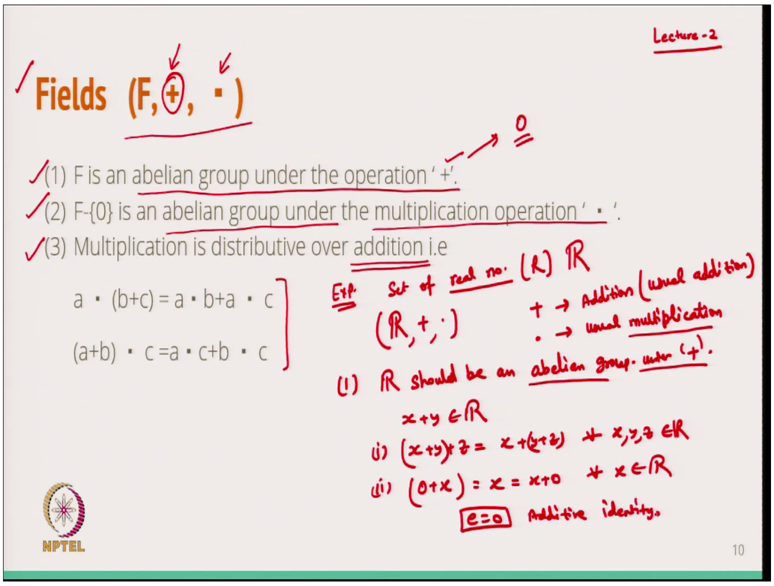


So, after that, we are going to start with another one that is a field. So, what is the field? So, in this case suppose we have a set F so, in this case, we have a set F and we are having two types of binary operation; one binary operation I will represent by addition and another binary operation will be represented by multiplication so, that we are going to have.

So, the first thing is that in the field, we are going to have two binary operations and one is represented by addition and other is represented by multiplication. So, what is the field? So, suppose we have a F is a non-empty set and 1st thing is that F is an abelian group under the operation addition. So, it should be clear that F should be an abelian group under the operation addition whatever the binary operation we are defining here.

Then, we know that under addition, it is an abelian group so, we are going to have the additive identity that we represent by 0. So, in this case, we just take F-{0}. F -{0} is an abelian group under the multiplication operation. So, it is also an abelian group under addition and also the abelian group under multiplication. The only thing is that I have to remove the additive identity from the set so, F-{0}.

And then, the third one is that multiplication is distributive over the addition so, what is that? a . (b + c) = a . b + a . c, or I can write (a + b) . c = a . c + b . c so, if this is satisfied, then we say that the multiplication is a left multiplication, it is a right multiplication so, we can say that multiplication is distributive over addition.

So, if these properties are satisfied, then we call that the given set (F,+, •) is a field. So, that is called a field. So, let us take some examples of how we can define a field. So, let us take the set of real numbers because this will be used when we define the vector spaces so, I will start with the simple one that is the set of real numbers or we also represent sometimes we also represent by R so, the set of real numbers.

Now, this is my set of real numbers, and I am defining addition and multiplication. So, addition I am defining as addition or I can take it as usual addition and multiplication I am defining as usual multiplication. Usual multiplication means whatever the multiplication we generally take for the set of real numbers that multiplication we are defining here.

Now, let us start with the properties. So, let us take the first property that R should be an abelian group under addition so, check this one, but we just we can define this very fast that ok so, because I know that if I take $x + y$ that is also going to be a real number so, that is also R so, I can say that addition is well-defined and then, so, this is a well-defined.

So, I can define the other properties of associativity.

(i) $(x + y) + z = x + (y + z)$ for all x, y, z belongs to the set of natural numbers N or the set of real numbers R.

(ii) Also $0 + x = x$ , true for all x belongs to the real numbers R. So, I can say that my $e = 0$ in this case so, this is my additive identity.

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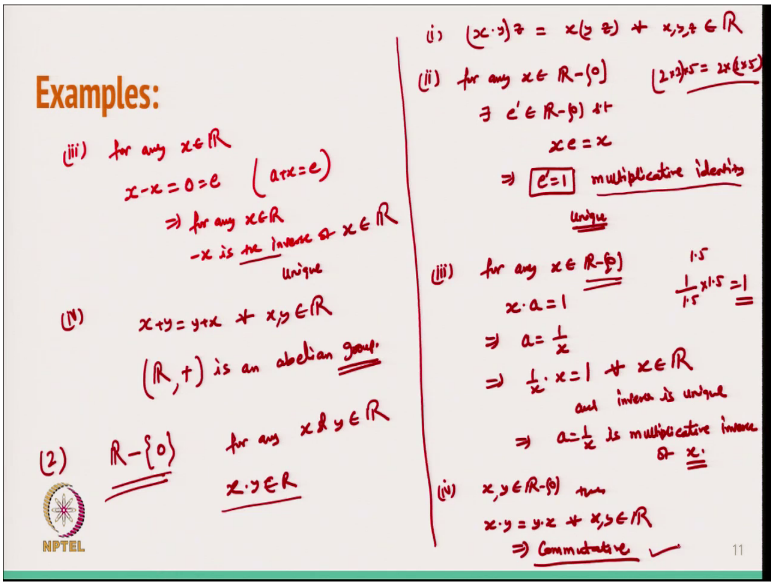


Now, after defining this one, we define the 3rd property.

iii x - x = 0 . So, from here I can say that because I need a + x = e, in this case, I can say that for any x belongs to the real number and is the inverse of x; the inverse of x in R so, it is the inverse and is unique.

So, x + y = y + x for all x, y belongs to the set of real numbers R. So, it is an abelian group. So, the first property is satisfied, then I define the second one. I have to take a set R-{0}, I know that for any x and y that belongs to R, x • y is also a real number so, belongs to R so, the multiplication is well-defined.

So, after defining this one, I can say that this is well-defined. Now, I can define that for any three element x, y, z it can be written as (x• y) • z = x • (y • z) for all x, y, z belongs to the set of real numbers R. So, it is associative.

Second one is that now, I have to find out the identity element e'. So, for any x belonging to R, it cannot be a 0 element. We need x • e' = x. So, in this case, you see that this is only true because it is a real number, then e' = 1 is the multiplicative identity and is unique.

The third one is so, in the third case, I need to find out the inverse, then for any x belongs to my set so, you know that if I take any real number 1.5, then (1/1.5) 1.5 = 1. So, for any x belongs to this, I need a, such that x • a = 1. So, from here, I know that a = 1/x because x is not 0 that is why we always exclude the value of 0. if I take 1 / 0 = ∞ and is not a real number, that is why it does not belong to this one.

So, that is why we always take R -{0} and that is. So, a = 1/ x so, from here I can say that 1 by a • x = 1 for all x belongs to the set of real numbers and this inverse is unique.

iv) Now, if I take any two x and y that belong to R -{0}, then because I am going to show that it is a commutative, we multiply any two real numbers. That is also always true for all x and y belonging to R. So, from here I can say that this is commutative. So, from here, we can say that this is true that it is commutative under addition and this is also commutative under multiplication.

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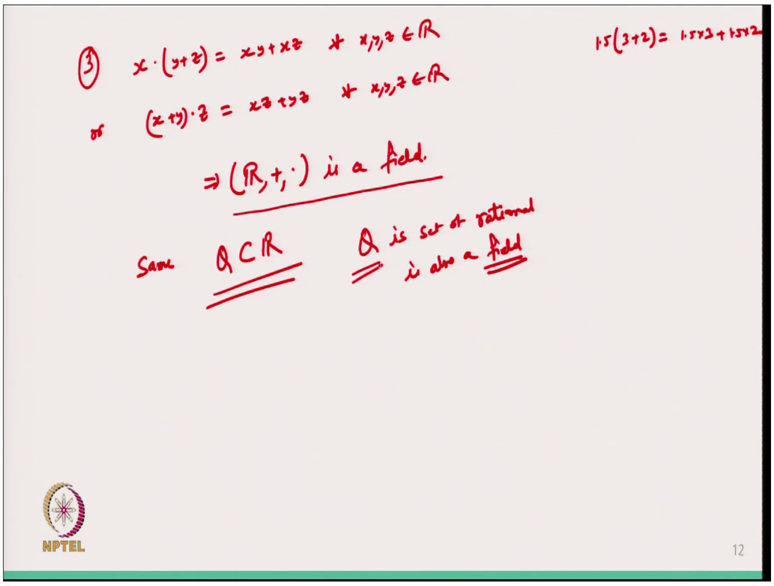


So, from here, now after satisfying this one, the next one is that third property I have to define. Now, I know that if I take 1.5 • (3 + 2) = 1.5 • 3 + 1.5 • 2. So, from here, I can say that x • (y + z) = x • y +x • z and this is true for all x, y, z belongs to the set of real numbers or I can write (x + y) • z = x• z + y • z with the multiplication sign and this is also true for all x, y, z belongs to R.

So, from here, I can say that the dot is distributive over addition. So, from here, I can say that the set of real numbers under addition and multiplication is a field. So, it is a field and clearly, it is also a group under the operation addition and the multiplication that we have already discussed. So, from here, we can say that the set of real numbers is also a field.

I can show that the Q is the set of rationals and is also a field that we can satisfy all the properties. Because I know that the sum of two rational numbers is a rational number and the product of two rational numbers is also a rational number. So, we can show that the set of rational numbers is again a field.

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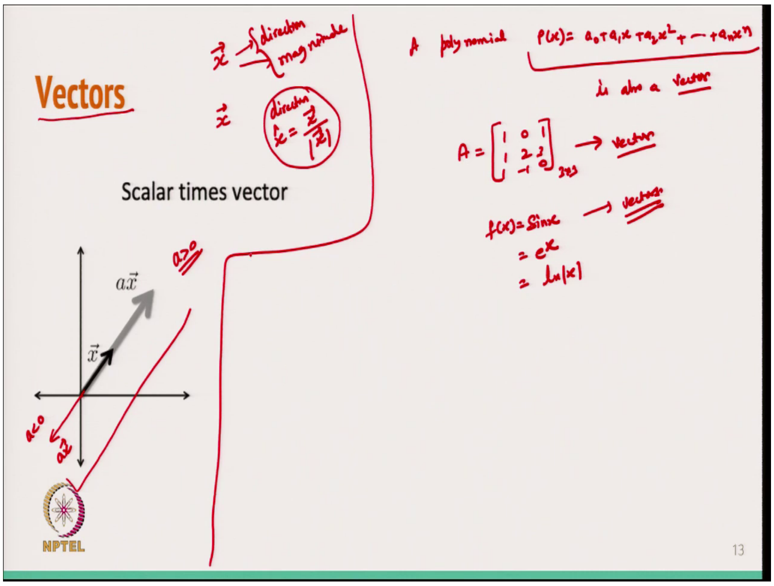


So, now after defining the groups and fields, let us start with the terms what are called the vectors. Because generally, we know that the vectors are represented by a sign like this one. So, this is a vector which has direction as well as magnitude.

So, these vectors we know are called vectors. So, let x be a vector and I multiply by some scalar a, then it becomes ax where in this case, a is positive, if a is negative, its direction will change and then, it will go in this direction. If it is my ax when a is negative and if a is 0, I will get this point.

Now, these vectors we already know, but in the vector spaces, we can define different types of vectors and what are these vectors? So, now, a polynomial

$p(x) = a0 + a1\ x + a2\ x^2 + a3\ x^3 + ... + an\ x^n$ so, this is a nth degree polynomial is also a vector or I take a matrix A = [1, 0, 1; 1, 2, 3; 1, -1, 0] so, it is a matrix 3 x 3 matrix. So, it is also a vector or I take a set of functions like suppose I take f(x) = sin x or $e^x$ or any function log x. So, these are also vectors.

So, now, a vector does not mean that whatever the vector we use to take in our schools, that is the only vector, even a matrix or a function or a polynomial, can also be a vector and more. So, from here, we will take the different types of vectors to define the, our next definition and that is called the vector spaces.

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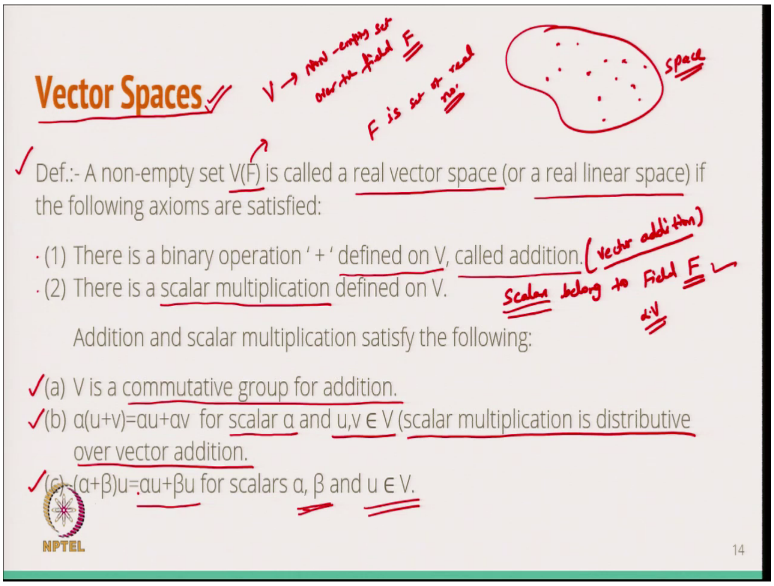


So, this is the main content of this introduction to vectors algebra first. So, in this case, we define what is called the vector space. So, as the name suggests, suppose we have some space, and that space is made up of different types of vectors. Space means in which some properties are defined, then only we can say that is a space and then, based on this one, we call it a vector space.

So, generally the space means where we can walk, we can move anywhere so, in the general sense, the space means that a person is having the freedom to walk in any direction so, that is called the space. So, let us define this one.

So, what is the definition? A non-empty set V defined on the field F. So, V is a set; a non-empty set over the field F. So, in a non-empty set V(F) is called a vector space. We start with the only real vector space first or it is also called a real linear space if the following axioms are satisfied. So, real vector space means my F is a set of real numbers. So, if the field is a real number, we are defining the V over the real numbers so, there and it is called the real vector space.

So, now, we have to satisfy the following axioms only then, we will say that it is a vector space, it is made up of vectors and it is a vector space. So, 1st one is that the first part is that there is a binary operation addition defined on V called addition or vector addition, addition or I also call it vector addition.

There is a scalar multiplication defined on V. So, scalar multiplication, the elements because in this space, we have vectors. So, where is the scalar multiplication? The scalars are coming from F. So, I can say that scalars belong to field F. So, whatever the field we are taking, scalars are coming from that field.

So, scalar multiplication means I multiply a scalar. Suppose we call it $\alpha$ is a scalar and $\alpha v$ is called the scalar multiplication, multiplication of a scalar with a vector. Now, after defining these two, addition and scalar multiplication satisfy the following condition. So, first we have defined the binary operation and then, we satisfy this condition. So, the first one is that V is a commutative group under addition.

So, it should be a commutative group under addition, second thing is that $\alpha$ is a scalar. So, we generally take $\alpha$, $\beta$, $\gamma$, like a scalar and u, v, w for the vectors. If $\alpha (u+v) = \alpha u+\alpha v$. We say that the scalar multiplication is distributive over the vector addition. So, this should be satisfied.

Now, $(\alpha+\beta)u = \alpha u+ \beta u$. Here we are adding two scalars and then, I multiply the u. So, this way we have the distribution.

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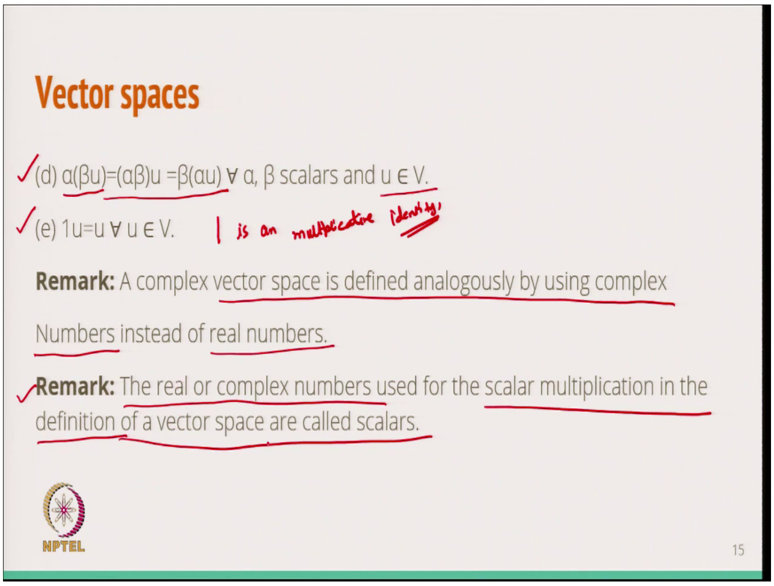


So, vector space is then, the next one is that if I take the (αβ) u=(α)βu = β(αu). So, just I multiply alpha beta u, then it does not matter, I multiply by alpha beta with the u or I just change the position beta here and alpha u. So, these conditions are satisfied for any type of scalars alpha and beta and for the vector V.

And the last property is that 1 u=u for all values of u where 1 is the multiplicative identity in F. We generally represent the multiplicative identity with 1.

So, this is the remark: a complex vector space is defined by using complex numbers instead of real numbers. So, the same way we can define the set of complex numbers and then, if you satisfy the given condition of the vector space, then it becomes the complex vector space.

And also this is the remark: the real or the complex numbers used for the scalar multiplication in the definition of the vector space are called scalars. Scalars are coming from the field. So, if that field is a set of real numbers, then it is a real vector space; if it is a complex number, then it is called the complex vector space.

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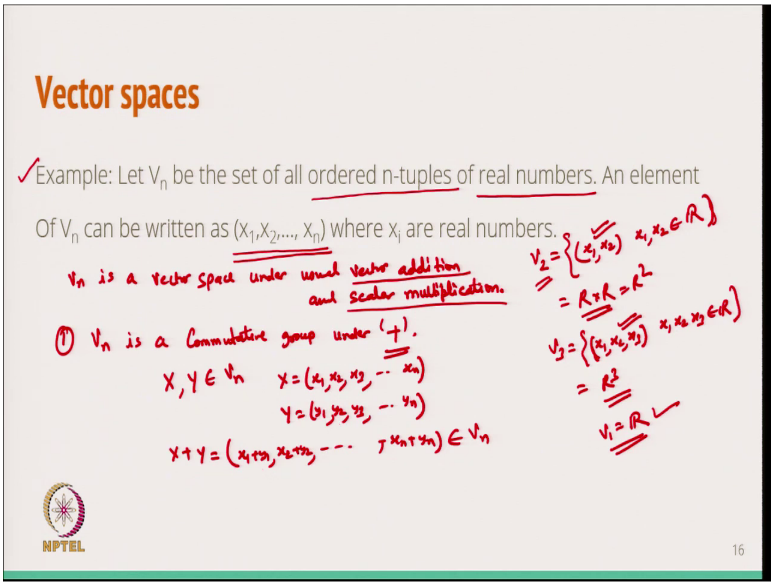


So, for example, that it is a very important vector space we are going to define. So, let Vn be the set of all ordered n-tuples of real numbers, I am defining them with the real numbers now. An element of Vn can be written as with this n-tuple where xi are reals.

So, for example, if I take V2 so, V2=( x1, x2) so, this is the 2- tuple and x1 and x2 belong to a real number. So, this is basically if you see, then this is R x R=$R^2$, xy plane, the whole plane we are defining. Similarly, V3 is the 3D, V3= (x1, x2, x3).

Now, the first thing is that we want to check whether it is a vector space or not. So, we can show here that Vn is a vector space under usual vector addition and scalar multiplication. Usual means the way we take the addition of the two vectors.

So, how can we define it? So, if we can show it by showing the properties that are so, now, we can check that the 1st property is that Vn is a commutative group under addition. So, this is a commutative group because we know that if I take a element so, let us take the element suppose I take X and Y belongs to Vn, then suppose I take X =(x1, x2, x3, . . ., xn) and

Y =(y1, y2, y3, . . . , yn). So, these are the elements that belong to Vn, then we can show that X +Y = (x1 + y1, x2 + y2, . . . ,xn+yn).

So, this is the way we generally add two vectors so, that is why it is called the usual vector addition, it is a component wise addition so, this also belongs to Vn. So, it is closed under addition and then, we can define that it also satisfies the other property. So, that we are going to discuss in the next lecture. So, we will stop here.

So, today, we have continued with the definition of the algebraic structure field and then, after the field, we have defined how a vector space can be defined for different types of vectors and then, in the next lecture, we are going to continue with this one.

Thanks for watching. Thanks very much.