Matrix Computation and its applications

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Lecture - 15
Linear functions

Hello viewers, welcome back to the course on Matrix Computation and its application. So, today we are going to start with the another topic that is the four subspaces, that are connected with the linear algebra. So, let us discuss that one.

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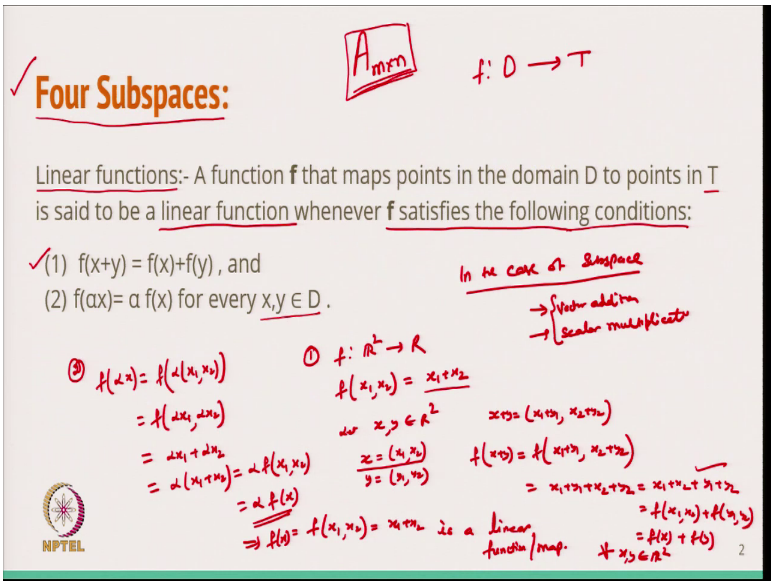

So, the today's topic is about the Four Subspaces. So, today we are going to deal with the matrix, A matrix of order m cross n. So, how we can find out the subspaces four subspaces are related to this matrix. So, before that we just want to discuss one function that is called the linear functions.

So, a function f that maps position in the domain D to the points in some T is there. So, I am defining the map f from some domain D to the set T if this is going from D to real number then I can say T is a set of real numbers or if it is a complex number or whatever it is. So, we are mapping the function f from the domain D to the co domain that is T.

So, a function f that maps position in the domain D to the point T is said to be linear function whenever f satisfy the following conditions. The first one is that if I take the function f of the addition of two numbers from the domain x plus y then it should be the sum of their images. So, it should be equal to f x plus f y.

And, if I take the scalar multiple alpha times x and then I taking the map of that one. So, that should be equal to the scalar multiple of that the image of that under the function f for every x and y belongs to D. So, this is type of functions are called linear functions. So, this type of functions you if you see from here then looking at this type of function you can also remember the definition of the subspaces.

Because in the case of subspaces; in the case of subspaces we also have to satisfy two condition. What is the vector addition and another is the scalar multiplication these two. And here also, we if you see it is also type of vector addition we are doing. So, instead of vectors we are dealing with x and y that is coming from the domain and this is also the scalar multiple. So, we can have some relations between the linear function and the subspaces.

So, for example, I just take one example that how the linear functions will look like. I can define the function f may be from R square to R or maybe I can define the function f x 1, x 2 is equal to x 1 plus x 2. So, in this case I just want to see whether it is a linear function or not then if you take f. So, this is a I am taking the function here.

So, let we take x and y belongs to R square, then x will be written like this one y is equal to y 1, y 2. Now, I can take the linear combination x plus y. So, that can be written as x 1 plus y 1, x 2 plus y 2. Now, I take the function f of x plus y. So, it will be equal to f of x 1 plus y 1, x 2 plus y 2 and this is equal to given here.

So, from here I can write as a x 1 plus y 1 plus x 2 plus y 2 and this can be written as x 1 plus x 2 because it coming from the real number and y 1 plus y 2 and this is also equal to x 1, x 2 plus f y 1, y 2 and that is equal to f of x plus f of y. So, this is true for all x y. So, true for all x y belongs to the domain R square. So, this is now from here we can check that this is satisfied.

So, this is the first one we are doing. Then the second one I can take f of alpha x. So, it can be written as f of alpha x and x I am taking here this. So, it can be written as a alpha x 1, x 2. So, it can be alpha x 1 alpha x 2 and this is equal to according to the transformation, I am taking alpha x 1 plus alpha x 2 and from here I can say that alpha is x 1 plus x 2 and this is I can write as a alpha f of x 1, x 2 and that is equal to alpha f of x. So, second property is also satisfied.

So, from here I can say that the mapping; whatever the mapping we have defined from x 1, x 2 that is equal to. So, this is I can write as x 1, x 2, x 1 plus x 2 is a linear function or map we also call it map. So, this is a linear functions or a map we can define.

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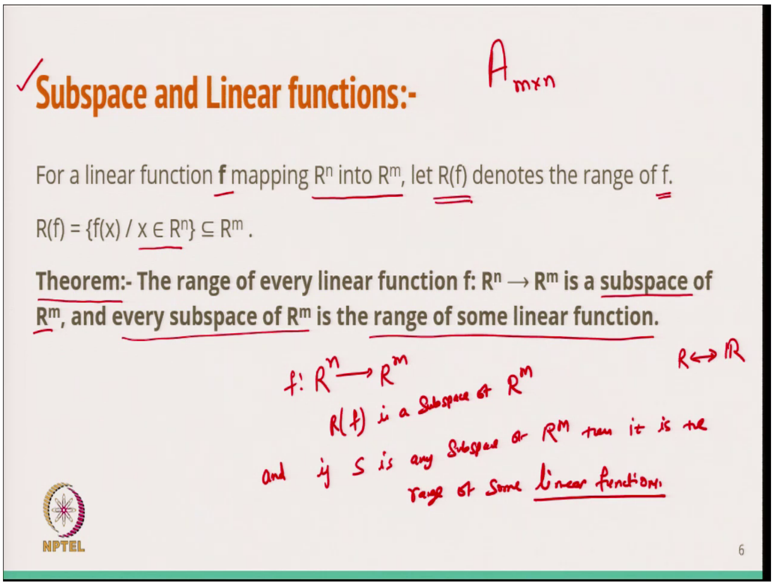

Now, so from here now we define the definition subspaces and linear functions. So, now we want to find the relation between these two. So, for a linear function f mapping from R n to R m. So, now, we are talking about the matrix A, that is of type m cross n. So, let R f denote the range of the function f. So, R f basically is set of all the images f x such that x belongs to R n. And we know that, this will be the subset of R raised to power R m because that is the functions coming to this one.

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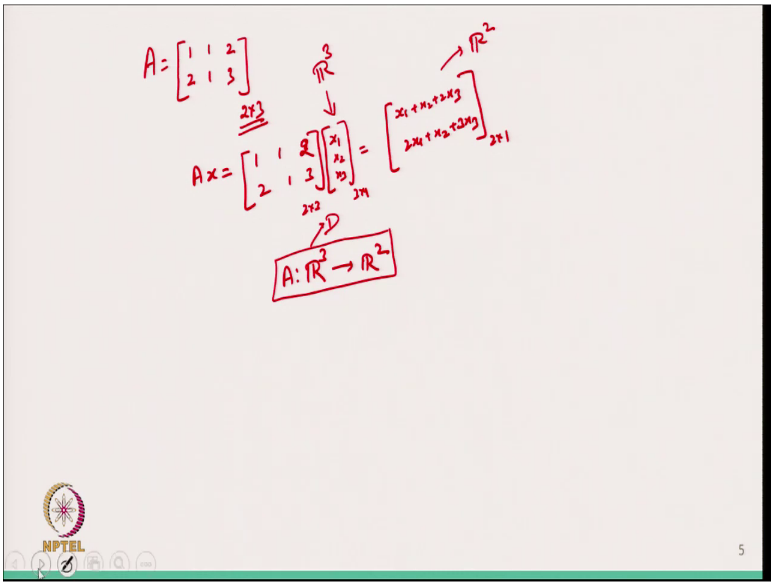

So, for example, I take a matrix A suppose I take a matrix like this one 1 1 2 and 2 1 3. So, suppose I have taken this matrix this is 2 cross 3 matrix. Now, if I take the transformation using this matrix then definitely I have to I want to calculate what is the A x. So, A is here 1 1 2 and 2 1 3. And I will taking a vector applying on this one. So, it is 2 cross 3. So, it should be 3 cross 1.

So, suppose I am applying it on the x 1 x 2 x 3. So, that is 3 cross 1. So, from here if you see this one I will get here x 1 plus x 2 plus 2 x 3. 2 x 1 plus x 2 plus 3 x 3. So, this is a vector I am getting. So, and this vector is 2 cross 1. So, this is a matrix from of order 2 cross 3.

And I am taking the vector x coming from R 3 and then the image is going to R square. So, I can say that this A represent a transformation that is from R 3 to R 2. So, R 3 mean the triplet I am taking and. So, that is coming from R 3 to R 2. So, I am taking the vector from R 3; that is coming from the domain. So, this is my domain here I can say there is a domain and then the image is going to the R.

So, this is the mapping we are basically applying here. So, this is what we have written. So, we have defined that for a linear function f mapping R n to R m, let range f denote the range of R f the range of the function f that is given by this one. So, then we have the theorem that the range of every linear function f from R n to R m is a subspace of R m and every subspace of R m is the range of the some linear function.

It means that if I take a map f from R n to R m; actually sometime we also represent R like this one. So, this is same equivalent. So, we have a linear function from R n to R m. So, it says that the range of f that will be is a subspace of R m and in this case that if s is any subspace of R m then it is the; it is the range of some linear functions.

So, this is the definition here we have defined the theorem. And now, we want to prove this theorem. So, let us start doing this one.

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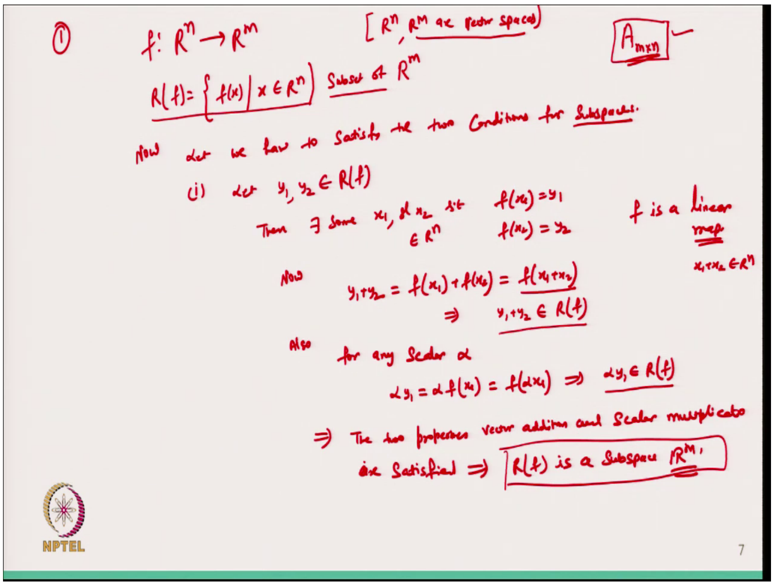

So, the first one. So, the first one says that I we have a map from R n to R m. So, this is same as I have showed the matrix of the order m cross n. Because whenever we deal with the vector space is having the dimension more than 1 that is like a R n to R m and we want to if we want to show the linear function then that come across the matrix.

So, that we will discuss in the future, but just now I am defining a map that can be also written in the form of the matrix A m cross n. So, I am writing this is the function f. Now, so R f the set of all the images such that x belongs to R n ok and definitely we know that this is subset of R m because if I take any element from the domain applying the function f then it will go to here.

So, it is a subset of R m now let we have. So, now we want to show that this R f is a subspace of R m. So, for this one we need to satisfied two condition vector addition and scalar multiplication.

So, now we have to satisfy the two conditions for subspaces. So, the first one is that vector addition. So, let we take y 1 and y 2 some belongs to R m. So, I am taking two elements from the range set R m not from R m I just take from R f. Then there exist some say x 1 and x 2 such that f of x 1 is equal to y 1 and f of x 2 is equal to y 2.

Because, I am taking two from two elements that is distinct elements from the range space. So, if it is in the range space then definitely there will be some x 1 and x 2 from R n such that f x 1 is equal to y 1 and f x 2 is equal to y 2. So, this x 1 and x 2 is coming from the domain that is R n.

Now, y 1 plus y 2. So, this I want to see that where this will lie. So, this can be written as f x 1 plus f x 2 and this can be written as f of x 1 plus x 2 because f is a linear map. So, it is a linear map or linear function. So, this is the property of the linear map. And now, from here x 1 plus x 2 belongs to the domain because x 1 we are taking from the domain x 2 is we are taking from the domain. So, and f is applying from the R n. So, it is a domain.

So, now from here I can say that which implies that y 1 plus y 2 also belongs to the range of f because it is coming as a function applying on R n and definitely its image are y 1 plus y 2 and it should be from the range space. So, it shows that y 1 plus y 2 is also belongs to the range space. Also, if I take for any scalar alpha. So, this is I am taking the scalar alpha alpha y 1 can be written as alpha y 1 I can write as a x 1 and since it is a linear map I can write it as a alpha x 1.

So, in which implies that alpha y 1 also belongs to the range space of function f ok. So, from here the scalar multiplication of y 1 is also belongs to the range space of f. So, from here we can say that the two properties vector addition and scalar multiplication are satisfied which imply that the range space of f is a subspace R m and R m we already know that is a vector space.

So, here we know that let R n and R m are vector spaces that we already know. So, range space of f is a subset of R m and then we also showed that it also satisfies vector addition scalar multiplication. So, it is a subspace of R m. So, the first one is satisfied conversely second one is because we have to show that the range of every linear function is a subspace of R m. So, that we have shown.

Now, we have to show that every subspace of R m is the range of some linear function. So, this one we want to show.

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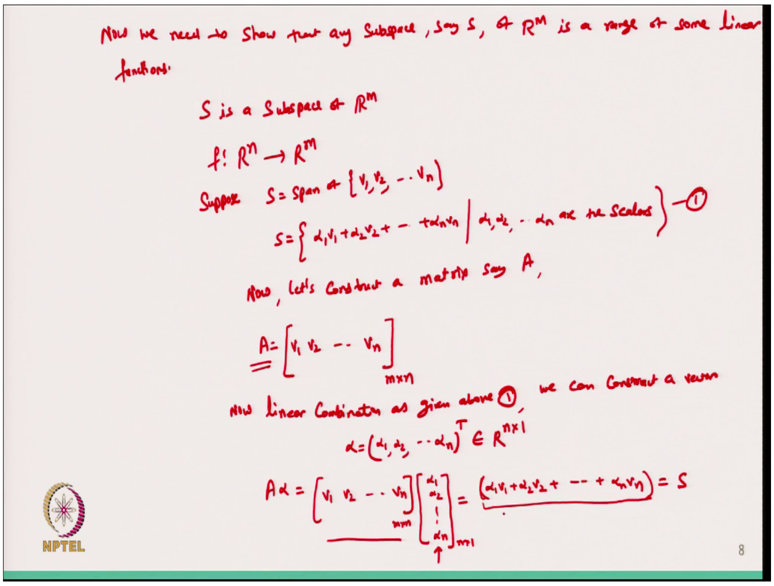

Now, we need to show that any subspace, say S, in this case of R m is a range of some linear functions. So, this one we want to discuss. Now, for this one let S is a subspace of R m. So, this one we have just defined. Now, the f is a map from R n to R m. So, that we already know. Suppose, S is equal to span of v 1, v 2, v n because S is a subspace of R m.

So, I know that it can be spanned with the vectors. So, say I have taken v 1, v 2 up to v n. So, these vectors are spanning the whole subspace S. So, I can write that S. So, if it is span of this then I can say that S is equivalent to the linear combination alpha 1, v 1 plus alpha 2, v 2 alpha n, v n. So, it is set of all the linear combination of this one where alpha 1, alpha 2, alpha n are scalars.

Now, so now, let us construct a matrix. So, I construct a matrix say A. So, let us take a construct a matrix A. So, what I am doing is that putting column vector as this v 1, v 2 up to v n. Now, I have taken the n number of vectors v 1, v 2, v n. So, I am putting this one as the columns of the matrix this one. So, now from here you can see that this is a n number of vectors.

So, it has the n number of columns and in this case it is coming from the R m. So, it has the component m components each of the vector v 1 has m component. So, you from here you can see that this will be a matrix of order m cross n. So, I have taken the n number of vectors that we are taking that it is spanning the S that is the subspace of R m and that each of the vector has a m number of component. So, from here I can write like this one.

Now, so this now the linear combination as given above that is in one we can construct a vector that is alpha 1, alpha 2, alpha n transpose it means the column vector. So, that belongs to R n cross 1 it means it has a n number of component and cross 1 means it is a column vector. So, this I am writing then what I could do I will write a into.

So, this I call it alpha. So, a alpha. So, it become v 1, v 2, v n and alpha is alpha 1, alpha 2, alpha n. Now, I have taken the alpha from the R n. So, this is containing the n number of component and this is the matrix A. Now, from here this one if i. So, it is a matrix of order m cross n and this is n cross 1 and now we can multiply. So, from here you can see that this will become alpha 1, v 1 plus alpha 2, v 2 and I can write alpha n, v n.

So, this one I am getting from the right hand side by multiplying this vector with this matrix and this can be written as alpha 1, v 1 and alpha 2, v 2, but what is this alpha 1, v 1 and alpha 1 it is a linear combination of v 1, v 2, v 3 and that is coming from the S. So, basically if you take from this is equal to S. So, from here I can say that.

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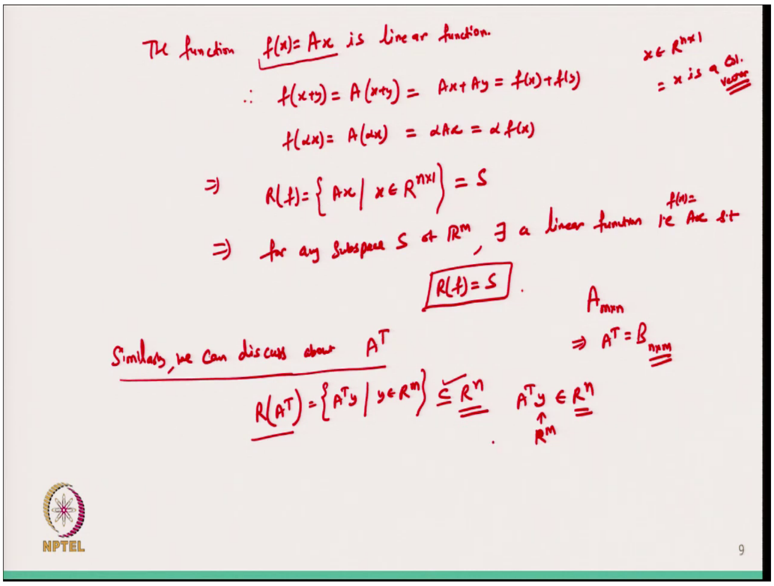

So, the function I am taking function f x is equal to A x is linear because f x plus y can be written as A x plus y and this we can write as A x plus A y and that is f x plus f y. So, I have taken this transformation as A x as a linear function. So, in this case it is satisfying this one and also f of alpha x it can be written as alpha A x because it is just a scalar I can take on the left hand side and that becomes f x. So, it is a linear function.

So, from here I can say that the range space of my f in this case is set of all a x such that x belongs to R n cross 1 n cross 1 whenever we write that it is a column space. So, x belongs to a R n cross 1 it means x is a column vector and this is become complete S. So, that shows.

So, which implies that, for any subspace S of R m, there exists a linear function that is A x in this case such that a range of that linear function f x is equal to x is equal to the S. So, this is the, the proof of the theorem. Now, the same thing. So, we have discussed this thing. Similarly, we can discuss about A transpose. So, if I take the A transpose and then I can define the range space of A transform. So, what is going to be there.

Now, if I my A is m cross n then we know that A transpose will be matrix that will be suppose it is matrix B. So, it is n cross m. So, m cross n that is n cross m. So, in this case I can say that the range space of A transpose is set of all. Now, we have to take A transpose y suppose I take the y and y in this case is coming from R m. So, this will belongs to R n ok.

So, I can say from the range space of this is the set of all A transpose y such that y belongs to R m ok. So, we have discussed the range space of matrix A. And so, just now we have discussed that we have defined the linear map from R n to R m and then we have defined the range space of f ok; so, from here.

Now, after that using this one we have shown that for any linear map from R n to R m there is a matrix involved that is A, such that we can have a linear transformation for each of the subspace of S, this one. So, now from here I have shown that, if we define the linear transformation with the matrix then we can also have the range space of R T.

So, this is the definition we have taken for the range space of A T that is taking the transformation of matrix A that is the transpose and putting the vector y applying on the y. So, that is the range space of A transpose, where y is coming from R and definitely this is subset of R n. So, it is just the converse of this one. So, these things we will.

So, let me stop today here. So, today we have started with four subspaces. So, in that case we have discussed the linear function and then we have discussed that how the linear function can be represented by the matrix and we have discussed about the theorem related to that one. So, in the next lecture we will continue with this one. So, thanks for watching.

Thanks very much.