Matrix Computation and its applications
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Lecture - 16
Range space of a matrix and row reduced echelon form

Hello viewers. So, welcome back to the course on Matrix Computation and its applications. So, in the previous lecture we have just started with the 4 subspaces are related to the a matrices. So, today we are continue with that one. So, this is lecture number 16.

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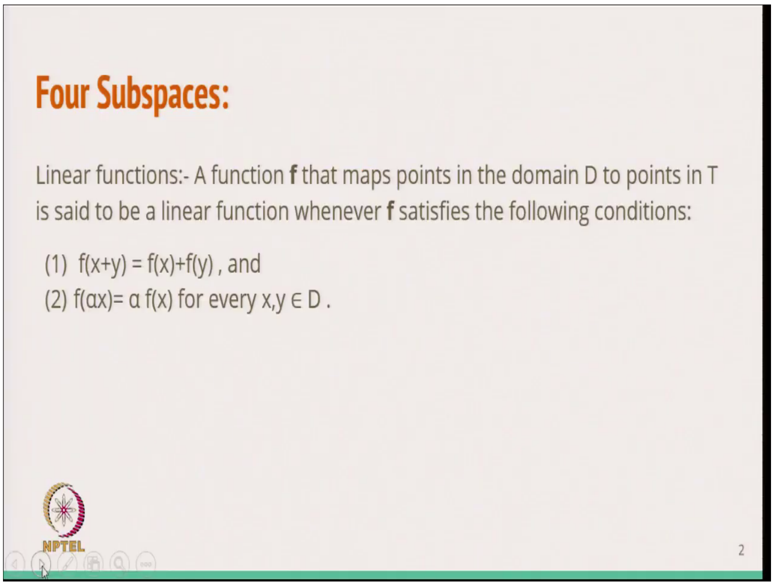

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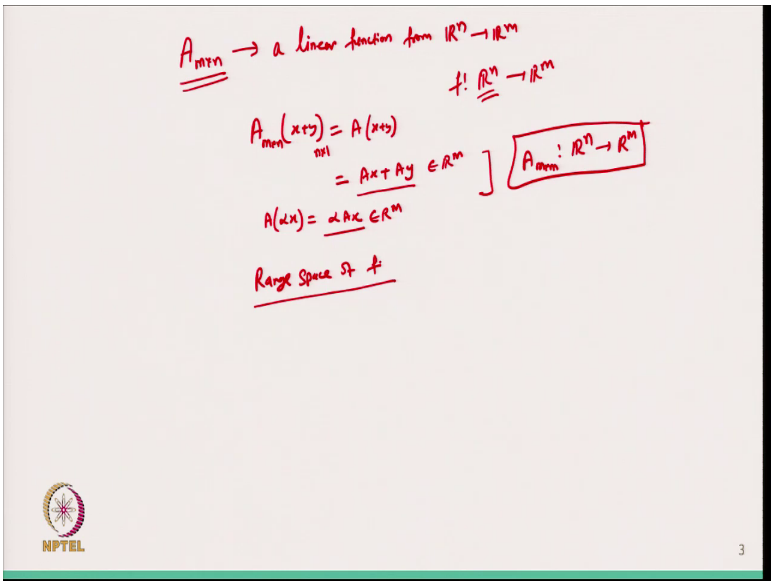

So, in the previous lecture we have shown that the matrix, if I define a matrix m cross n, so, it can be represented as a linear function from R n to R m; that we have a map f from R n to R m and that can be represented by the matrix A. Because in this case we have shown that if I take the matrix A that is m cross n and then I apply on the element x plus y then and this x plus y is coming from this n cross 1. So, it means it is a column vector coming from this domain R n.

So, we can write this as A x plus y and this can be written as A x plus A y, ok. So, and this will belongs to R m. Similarly, A alpha x can be written as alpha A x and that is belongs to R m.

So, because these two properties are satisfying the addition and the scalar multiplication then from here we can say that A that is matrix m cross n is a linear function. And when we have define the linear function then we have started with the range space of function f. Now, we have discuss the range space of f.

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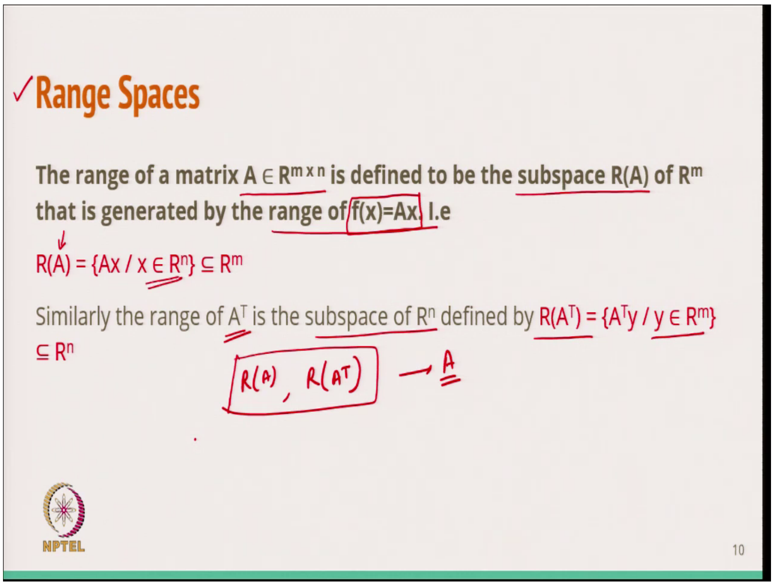

So, now we will discuss the range space of the given matrix. So, what is the range space? So, the range of the matrix A, now we are writing the A belongs to R m cross n. So, we can call it this matrix belonging to the space R m cross n, where m is the number of rows and n is the number of column. It defined to the subspace R A of R m that is generated by the range of f x is equal to A x.

So, here we are taking the linear function f x is equal to A x. So, that is the range space of A. So now, we instead of the linear function f we are started with taking the matrix A. So, the range space of A is equal to A x, all A x such that x belongs to R n and that is also belongs to the subset of R m.

So, once we have defined this one then we can take the transpose of the matrix A and we can show define the subspace of R n. So, that is defined by range space of A transpose is equal to A transpose y, where y belongs to R m and that will belongs to the R n. So now, we have a two range spaces, R A and range space of A transpose. So, this one are the two subspaces that is connected with the matrix A. So, these are the two subspaces we can define.

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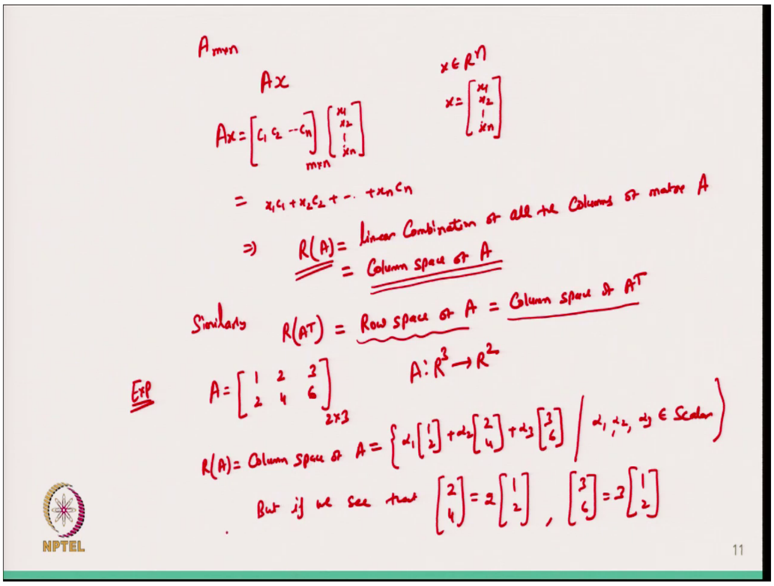

Now, if you see from here, I take a matrix A that is m cross n. Now, A suppose I take A x and x is coming from R n. So, let my x is I can write as x 1, x 2, x n. So, I can write my A x is equal to and A is have the columns. So, it has column c 1, c 2, c n m cross n and that is multiply by x 1, x 2, x n. And if I multiply I can write this as a x 1 c 1 plus x 2 c 2 x n c n. So, from here; and this is the linear combination of all the columns of the matrix A, and we know that this is a range space basically.

So, from here I can say that that the range space of A is a column space I can say that R range space of A is a equal to the linear combination of all the columns of matrix A. So, from here I can call this as the column space of matrix A. So, the range space of A is equal to the column space of A. It means the range space of A is span by the column space by the columns of the matrix A.

Similarly, if I want to define the range space of A transpose; so, when you take the A transpose these columns will become the row. So, from here I can say that it is equal to the row space of A, because I can also write as a column space of A transpose. Because whenever, whatever the column will be here I just take the column space of A transpose that will be the range space of A transpose and it will similar to the row space of A.

So, I can have a column space and the row space. So, let us take one example. Suppose I take example. I take a matrix A maybe I just take 1 2 3 the simple one 2 4 6 just I take this matrix and this is 2 cross 3. So, basically I can say that the A is a linear map from R 3 to R 2. So, this is my linear map.

Now, I want to discuss about the range space and the range space of A and range space of A transpose. Now, the range space of A that is equal to column space of matrix A that we have just shown. And this can be written as I can write here maybe alpha 1 1 2 alpha 2 2 4 alpha 3 3 6. So, I am taking all the linear combination where alpha 1, alpha 2, alpha 3 belongs to the scalar and scalar in this case is a real numbers. So, coming from the field.

So, I just taken the column space of A means the linear combination of all the column, so that will become the range space of A. But, if we see that the vector 2 4 can be written as 2 times the first vector first column and also 3 6 can be written as 3 times the first vector 1 2.

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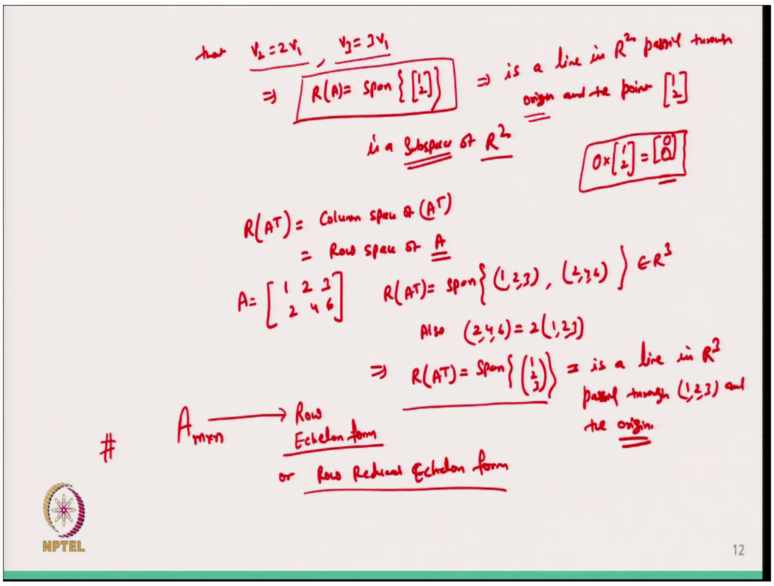

So, now from here we can see that that the second vector; so, I can just call it V 1 V 2 V 3. So, I can say that since V 2 is 2 times V 1 and V 3 is 3 times V 1. So, from here we can say that the range space of A can be span by the vector 1 2 itself, because all other vectors are linear combination of the this one scalar multiplier. So, we can remove those vector that we have already done in the previous theorems that there is no need to keep this vectors which are the linearly. So, it become basically linearly dependent vectors.

So, range space of A is a span of this one and this is even a first column. So, that become the range space of A. So, from here you can see that the range space of A is a line of line in R 2 passing through origin and the point 1 2. Because we already know that this range space of A is a subspace of R 2, because it is a linear map from R 3 to R 2.

So, it is a subspace of R 2 and each subspace must contain the 0 element the additive identity of that vector space. So, that is a 0 in this case. So, it should definitely will come from the origin should pass through the origin and definitely it is going from the point 1 2, because the span of a single element is just the scalar multiple of that vector.

So, if somebody says that how the origin is passing, you can just take element 0 into any other element maybe I can take 1 2. So, it will be 0 element. That we have already discussed in the some facts or the property of the vector space. So, always the 0 0 will be always there in that vector space in that subspace. So, this is the range space of A that is span by 1 2. Now, I want to see what is the range space of A transpose.

So, range space of A transpose is also the column space of A transpose and also equal to row space of A. Now, we have A that is 1 2 3 and this is equal 2 4 6. Now, I can say that range space of A transpose is a span of the vector 1, 2, 3 and 2, 4, 6. So, in this case we are just taking the row vector, but that vectors belongs to, so this vector belongs to R 3, ok.

So, these vectors we can write like this one, span of these vectors. First element is 1, 2, 3 maybe I can just remove transpose. It is understood that this is vector belongs to R 3 having the 3 component 1, 2, 3 and 2, 4, 6. Now, also the vector 2, 4, 6 is again 2 times 1, 2, 3. So, from here I can say that the range space of A transpose is a span just a single element 1, 2, 3. And this is also is a line in R 3 passing through the point 1, 2, 3 and the origin.

So, it is another subspace of R 3. So, this way we can define the range space of A and the range space of A transpose. Now, so now, we suppose I have a matrix A that is of order m cross n. Then if we remember then we can transform this matrix into the matrix that is called row echelon form or row reduced echelon form. So, this echelon forms are very important sometimes to show the range space their dimensions and everything.

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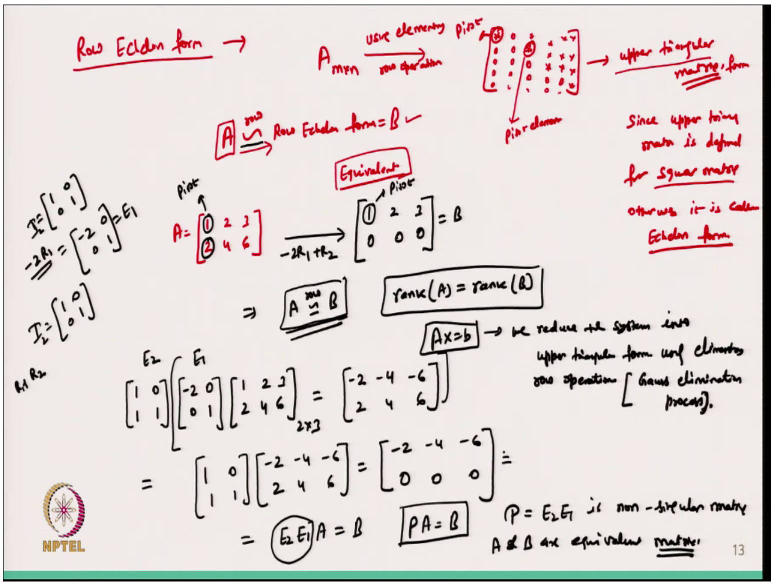

So, let us discuss this thing that what do you mean by the row echelon form. So, the row echelon form means that we have a matrix A that is of order m cross n and suppose using elementary row operations we transform this matrix into this form. Suppose we have a non-zero entry here, then all other elements are 0 then suppose I get 0 0 0 0 here again then maybe I have non-zero entry here.

So, all the elements below this 0 and suppose I get non-zero entry here and non-zero entry here, so all the elements here it is 0 and this one is some non-zero elements. So, it is type A upper triangular matrix. So, this is type upper triangular matrix, where all the. So, this is where it is non-zero, it is the pivot element. So, pivot element, this one is the pivot element.

So, it will reduce the matrix in the upper triangular form, because this words comes upper triangular when the matrix is; now since upper triangular matrix is defined for square matrix. So, definitely that is why I am saying that upper triangular form. So, if it is square matrix then it will become the upper triangular, otherwise it is called echelon form. And we are taking only elementary row operation. So, it is becoming the row echelon form.

So, now, we can say that from here that a matrix A is equivalent to its row echelon form. It means by the using the elementary row operation we can reduce the matrix A into the matrix row echelon form or maybe I call it B. So, then the matrix A and B are said to be equivalent.

So, equivalent means I start with the A applying the elementary row operation and reduce the matrix into the echelon form that is B. So, then the matrix A and B are said to be row equivalent or equivalent or maybe I can say that is the row equivalent. For example, suppose I take the matrix A the similar matrix I just take 1 2 3 2 4 6 the same matrix I have taken in the previous example.

Then what I do is that I have to reduce this one into the echelon form. It means, in this case this is my first non-zero entry. So, this is pivot. Now, I have to reduce this into 0. So, this one we can do. So, what I do is that I apply minus 2 times R 1 and I add to R 2. So, this is my elementary row operation I am applying.

So, in this case I will get 1 2 3 and this become 0 0 0. So now, this becomes a echelon form and this is my non-zero entry coming here. So, this is only pivot element. So, from here I suppose this become B. So, I can say that A is row equivalent to B equal sign also there. Row equivalent means; why they equivalent because every matrix is a equivalent to itself, no problem. So, row equivalent means I have taken only the elementary row operation and got this matrix. So, I will call it that A is row equivalent to B.

And the important property if you remember the basics of matrix analysis or matrix theory then from here we know that the rank of A is equal to rank of B. So, generally we apply this echelon form row echelon form whenever we have a system A x is equal to b and we want to solve the system.

Then what we do? To solve this one we reduce the system into upper triangular form using elementary row operation. So, and we also know that this method is called Gauss elimination process ok.

So, this can be done because if you see from here what I am doing here that I am first multiplying by minus 2 to R one and then adding to R 2. So, this one I can just write like this also. So, my elementary this is my matrix A 1 2 3, 2 4 6. So, it is 2 cross 3. Now, I take a elementary matrix E 1 that is 1 0, 0 1 and I just multiplied by minus 2 E 1, because I am applying here minus 2 R 1. So, I am multiplying.

So, the same operation I am applying here. So, I am applying minus 2 R 1 in this case or maybe I should just write it I 2 and applying this one. So, that become minus 2 0, 0 1. So, this become, E 1 ok. And I will pre multiply by this, so minus 2 0 0 1. So, if you see this one and you multiply here, so, 1 minus 2, 2 0. So, it will be minus 2. Similarly, it will be minus 4 and it will be minus 6 and then applying here.

So, it will be same 2 4 6. So, this is what we have done. Now, I take again I 2 because it 2 by 2 matrix 1 0 0 1 and then what we have doing we are adding to R 2, ok. So, in this case I will applying just adding here and adding to R 2 R 1 plus R 2. So, it will be 1 0 and adding here 1 and 1. So, this will be come.

Now, after getting this value I am doing this one. So, from here you can just check it would be 1 0, 1 1 and applying on the matrix minus 2 minus 4 minus 6, 2 4 6. And if you see from here it is just minus 2 minus 4 minus 6; so, minus 2 minus 4 minus 6. And then applying on the second row, so it will minus 2 plus 2; so, 0 0 0 ok. So, this way you are getting the from here and then maybe I can just take remove minus 1 and you will get the matrix like this one.

So, in this case you can say that I have taken this E 1 first and then E 2. So now, from here and I got this matrix. So, what are the operation you are applying here? I have applied on the elementary matrix and pre multiply this one. So, from here you can say that I will get E 1 E 2 applying on A and I get this matrix B. So, this matrix whatever we are doing the pre multiply, because we are doing the pre multiply because we are dealing only with the row vectors the row operations.

So, this become a matrix PA is equal to the matrix B, where P is just a matrix made up of E 1 and E 2 and is always nonsingular; nonsingular matrix. And from here you can say that P is the transformation matrix that is a nonsingular matrix and I got this one. So, from here this is the way we can find out the equivalent matrix and also A and B are equivalent.

Similarly, so in this case you can see that we are going to get the matrix we are going to get the matrix in this form. Now, what will happen if I want to reduce this matrix with the entries only 1 at the pivots and all other element 0? So, that we comes with the help of.

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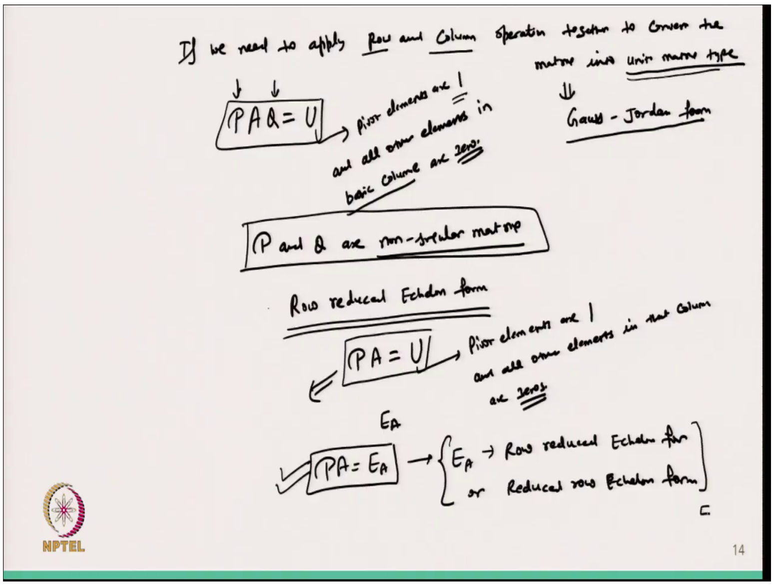

So, in the sometimes if we need to apply row and column operations together to convert the matrix into unit matrix type. It means, the elements if the pivot element should be 1 and all the elements in that column should be 0. So, that is called it unit matrix type or we also know it the Gauss-Jordan form.

So, in that case we need to take elementary row operation as well as column operation and what are the column operation we are taking that we have to multiply the that is call the post multiply. So, in that case we have a matrix A and suppose I apply elementary row operation and then I apply elementary column operations and then we reduce this into the matrix of unit type matrix in which.

So, this is a unit matrix type unit type matrix in which pivot elements are 1 and all other elements in column in, so that is called basic column; basic column means, which has the pivot element and all other element in the basic column R 0. So, that is call the unit type matrix.

It is definitely the upper triangular matrix echelon form definitely, but only the thing is that the in the echelon forms we have put two more condition that the pivot element should be 1 and all the basic column should be 0. So, then it become this one. So, here we are applying by P and Q, so where P and Q are nonsingular matrix. So, this is a nonsingular matrix.

So, for example, in the previous one we after applying this row echelon form may be I just multiply by minus 1 and I will get rid of this. I will get this matrix B. So, here the pivot element is coming 1. So, this is ok. So, this is I just I can call it that in this case it is also of; there is another term we want to discuss that is called row reduced echelon form.

So, this U, I can say that in the form of a row reduced echelon form. Only thing is that so, in this case not like this one. So, in the row reduced echelon form is that we apply row operation on A and we get the matrix U. So, U is a matrix with the same property that the pivot elements are 1 and all other elements in the column in that column; in the column which has the pivot element are 0.

So, if that is a then whatever the matrix we get is called the row reduced echelon form and that sometime is also represented as E A. So, in this case I can write PA is equal to can be written as E A. So, this is called where E A is called row reduced echelon form or sometime also called reduced row echelon form. Actually these things we will discuss later on when we will discuss about the change in basis. So, at that time it will be used heavily.

So, just we just keep in mind that either we are going to use this matrix where we are reducing the matrix into the row reduced echelon form or in the echelon form. Generally, we go only for row operation elementary row operation. So, that is why we are only using these two type of matrices and we are not going to deal with the column elementary matrices. So, let me stop here.

So, today we have discussed about the that the how the linear function can be represented by the matrix. And then we have discuss the range space of the matrix A and the range space of the matrix A transpose. So, in the coming lectures we are going to discuss about the other two subspaces related to the matrix. So, I hope you have enjoyed this one, thanks for watching.

Thanks very much.