Matrix Computation and its applications

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Lecture - 19

Null space of a matrix

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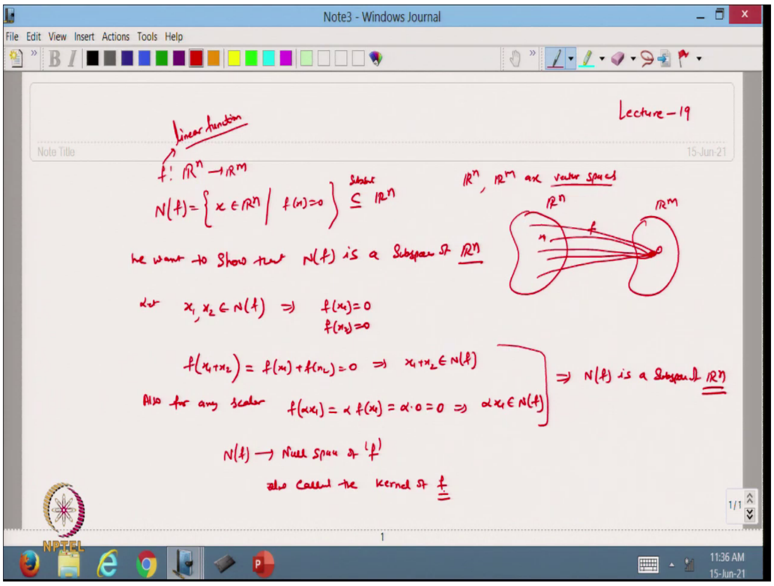

Hello viewers, welcome back to the course on Matrix Computation and its application. So, today lecture, we are going to introduce two other subspaces based on the given matrix of order m cross n. So, let us start with that one. Now, suppose we have a linear function f from R n to R m and in this case I will define a set N of f.

So, this is set of all the elements x from the domain that is R n, such that f of x becoming 0; because I know that R n and R m are the vector spaces. So, that is going to have a zero element also. So, in this case we are going to introduce taking all the elements from this domain, which maps to 0.

So, maybe I have some this is my R n, this is my R m and I define if linear function f. So, it maps that may be, this is my 0 element. So, I will take this element may be x 1, then x 2 is also going, x 3 is also going; this element is also going, this element is also going to the same element 0.

So, I want I just taken these elements belongs to the R n and that is my set I call it N f, ok. So, now, the question is that, this is a subset of R n. Now, I want to check, we want to show that N f is a subspace of R n. So, let us do that one. So, let x 1 and x 2 belongs to the set N f, which implies if they belongs to the N f; then f of x 1 is 0 and f of x 2 is 0.

Now, let us see what about x 1 plus x 2? Now, what about f x 1 plus x 2, let us check this one. Now, this is a linear map. So, it should be equal to f x 1 plus f x 2, because f is a linear function. So, if it is linear function, then f x 1 plus f x 2 and that is going to be 0. So, which implies that, x 1 plus x 2 belongs to the set N f; also, for any scalar f of alpha x 1, so it is a linear.

So, I can take this scalar outside and that becomes x 1 and this is alpha into 0 and that is 0; which implies that alpha x 1 also belongs to N f. So, these two properties are satisfying, then other properties we know that it is automatically satisfied to check that it is a subspace or not. So, from here I can say that, N of f is a subspace of R n, ok. So, that is a subspace of R n.

Now, what is the; so, N of f is a subspace and it is called null space of f or also called kernel of the function f. So, this is also called the null space of f or the kernel of f.

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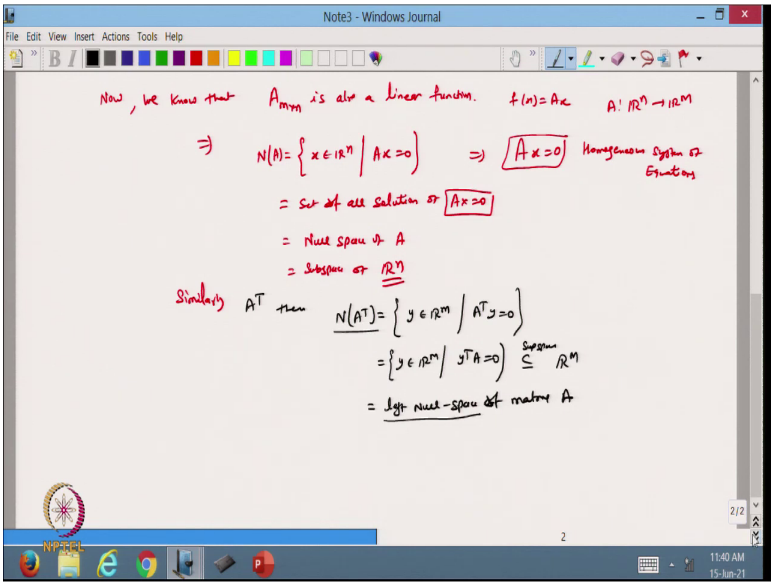

Now, we know that the matrix A of order m cross n is also a linear function. So, I have taken f x as A x, where A is, so A we know that one and it means that A is a linear function from R n to R m. So, which implies that, I can take N of A is the set of all x belongs to R n, such that A of x is equal to 0.

It means which implies that, I know that A x is equal to 0 is a homogeneous system of equations, ok. So, that shows that, this will be equal to set of all solutions of this one, equal to this one; it means we are taking the homogeneous system equation and we find this different solution and that solution make the set N A. So, that is a called null space of matrix A and this is subspace of R.

So, similarly I just take A transpose; then null space of A transpose is a set of all y belongs to R m, such that A transpose y is equal to 0, ok. So, that is also we can take and this can also be written as y belongs to R m, such that y transpose A is equal to 0; I have just taken the transpose of this one, ok.

So, this is a of course, it is subspace of R m; because y is coming from R m here. So, it is a subspace of R m and this is called. So, this is called left null space of matrix A, ok. So, this is called the left null space of matrix A and that is equal to N A transpose. So, this is how we are defining this one. Now, so from here I can draw the picture of the subspaces, that is attached with the matrix A. So, we have a matrix A.

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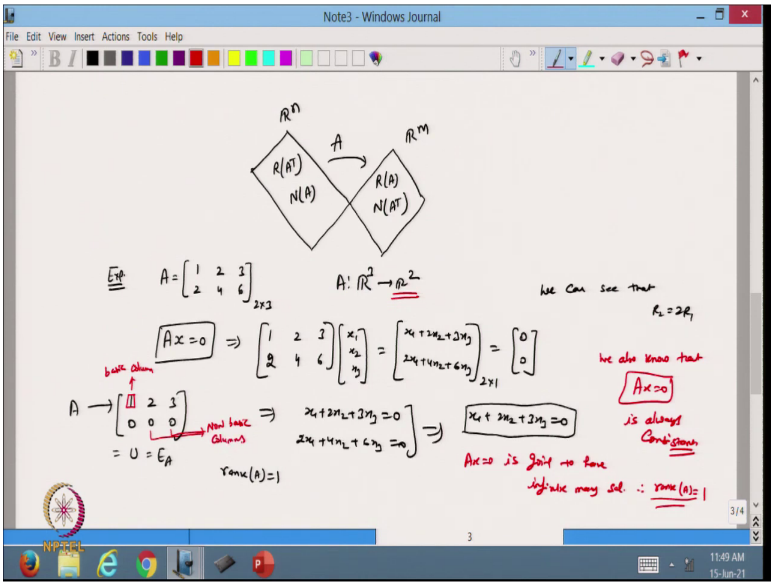

So, now suppose I take, just I will construct one picture, this one. So, I suppose I take this is my R n and this is my R m and I take a linear map A, so that is depended to the matrix m. So, I know that the range space of A is here, it is a subspace of R m and range space of A transpose is here. And now from here, this null space is a subset of R n, subspace of R n.

So, it is a null space of A and here is the null space of A transpose. So, these are the four subspaces that is always attached with the given matrix of order m cross n, ok. So, from here you can check that and we have already discussed that all are the four subspaces and two subspaces this belongs to the R n and two other subspaces belongs to the R m, ok. So, this way we can find out the subspaces.

So, let us take one example; I have the matrix A, it is 1, suppose I take the simple matrix 1, 2, 3 and 2, 4, 6, suppose I just take the simple matrix and this is of course, 2 cross 3 matrix. So, I have a linear map R from R 3 to R 2. Now, so using this matrix, I just want to find out its null spaces, ok. So, let us do that one.

Now, first what I will going to do is that, I just take A x equal to 0, the homogeneous system I will take, ok. So, that gives me 1, 2, 3; 2, 4, 6 and that is x 1, x 2, x 3; because it is coming from the R 3 and that will be equal to x 1 plus 2 x 2 plus 3 x 3 and 2 x 1 plus 4 x 2 plus 6 x 3.

So, this is a vector I am going to get and that is of course, it will be 2 cross 1. So, it is just belongings to the R 2. From here you can see that, now I put this equal to 0; because I have to take it equal to 0. Now, we can see that the R 2 is 2 multiple is. So, R 2 is 2 times R 1, that we can see from here, ok.

So, from here I will get x 1 plus 2 x 2 plus 3 x 3 equal to 0 and I will get 2 x 1 plus 4 x 2 plus 6 x 3 equal to 0. And from there I can just multiply this with the minus 2 and add here. So, I will get this system x 1 plus 2 x 2 plus 3 x 3 equal to 0, all other things will be 0.

So, basically if you take the matrix A and convert into the row echelon form, so, you will get the matrix 1, 2, 3 and 0, 0, 0. So, this is my U or may be reduced echelon form, that is E A. So, based on this one, I get this value. Now, from here you can see from here; now rank of A is also coming 1, right. And I also know that this x 1, x 2, x 3 belongs to R 3.

So, from here I will just take the help of my knowledge based on matrix theory. Now, one thing is true that, we also know that the system A x is equal to 0 is always consistent. Why it is always consistent? Because I know that 0 element is always there in the R 2, because that is a vector space.

So, it means I am taking the image of A and this is just the linear combinations of the columns and that is equal to 0. So, 0 is always there in the given subspace, any subspace, so it is always consistent. So, it is a consistent and rank is coming 1, so it will have infinite number of solutions, ok.

So, from here I can say that, if the system A x equal to 0 is going to have infinite many solutions, because rank of A is equal to 1 and the number of variable is 3. So, by the way for any value; even if the rank is 2, the still it is going to have a infinite many solutions.

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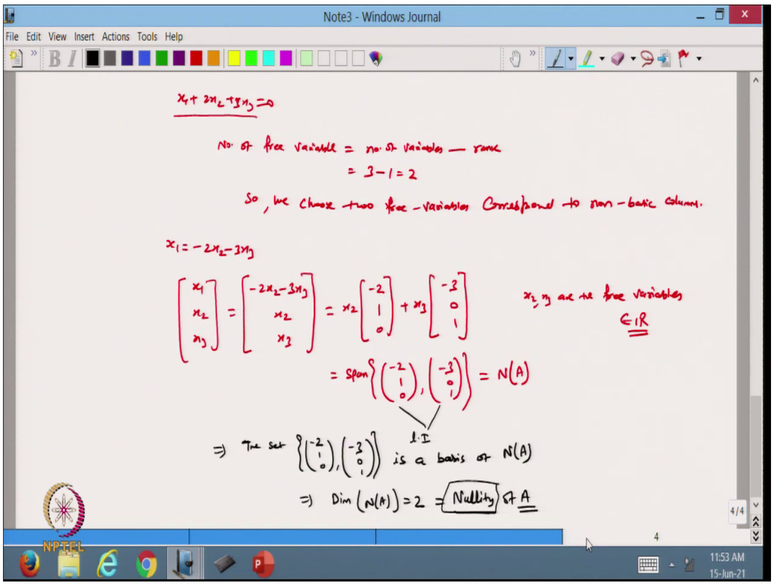

So, based on this one, now I get x 1 plus 2 x 2 plus 3 x 3 equal to 0. So, from here I can take now from the previous one it is my, so, this is my pivot. So, this is basically a basic column and these are the non-basic columns, non basic columns. Now, we need to find out the free variable. So, number of free variables is always equal to number of variables minus rank.

So, number of variable is 3, because coming from the R 3 minus the rank is 1, so it is 2. So, we choose two free variable corresponding to non-basic columns. So, the non-basic column is coming from here that is the second and third one. So, from here I can say that. So, my A x is going to be; now from here, so what I can write down is, I can write my x 1 is equal to minus 2 x 2 minus 3 x 3.

So, my solution x 1, x 2, x 3 can be written as x 1 is minus 2 x 2 minus 3 x 3 and x 2 is x 2 and x 3. So, now, you can see that there are two free variable x 2 and x 3. So, this is the solution. Now, this can be written as, I can take x 2 common. So, it is minus 2, 1, 0 plus x 3 that is minus 3, 0, 1, where x 2 and x 3 are the free variables and belongs to the real line; because we are taking coming from the vector spaces based on the field that is real line.

And from here I can say that, this is equal to the span of minus 2, 1, 0 and minus 3, 0, 1 and this is basically if you take the span, that is equal to the null space of. So, now, from here, I am taking from the corresponding rows. So, that is the null space of A, because coming from the A. And now these two vector if you see. So, this is basically linearly independent, that you can check from there.

So, now from here I can say that the dimension. So, I can; from I can say that, the set minus 2, 1, 0 containing two vectors minus 3, 0, 1 is a basis of N A. So, that is the basis. And from here it is, only has the two independent linear independent vectors. So, from here I can say that the dimension of N A is 2 and that is called nullity of A, ok. So, this is called the nullity; like the rank, so it is a nullity.

So, the dimension of the null space is called the nullity. So, that is 2 and it is coming from the by solving this way. So, I know that rank space of A and N A is coming from this one. Now, the next question is that, we are able to find the nullity of A; then how we can find the null space of A transpose? Now, we want to find null space of A transpose.

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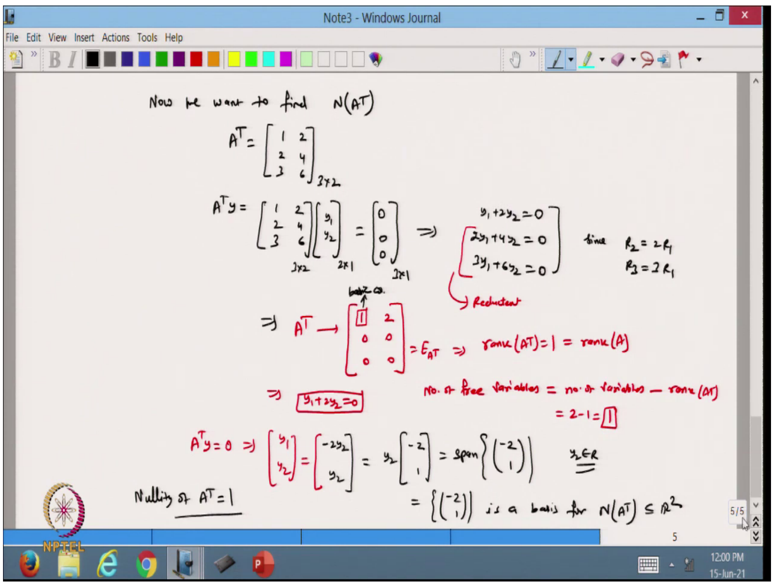

Now, the null space of A transpose, so in that case first I will define the A transpose. So, A transpose will be 1, 2, 3 and it is 2, 4, 6 and it is 3 cross 2. Now, I need to take A transpose y. So, this one I can write as 1, 2; 2, 4; 3, 6 and y will come from y 1, y 2 that is it. So, this is basically 3 cross 2 and this is 2 cross 1 and that should be equal to 0, 0, 0; 3 cross 1.

So, now this y will definitely come from R 2. So, from here you can check that, this become y 1 plus 2 y 2 is equal to 0, 2 times y 1 plus 4 y 2 is 0, 3 times y 1 plus 6 y 2 is 0. Now, from here you can check that, since R 2 is 2 times R 1 and R 3 is 3 times R 1; so from here you can see that these two rows are redundant.

So, this is redundant rows; it means that does not make any sense. So, from here if you convert this matrix A transpose into the echelon form. So, my A transpose convert to. So, it become 1 2; 0 0; 0 0. So, that is basically my E A transpose or the echelon form, and from here you can check. So, now, the rank of A transpose is 1 and that also I know that the, this is equal to rank of A.

So, from here I can say that only one equation y 1 plus 2 y 2 is equal to 0, that is going to be to play the role to finding the solution. Now, from here; so and definitely it is going to have, because we are now dealing with two elements. So, it becomes belongs to R 2, so y 1, y 2 and only rank is 1. So, from here I can say that, the number of free variables is equal to number of variables minus rank.

So, it is 2 minus 1 that is 1. So, one free variable is needed. So, from here I can say that A transpose y is basically equal to 0, that gives me y 1, y 2 becomes. So, y 1 I can write; now from here it is my basic variable and this is non-basic. So, this is my basic column and this is non-basic column. So, from here I can write my y 1 in the terms of the non-basic, so y 2. So, I can write my 2 y 2, because coming from here and y 2 is always there and that I can write as y 2 minus 2, 1.

So, that is always I can write and y 2 belongs to R 2. So, I can write this is equal to the span of the vector this one, where y 2 belongs to the real line. And now it is linearly independent, so from here I can say that, this is equal to the. So, the set with one vector is a basis for null space of A transpose and that also we know that it is a subspace of R 2; because we have started with the matrix this one, so it was from R 3 to R 2.

So, from here I can say that, this null space of A transpose belongs to this one and it is spanned by this. Now, so based on this we found that, the null space of A is dimension 2 and the null space of A transpose is of dimension 1. So, I can say that, its nullity of A transpose is 1.

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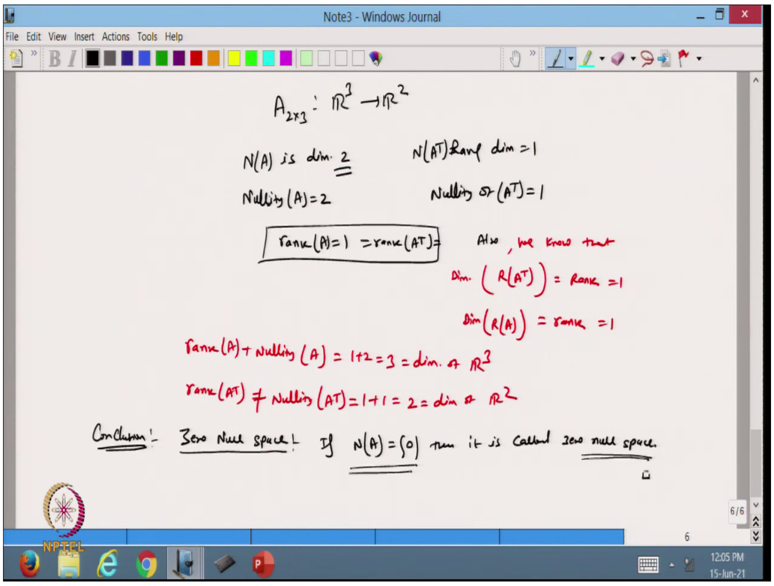

So, now, from here, I have a A that was my 2 cross 3. So, moving from R 3 to R 2, we found that the null space of A, so that is the null space of A we have found and that has the dimension 2. So, null space of A is dimension 2, null space of A transpose having dimension 1.

So, it means nullity of A is 2, nullity of A transpose is 1. So, that we have seen. Now, we have also seen the rank, rank of A is 1; that we have already seen, that this was the matrix coming from here and then we found that is rank is just 1. Now, the rank of A is 1 and that is also equal to the rank of A transpose. So both, in the both the case that is becoming 1 and also we know that that the range of A transpose.

So, dimension of this one, dimension of range of A transpose is equal to the rank or dimension of range space of A is also equal to the rank; because the rank of A and A transpose is same and from here you can check that, this is equal to 1 and this is equal to 1. So, it means, I have now from here you can check that, rank of the matrix A plus nullity of the matrix A.

So, nullity of the, rank of the matrix A is 1 ok and the null nullity of the matrix 2. So, that is equal to 1 plus 2 that is equal to 3 and that is equal to the dimension of R 3. Also rank of A transpose plus nullity of A transpose; so the rank of A transpose will be same and nullity of A transpose just we have seen, that is equal to 1. So, that is equal to 2 and that is the dimension of R square.

So, the picture I have made this one. So, from here you can see that in the case my R A is 1 and N A T is 1, so it is 2 and this because this will be same, so it is 1 plus 2 is 3. So, that is this one. So, now, from here after doing this one, we can make some conclusion.

So, now, some conclusion is there that, I will discuss another term zero null space. So, if null space of a matrix A that contain only one element that is 0 element; then it is called, then it is called zero null space. And now from here, you can check from here in the from the this one; now in this case my null space is not the 0 element, containing not the 0 element, not even here, not even here.

But what will happen if the rank of the matrix becomes 2? Now, if the rank of the matrix become 2, so in that case you will see that it becomes 2; then the, we have to make the changes in the nullity of this one to satisfy this condition. So, we will see this one in the next lecture that, how we are going to take this zero null space, whenever we are dealing with the subspace, null space of the matrix A.

So, let me stop here. So, today we have started with the other two subspaces, that is the null space of the matrix A and its transpose, that is A transpose. And then we have taken one example and show that, how we can find the null space of A and null space of A transpose. So, in the next lecture, we will continue with this one. So, thanks for watching.

Thanks very much.