Matrix Computation and its applications

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Lecture - 17
Row equivalent matrices

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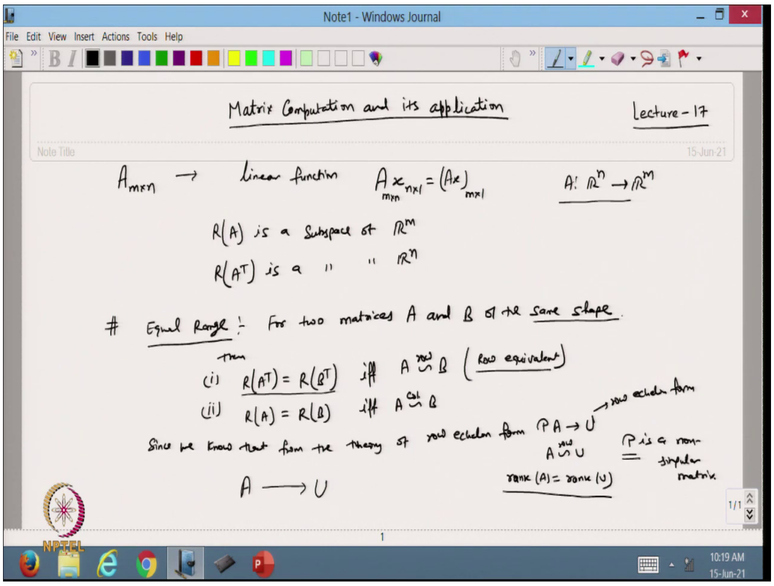

Hello viewers. Welcome back to the course on Matrix Computation and its Application. So, in the previous lecture, we have introduced the started with the 4 subspaces connected with the given matrix of order m cross n, and we have discussed about the range space of the matrix and its transpose. So, in this lecture we will continue with that one.

So, in the previous lecture, we have discussed that suppose we have a matrix A, that is of order m cross n, and then from here we can define a linear function that is A x that is m cross n and this is if vector coming from n cross 1. And suppose, this is after getting this value it is becoming m cross 1.

So, it is a map from; so, A we can define a map, a linear map from R n to R m because my function x is belongs to R m. And then, we have discussed that the range space of the matrix

A, it is a subspace of R m and the range space of A transpose is a subspace of R n. So, this one we have discussed.

And now, so after discussing these two things, we want to go further and we want to find out what will happen if we have a equal range. So, in this case, suppose we have for two matrices A and B of the same shape.

Same shape means that should be of the same order. Then, the first is that the range space of A transpose is equal to the range space of B transpose, and this is possible if and only if A is row equivalent to B. So, that means, row equivalent. It means by taking the elementary row operation we should be able start from A and we should be able to get the B, then it is called the row equivalent matrices.

Second one is, then the range space of A is equal to range space of B if and only if A is column equivalent to B. So, it is the column equivalent means, in this case we are taking only the column operation and map transforming from A to B by the elementary column operation. So, this is the column equivalent.

So, this is because; why we are doing this one? Because in this case suppose we have two matrices and one matrices row equivalent to the another matrices, then they are going to find, going to produce the same range space of A transpose is equal to range space of B transpose and range space of A will be the range space of B. So, these things you have to keep in mind.

And now we know that, since we know that if I from the theory of row echelon form, we are always transferring the matrix A into some matrix U by the transformation P, so where this is a row echelon form. So, then we say that A is row equivalent to the matrix U and where P is a non-singular matrix. So, P is basically made up of all the elementary operations that we multiply pre-multiply the matrix A. So, this will be more clear in the example.

So, from here we know that A and U is row equivalent and from there also we can say that the rank of A is equal to the rank of U. So, that we already know. So, now if you see very sincerely from here that when we start starting from the A, then we always take the linear combination, some linear combination of the rows of the matrix A, and then after doing this linear combination we get U.

So, from here one thing we can see that U, the rows of U are basically made from the linear combination, some linear combination of the rows of A. So, that is why we are able to discuss these things for range space of A transform, A transpose is equal to range space of B transpose.

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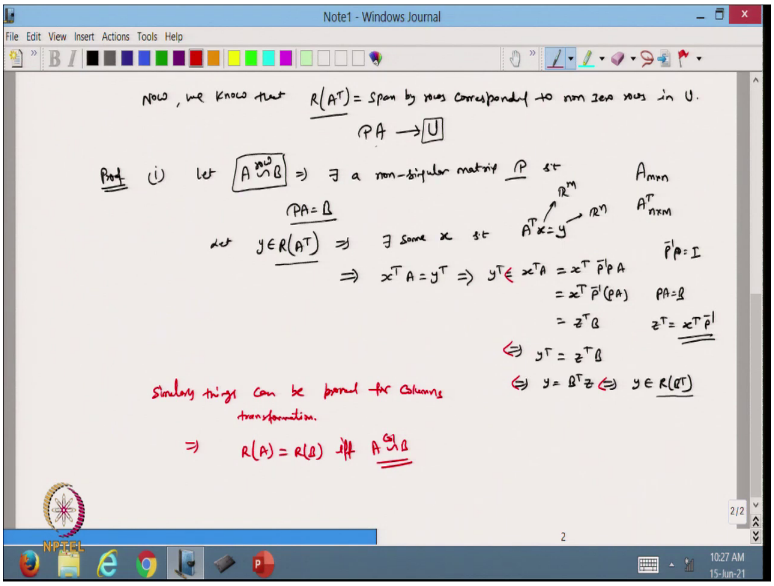

Now, we know that the range space of A transpose is span by non-zero rows, span by the non-zero rows by span by rows corresponding to non-zero rows in U. It means we start with A, transform into U by some elementary row operation P.

And in this U whenever we have a U, and in this whatever the non-zero rows are there, so the corresponding non-zero rows I will find out the rows in A. So, that will be; so, these rows will span the range set of A transpose, ok. So, these things is there. So, let us prove this one, the first one. So, we only prove the first one, the second one will follow the same.

So, proof for first one. So, let A is row equivalent to B, implies there exists a non-singular matrix P, such that PA is equal to B. So, this is given to me, and from there I can find PA is equal to B.

Now, let I take some vector y belongs to the range space of A T, so which implies that there exists some x, such that A transpose x is equal to y, because y is belonging to the range space of this one. For some x, I will get this value. You know that this A is m cross n matrix, then A transpose will be n cross m. So, this x will coming from R m and this y is coming from R n.

Now, so this one is there. So, from here you can find that I just take the transpose, so I can write x transpose A is equal to y transpose. Just taking the transpose of this one. And now from here, I can write from here that y transpose is equal to x transpose A and A is already there.

So, I can write this as a x transpose and A instead of A, I can write P inverse PA because P is a non-singular matrix. So, P inverse P will be identity matrix. And from there I will write x transpose P inverse PA and PA is equal to B from the. So, from here I can write one vector z transpose and then it desire by B, where z transpose I am taking x transpose P inverse. So, it is the new vector we are introducing z transform.

So, this is vector that is coming from x transpose P inverse and from here you will see that; so, from here you can say that my y T is becoming z T B or I can say that my y become B transposed z. So, that show that y belongs to the range space of P transpose, ok. So, from here I started with the vector from the range space of A transform and I showed that the same vector belongs to the range space of B transform.

So, the same way I can go. I can take from y, and then moving the same way in the opposite direction, and I can, from here I can go back the same way and from there I can show that the y will belongs to the range space of a transform. So, from here that first is, ok that A is row equivalent to B, then we have showed that the range space of R A T is equal to the range space of B T that will be same.

Now, so this is converse is also true. So, similar things can be proved for the columns transformation and that is we can prove for the second one. So, now, based on this one we are able to say that if the two matrices are row equivalent then the range space of A transpose is equal to the range space of B transpose. And so, similarly things can be proved for a column transformation which implies that the range space of A will be same as the range space of B, if and only if A is column equivalent to B. So, this is the way we can do.

Just we can take the transformation also of A transpose and B transpose, and then we can do the row operation, the similar things will come. So, this is the proof we can have. So, after this one, there is a one more.

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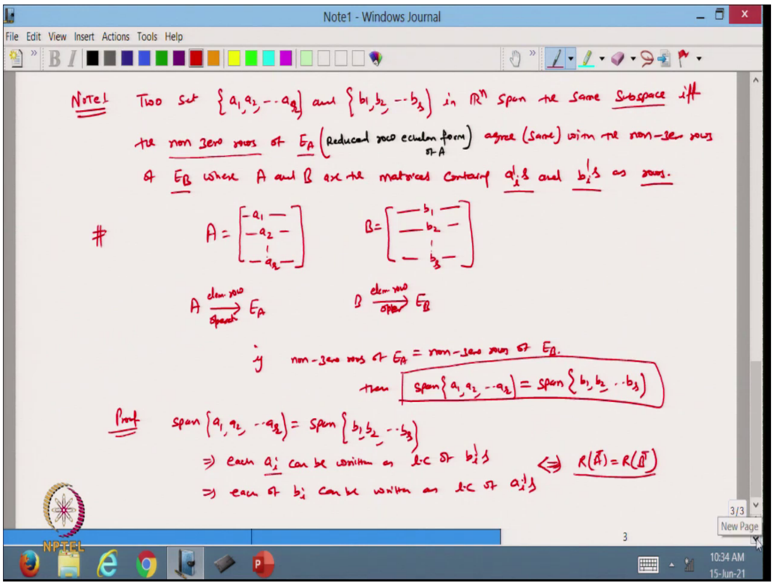

So, one note that, two sets suppose I take two sets a 1, a 2, a n or maybe I should take R to set this and suppose I take the another b 1, b 2, b s that belongs to R n. So, I am taking the R number of vectors here and s number of vectors here, ok; that is lying in the n R, that span the same subspace if and only if the non-zeros rows of E A. So, E A I am writing here. So, what is the E A? It is reduced row, reduced row echelon form of A.

So, if the non-zero rows of E A agree; agree means same with the non-zero rows of E B. So, that is the reduce row echelon form of B, where A and B are the matrices containing a i's and b i's as rows. So, span the same subspace if and only if non-zero rows of these agree with the non-zero rows of these when A and B are the matrices containing a i's and b i's as rows.

So, it means, what is the meaning of this? It means that I will take a matrix A and this will be a 1, a 2, a r. So, I will take this as a row. And another matrix I will take B and that will be b 1, b 2, b s. So, it contains s number of rows. Then, I will transform A into E A by elementary row operations.

B, I will transform into E B, so that is also row equivalent elementary, row operations. Then, it says that if E A is equal to E B, if you know non-zero rows, so we talking about only non-zero rows, so if non-zero rows of E A is equal to non-zero rows of E B, then, then they will span the; then the span of a 1, a 2 up to a r will be same as span of b 1, b 2 up to b s. So, that is our.

So, these things is will take one example. But here we can just do the proof. So, proof is very easy because just we have to take that. So, the span of a 1, a 2, a r is equal to span of b 1, b 2, b s. Now, suppose it is we; suppose we have a span of this is equal to span of this, so definitely it implies that each a i can be written as linear combination of b i's because the ranges span of this one is basically contains all a 1, a 2, a r and that also contain b 1, b 2, b r.

So, each of a i's can be written as a linear combination of b i's. Similarly, each of b i can be written as linear combination of a i's because that we have taken, and from there I can say that in the range space a of A transpose will definitely equal to the B transpose, and if it is there then this will also converse (Refer Time: 20:20), ok. So, these things we can check easily. So, let us to take one example and then the things will be more clear to you.

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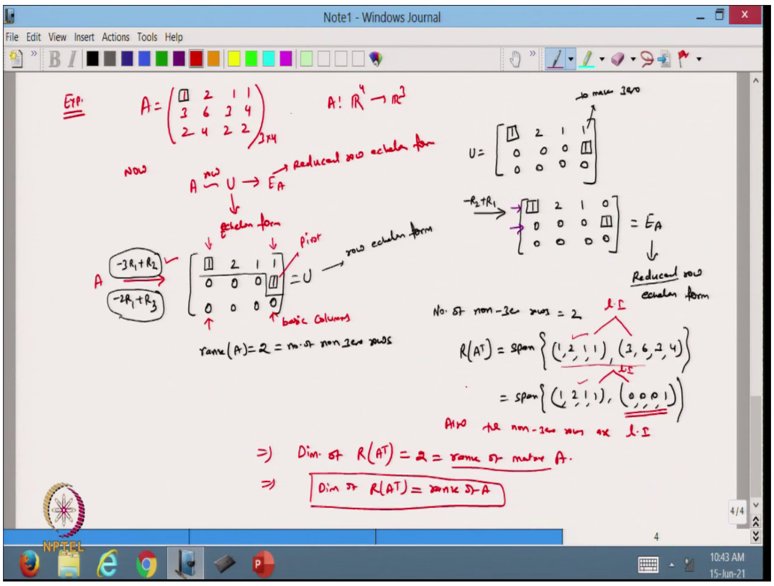

Now, suppose I take the example. So, let us I take the matrix A, suppose I take first this matrix 1, 2, 1, 1; 3, 6, 3, 4; 2, 4, 2, 2; now, this is a 3 cross 4 matrix. So, A is a transformation from R 4 to R 3, and is a linear transformation or linear function.

Then, now we will convert the A into the raw echelon form. So, first I will convert this into the row echelon form and then from this one I will convert to E A, that is reduced row echelon form and this is a row echelon form, ok.

So, from here, now this is there. So, from the A, what I do, I take minus 3 R 1 and add to R 2 and I take minus 2 R 1 and add to R 3, just to make this element 0 because this is the pivot element. Now, if you see from here I am taking linear combination of R 1 and R 2, here I am taking the linear combination of R 1 and R 3, this one, and I get this matrix from here that is 1, 2, 1, 1 and this is 0, so because I am multiplying by minus 3. So, minus 3 plus 3 0 minus 3 into, so it will be again 0. This will be also 0 and this will be minus 3 plus 4, it will be 1.

Now, minus 2 to 0, so it will be 0, 0, 0, ok. So, in this case now you can see that this is my pivot element and next pivot element is this one. So, from here I can say that, so you can see from here that I get a step like matrix and this is my U. So, that is a row echelon form. And from here I can also say that the rank of A is equal to 2 and that is equal to number of non-zero rows.

Now, from U, so this is my U; 1, 2, 1, 1, and where in the columns where this pivot elements occurring, so it is pivots, so this is called, this and this is called basic columns where the pivot element is appearing. So, in this row echelon form. So, I get this one, 0, 0, 0, 1; 0, 0, 0 0. Now, this is my pivot element, this and this. So, I have to make this element 0, to make 0.

So, what I do is that I will just minus R 2 plus R 1. So, in this case also I am taking the linear combination of the second row and the first row, and I will get this matrix. So, it is 0 nothing will change, 1, 2, 1, I will multiply minus 1 and adding here, so it will be 0. And this is 0, 0, 0, 1; 0, 0, 0, 0. So, here this is the pivot element and this is the pivot element, so all other elements in that pivotal column are 0, here also it is 0. So, this become E A. And this E A is called reduced row echelon form.

Why it is called reduce? Because the number of elements are reduced, non-zero elements are reduced in the given echelon form, so that is why it is called the reduce row echelon form. So, this is my E A. So, now, from here I can say that the number of non-zero rows are 2, this is a non-zero row and this is the non-zero row. So, this is non-zero and this is non-zero, ok.

So, from here I can say that the range space of A transpose, if I take the range space, so it is span by the non-zero rows corresponding to non-zero row, so it is 1, 2, 1, 1 one vector this one, first row and the second row 3, 6, 3, 4, this one. Now, it is the span of this one, and from here I can also say that this also span by 1, 2, 1, 1 and 0, 0, 0 because if you see from here then you will come to know that this vector, the last vector is coming by taking the linear combination of these two vector.

First vector is same, this and this is same, so we take the linear combination of these two vector whatever the I have taken; I have taken this linear combination. Then, I got the second one, so second one is here. So, this linear combination I have taken and I got this one. So, basically the linear combination of these also contain this vector. And also from here, also the rows the non-zero rows are linearly independent.

So, basically, these and these two vectors are l I, and also this and these two vector are also l I and this vector is a linear combination of these two vector. So, from here very easily we can check that the span of this vector is same as the span of this vector.

And from here I can also say that the dimension of range space of A transpose, so that is the dimension of the range space of A transpose, so that is equal to the number of known, this linearly independent vector appeared in this one, so that is equal to 2 and that is equal to the rank of matrix A. So, that is equal to the rank of the matrix A. So, from here we can say that the dimension of R A transpose is equal to the rank of A, this one, ok. So, based on this one, we are able to show.

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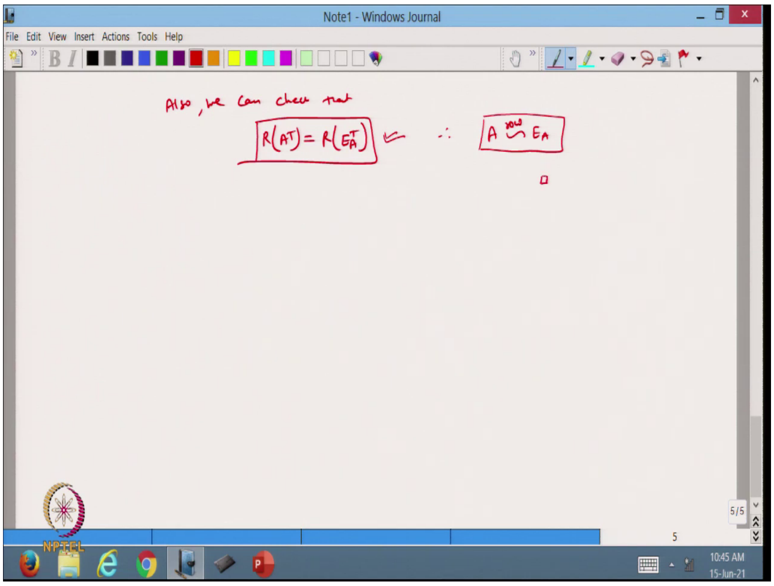

And this can also be written as also, we can check that range space of A transpose is same as the range space of E A transpose because the matrix A is row equivalent to E A because E A is coming by taking the elementary row operation. So, that is coming from here. And just now we have done one theorem in the starting one that first one, so it shows that if A is row equivalent to B, then there the range span by the rows is same. So, these things we can implement here from this one.

And that we have also seen that how the vectors of E A are coming by taking the linear combination of the column, the rows of E A are coming from the taking the linear combination of the rows of A. And we also check that the rank is same and their dimension is 2. So, that also we can see from here, ok. So, we stop here today.

So, in the today lecture, we have discussed that if the two matrices A and B, they are row equivalent then the range space of A transpose is equal to the range space of B transpose. And if they are column equivalent then the range space of A will be same as the range space of B, and then we have discussed one example based on that. So, in the next lecture, we will continue with this one. So, thanks for watching.

Thanks very much.