Matrix Computation and its applications

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Lecture - 13
Basis and dimension of a vector space

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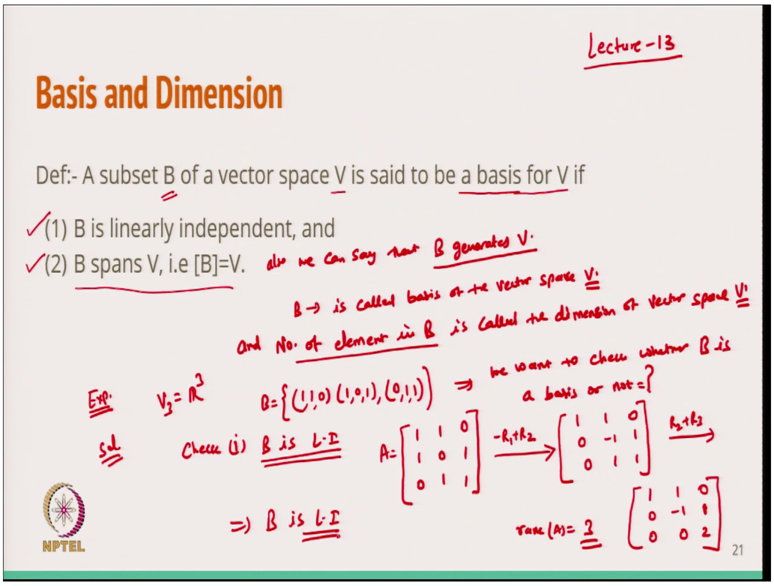

Hello viewers. Welcome back to the course on Matrix Computation and its application. So, today we are going to discuss the next thing, that is Basis and Dimension and this is the lecture number 13. So, let us start with this one. So, now, we are going to define the terms basis and dimension. So, what is the meaning of that?

Now, suppose I have vector space V and A subset B of the vector space V is said to be a basis for the vector space V; if, the first one is that, B is linearly independent, because this it is subset and it contains the vector. So, the B is linearly independent and the second one is that, the B spans V, it means that the span of B is equal to the complete V.

So, in that case we say that, this B is a basis and here we can write, also we can say that B generates V. So, the set B is called basis of the vector space V. And, number of elements in B, so the number of elements in B whatever the number of element is there is called the dimension of vector space V. So, two things we have to perform, we have to check it is a linearly independent or not and second one it should span this one.

So, let us do this one. So, let us take one example. So, suppose I just take vector space as V 3 that we are taking as R 3 ok. So, in this case I just take a set B is made up of suppose I take 1, 1, 0. So, I just 1, 1, 0; 1, 0, 1 and then I take 0, 1, 1. So, I just take three elements in B. Now, we want to check, whether B is a basis or not. So, this one we want to check.

So, the first thing is that, check first B is L I. So, this one we have to check. Now, this can be done with the help of the matrix. So, because we know that, when we write the linear combination this can be written as 1 1 0, 1 0 1, 0 1 1. So, we come across this matrix. So, whatever the vectors are there we will put it as a column vector in the matrix A and then I will reduce this matrix to the echelon form.

So, I will apply first I will make this 0. So, I will apply minus R 1 plus R 2. So, this we written 1 1 0, 0 and it is minus 1 1, 0 1 1 and then I will take R 2 plus R 3. So, from here I will get 1 1 0, 0 minus 1 1, and then 0 adding here it become 0 it become 2.

So, from here I can say that the rank of A is 3. So, it is invertible matrix. So, from here I can say, that set B is linearly independent. So, that is there, that it the B is linearly independent set.

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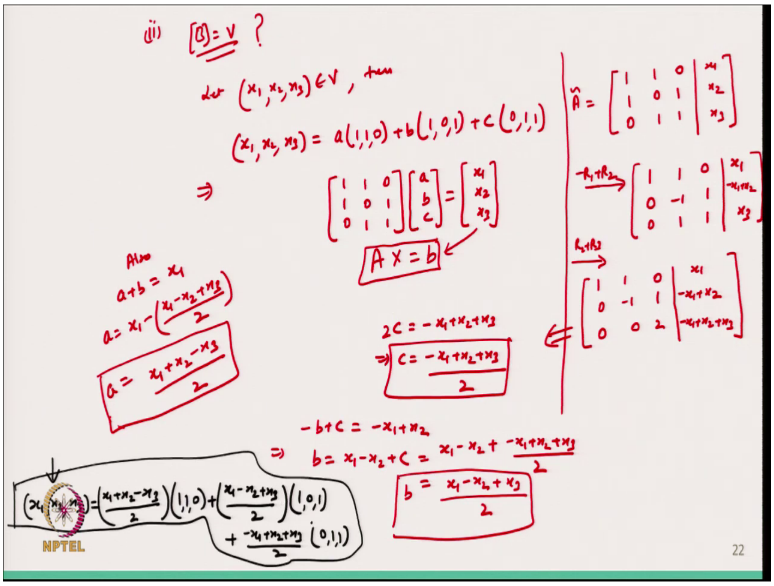

So, now from here this one, now I want to check the second property. So, that B spans V. So, this one I want to check. Now, what I do is that, let we take a vector x 1, x 2, x 3, belongs to V. Then, we just try to write x 1, x 2, x 3 as a linear combination of maybe I can take this as a and taking the 1, 1, 0 plus b 1, 0, 1 plus c 0, 1, 1. And, now from here I can write this as so, from here you know that you will get a matrix 1 1 0, 1 0 1, 0 1 1, a b c and it is x 1, x 2, x 3.

So, this become matrix A into some vector maybe I call it x is equal to b. So, this is my b. So, this is my system. Now, I want to solve this system. So, we know that how to solve this one? Because, it is a 3 by 3 matrix so, we can find our inverse also, but generally we go by writing the reducing this into the echelon form.

So, now, from here I can write it is augmented matrix. So, that is 1 1 0, 1 0 1, 0 1 1, x 1, x 2, x 3. So, this is my augmented matrix. And, then I will apply this to make it row echelon form as we have done in the checking that, this is linearly independent or not.

So, I have just taken minus R 1 and adding to R 2. So, it is 1 1 0, x 1. Now, it is becoming 0 because I am multiplying by minus 1 and adding so it is 0. Now, I multiply minus 1 and so, it is minus 1 it is 1. And, minus 1 so, it will become minus x 1 plus x 2 and then x 3, 0 1 1. Then, again I am applying this because I want to make this element 0. So, I am applying R 2 plus R 3.

So, I will get 1 1 0, 0 minus 1 1, 0 0 2. So, it become x 1, minus x 1, plus x 2 and I am adding here now. So, it become minus x 1 plus x 2 plus x 3. And, now it is a echelon form. So, from here using this one we can say that 2 c is equal to minus x 1 plus x 2 plus x 3. So, my c will be minus x 1 plus x 2 plus x 3 by 2. So, this is the c i got.

Now, from the second equation I will get minus b plus c is equal to minus x 1 plus x 2. So, from here my b will become. So, I take the c here. So, that will become x 1 minus x 2 plus c.

So, this one can be written as, so I can write is a x 1 minus x 2 plus. So, c is this one, minus x 1 plus x 2, x 3 by 2. And, from here I can write 2 x 1 minus x 1 so; it will be x 1 minus 2 x 2 plus x 2 so, minus x 2 plus x 3, by 2. So, this is my b.

And, from the first equation I get a plus b is equal to x 1. So, my a will be x 1 minus b so, minus of this one. So, that gives me 2 x 1 minus x 1, so, x 1 plus x 2 minus x 3 by 2. So, that is my a; now you can write down that from here we can write. So, I can write from here that my x 1, x 2, x 3 can be written as, so a is this one. So, it can be written as x 1 plus x 2 minus x 3 by 2, 1, 1, 0, b is this one, x 1 minus x 2, x 3 by 2, it is 1, 0, 1 and plus minus x 1 plus x 2 plus x 3 by 2, 0, 1, 1.

So, this can be written like this one. So, from here you can see that for any value of this one I can get the linear combination, you just give me the vector from, any vector from V 3 and then correspondingly this coefficient can be found.

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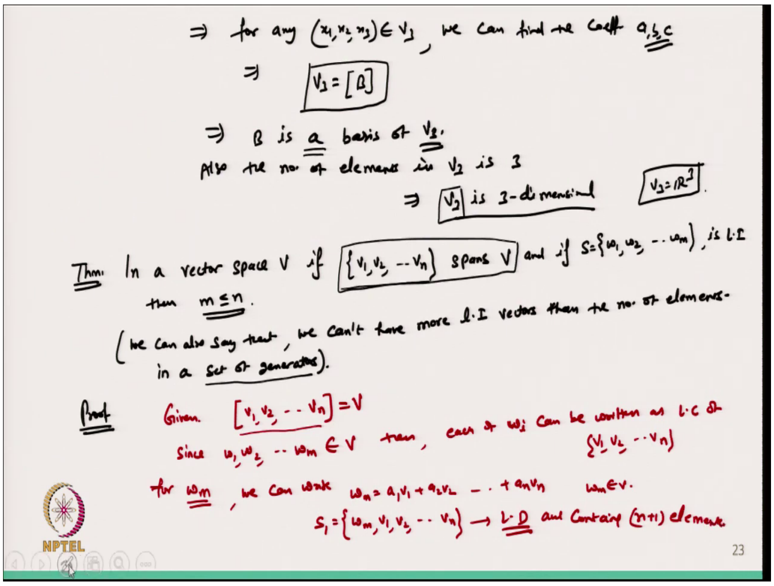

So, from here I can say that. So, from here for any x 1, x 2, x 3 belongs to v 3 we can find the coefficients a, b, c. So, from here I have taken any element and that can be written a linear combination of this one. So, from here I can say that the v 3 is written, can be written as the span of the vectors, span of the so, I will B. So, I can write this here. So, this is equal to the span B. So, it is spanning v 3 and the B is linearly independent.

So, from here I can say that B is a basis of v 3. Also, the number of elements in basis v 3 is 3 which implies that v 3 is 3 dimensional. So, it is a 3 dimensional space v 3 and that we already know, because v 3 is equal to R 3. So, this way we can find, that it is a basis for v 3. Now, if I change the elements we can get the another basis. So, that is why it is only I am writing a basis.

Now, there is a very important theorem regarding this basis, because here what will happen if I say that, the set B is not linearly independent, but still it is spanning the set. Then, what you can say about the number of elements in the basis or number of element in that space.

So, for this one there is a very important theorem. So, that theorem gives the relation between the spanning set and the linearly independent set, number of elements in the linearly independent set. So, this theorem says that in a vector space V, if v 1, v 2, up to v n spans V ok. So, it spans V and if I take a set S, that contain the element w 1, w 2, w m. And, this is if

S is linearly independent containing the m elements and linearly independent, then it says that m is less than equal to n ok.

So, it says that, this m will be always less than equal to n. In this case, where this is the spanning set, it need not be a basis, but only condition is that it spans V, that is it ok. So, in this case we can also say, we can also say that, we cannot have more linearly independent vectors, then the number of elements in a set of generators or in the set of span.

Like here we have a n number of elements in this generator because it spans V, it generates V, then we can say that if we have a set of linearly independent vectors, then m cannot be greater than n. So, this is the way we are defined.

So, here it is spanning basically. So, proof: now, it is given to me. So, that is just it is given, now given the span of v 1, v 2, v n, that is equal to V ok. So, that is given to me. Now, since w 1, w 2, or w m also belongs to V, then for then I can write, then each of w i can be written as linear combination of v 1, v 2 up to v n, that we can write [FL]. So, that is why we are writing like this one.

So, for example, so, then for I just to take w m. So, for w m we can write, w m as let I take a 1, v 1 plus a 2 v 2 up to a n v n, I am writing like this one. Because, w m belongs to v and v is completely spanned by this one, it means I can write like this one. So, from here I make a vector set S 1. So, I S 1 I will take first element w m and then I write v 1, v 2 up to v n.

So, definitely it is linearly dependent, because one vector is a linear combination of the other vectors. So, definitely these are linearly dependent and linearly dependent and containing n plus 1 elements. So, that is there.

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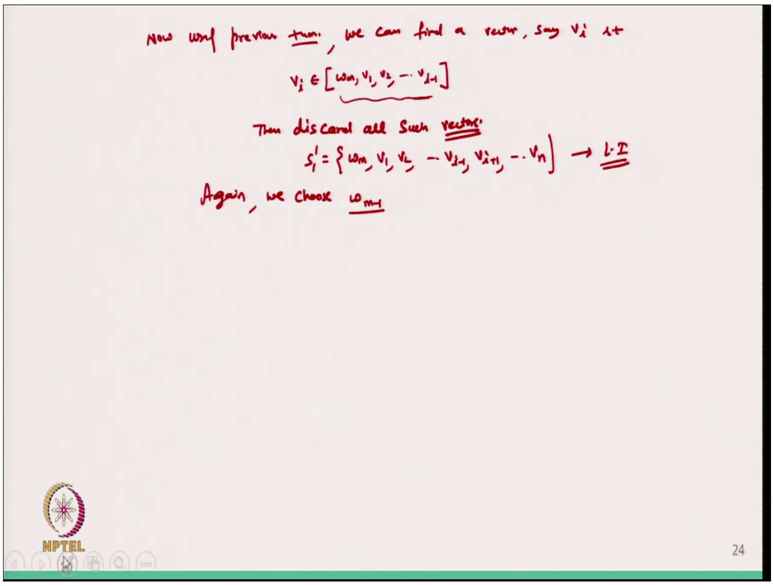

Now, what we are going to do is now using previous theorem. So, we can find a vector, say v i such that, v i belongs to the span of w m, v 1, v 2, v i minus 1. As we have seen in the previous theorems, that we start with the first element and then keep going up till we get a vector which is a linear combination of the previous vectors.

So, this theorem we are using here ok. Then, discard all such vectors, at least 1 vector will be there which is a linear combination of the previous one, we remove that one, and then we continuing for the whole set and then we remove all the vectors which are making this linearly dependent. So, we discard all such vectors and let us we call it S 1 dash.

So, S 1 dash is w m, v 1, v 2, v i minus 1, v i plus 1 and suppose going like this one. So, definitely this is linearly independent. So, this is linearly independent ok. So, we are done till here. Now, again we choose w m minus 1.

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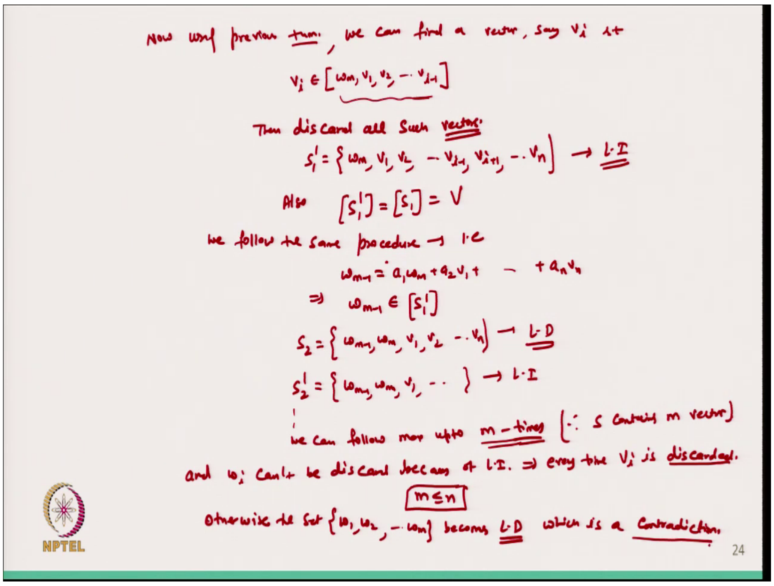

So, I can write and also so, this one is also S 1 dash span that is same as S 1, because this is my S 1. And, I have removed the linear dependent vector from here. So, span of S 1 dash is equal to span of S 1.

So, this is there and also the span of S 1 is basically the span of V and that is a span equal to V completely, because S 1 is just the putting one element, which is linear dependent. Now, so, we follow the same procedure that is I take w m minus 1 as a linear combination of like this one a 1, w m, plus a 2 v 1 up to this one.

Suppose last element is coming still n. So, we will take this one or I can say that we can write w m minus 1 belongs to the span S 1, we can write no problem. So, now, I choose, I write a line set S 2 the same way writing w m minus 1, w m, v 1, v 2, this one.

Whatever the elements remaining here; so, out of this one I am collecting this up to v n and it. So, this is again the linearly dependent. So, if it is linear dependent then the same procedure can be followed and I can write S 2 dash, which contains the elements w m minus 1 may be w m, v 1 and so on such that this is L I.

The same procedure we have followed removing the linearly dependent vectors ok. So, this way, we can follow maximum up to m times, because there is m number of elements involved, because S; because this set I am taking S contains m vectors. So, I can follow this only.

So, it can be only m times and definitely the w m; the w i s cannot be moved, because w i are linearly independent only v i's will be removed. So, that should be there, that only we can remove the v i's cannot be remove the w i's.

And, w i's cannot be discarded, because of linearly independent which implies every time V i is discarded. So, every time V i is discarded. So, from here I can say that, if I follow this 1, then definitely m will be less than equal to n because start with the n elements in that set and then adding 1 w m it become n plus 1 elements, and then we start removing the elements to make it linearly independent.

So, and then again we are putting one element and discarding the other elements. So, definitely in the end we are left with the w m elements or this is the maximum we can go. Even the before of that one also it may happen, that all the V i s will be removed. So, from here that the m is definitely will be less than equal to n. Otherwise, the set w 1, w 2 up to w m, becomes linearly dependent, which is a contradiction.

Because, we have assumed that this w 1, w 2 up to w m is linearly independent ok. So, after once we reach this one and all the v i’s as removed, then it becomes linearly independent we cannot go after that. So, in this case definitely the m will be less than equal to n and, so, we able to finish this one.

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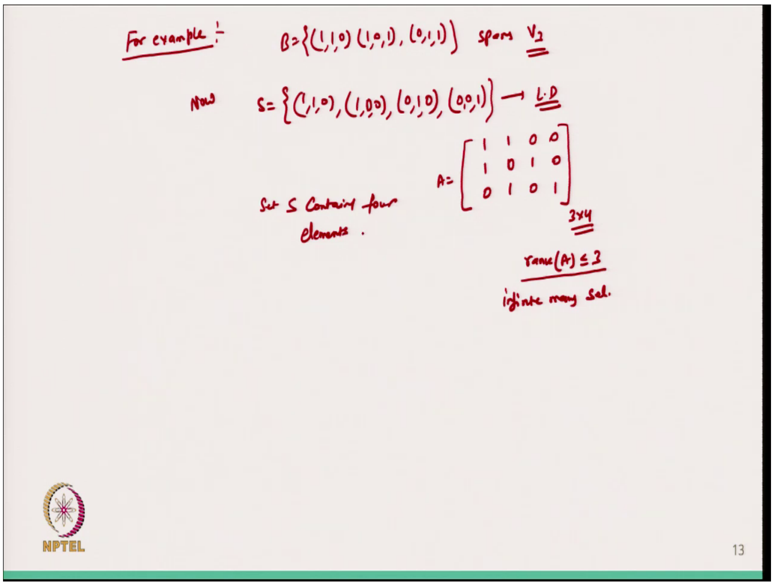

So, for example, just now we have seen in the previous examples, that is, that the set I have taken B 1, 1, 0; 1, 0, 1 and 0, 1, 1. So, it spans whole V 3 forget about that whether it is a linearly independent or not, but definitely it spans V 3. Now, suppose I take the set S, which contains any 4 elements, let us take the set S which contains 4 elements. Just I take, maybe I can take the same element 1, 1, 0; 0, 1, 0 and suppose I take 0, 0, 1 so, four element I am taking.

Then, definitely it will be linearly dependent, because we are going to have a matrix 1 1 0, 1 so, I have taken the same element. So, 1 0 1, then I am taking 0 1 0; 0 0 1, suppose I have taken these elements. So, this is 3 cross 4. So, in this case I know that the rank of this matrix A, suppose I take it A is always less than equal to 3.

So, definitely it cannot be a non singular matrix. So, rank is always less than equal to 3 and the number of variable will be 4. So, infinite many solutions and from there we know that this will be linearly dependent. So, using this one I can say that set S containing four elements, four or more.

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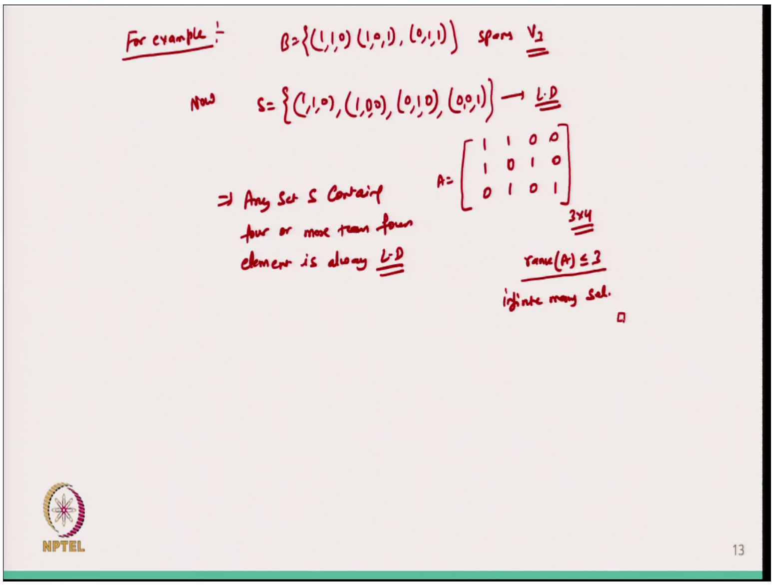

So, from here I can say that, any set S containing four or more than four is always linearly dependent ok. So, it will always linearly dependent. And, 3 we know that it is, it can be shown that it will be linearly independent or 2 can be linearly independent, but more than four, more than 3 is always going to be linearly dependent. So, that is the one of the way we can use the previous theorem.

So, let me stop here. So, let us stop here. So, today we have discussed some other properties about the vector spaces, we have defined the basis the dimension and in the next lecture we are going to be continue from this one. So, thanks for watching.

Thanks very much.