Matrix Computation and its applications
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Lecture - 11
Properties of linearly independent and dependent vectors

Hello viewers, welcome back to the course on Matrix Computation and its application. So, today we are going to discuss some more facts about the linearly dependent or independent sets. So, let us start with this one. So, this is lecture number 11.

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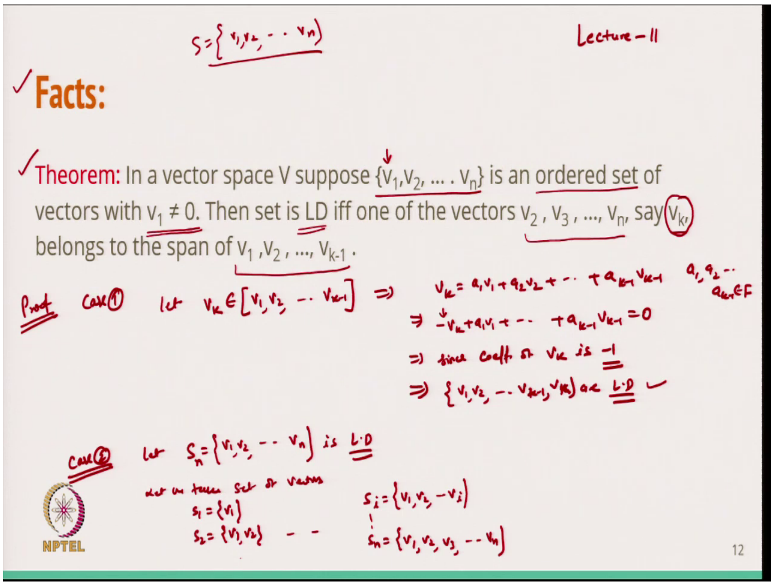

So, this is another fact we are going to discuss or maybe it is a theorem. That in a vector space V suppose we have n number of vectors and that is given to us in the ordered set, order set means they are given in the order.

So v1 is definitely coming before v2 and v2 is coming before v3. So, we cannot change the order of this set. So, this is a set of vectors with v1 is non zero then the set is linearly dependent if and only if one of the vectors say vk. So, one of the vectors coming from these sets belongs to the span of the vectors $\{v_1, v_2, \ldots, v_{k-1}\}$.

It means in this case the vector vk is coming and that is a linear combination of the previous vectors $\{v_1, v_2, \ldots, v_{k-1}\}$. So, that is what we are going to discuss. And, what is the use of this theorem is that, suppose I have a set of vectors with a large number of vectors $\{v_1, v_2, \ldots, v_n\}$. And, ask me to check whether it is going to be linearly independent or dependent, then we have to take the linear combination.

And, in that case it becomes very difficult when the number of vectors are large, to check whether they are linearly independent or dependent. Like we have seen in the previous examples also where we have a 5 number of factors and in that case we got the matrix that is 4 cross 5.

So it becomes very difficult, because we have to convert that matrix into the echelon form and then only we are able to check. So, if I have a 5 cross 5 matrix, or maybe 10 cross 10 matrix. Then, it is very difficult to check whether the set is going to be linearly independent or dependent, then we can use this theorem.

So, in this theorem we take the vectors and one thing is there that $v_1 \neq 0$. So, starting with this one and that is non zero. Because, if it is coming 0 then itself we have the theorem that if one of the vectors is 0 then it is going to be linearly dependent.

So, we are starting with v1 is that is coming non zero and then we say that the set is linearly dependent if one of the vectors say vk is a linear combination of the previous vectors. So, this is the theorem and let us discuss its proof. So, let us take case 1.

So, the case one is that, let vk belong to the span of v1 v 2 up to v k minus 1. So, that is given to us. So, I am taking this. Now, from here it implies that I can write my v k as some a 1 v 1, a 2 v 2 up to a k minus 1, v k minus 1, for all the scalars a 1, a 2 up to a k minus 1 belongs to the field.

And, from here I can write; now this one can be written as I can write as minus v k plus a 1, v 1 up to a k minus 1 v k minus 1 that is equal to 0. So, this one we can write. And, from here you can see that, this value of a 1, a 2 up to a k it can be any value, but the coefficient here is always minus 1.

So, from here I can say that, since coefficient of v k is minus 1, which implies that the set of vectors v 1, v 2, v k minus 1 and v k so, this set of vectors are linearly dependent. So, this is the vectors are linearly dependent and that is what we wanted to show, that if this is there then the set is linearly dependent. So, one part is ok. So, I take the converse of this one, the second part now let we take. So, let I assume that, the setS is LD, let I take the set s n, v 1, v 2 up to v n. So, this is a set I am taking of n vectors. Henceforth this is L D. So, the set is L D. Set is L D means the vector belong to these are L D linearly dependent. So, let this set is a linearly dependent set.

Now, so, now, I want to check show you that, if we get a vectors linearly dependent vectors, then one can be written as a linear combination of the previous one ok. So, this is what we want to show. So, let we take the vectors s 1. So, let us take set of vectors as I just to choose s 1 as containing only one element, s 2 contains v 1, v 2 like this one. So, you keep going this one.

So, I can have s i this is set of vector v 1, v 2 up to v i and then s n is the whole v 1, v 2, v 3 upto v n. So, this is the sets I can define from the set s n. And, I am keeping this in the order so, that is the most important thing.

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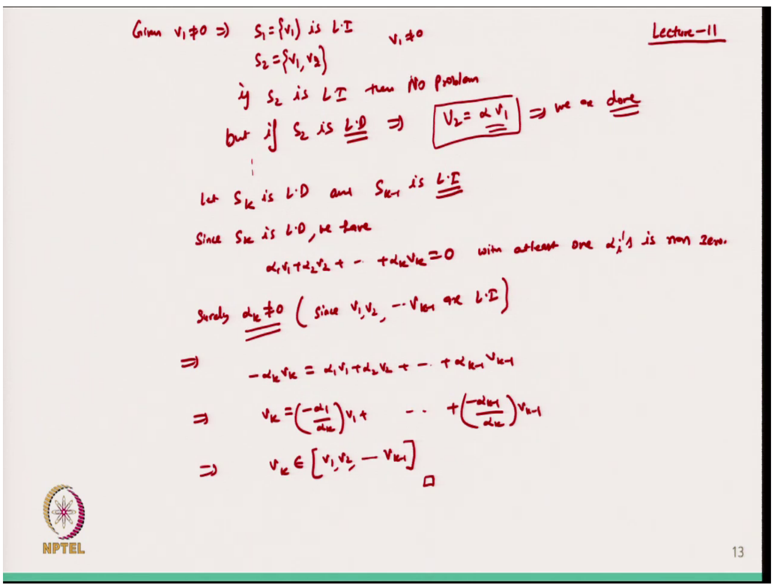

Now, given v 1 is not equal to 0 that is already given to us. So, from here and from the previous example we can say that s 1, which is containing only v 1 is linearly independent. Because, we have seen that, if set contains only one element and that is a 0 element, only then it is linearly dependent. Otherwise, it is going to be always linearly independent ok.

So, this is linearly independent the same way I just take s 2. So, in the s 2 we have v 1 and v 2. So, now v 1 is not 0. So, v 1 is not equal to 0. So, that is definitely is there. Now, we have discussed that we have a two elements v 1 and v 2 and suppose it maybe a collinear vectors. So, in this case this is also going to be linearly independent.

So, if it is going to be linearly independent that is ok. If, it is going to be linearly dependent now this is the two cases. So, just take now, if s 2 is linearly independent then no problem, then no problem, but, if s 2 is linearly dependent.

So, in this case if s 2 is linearly dependent it that implies that v 2 can be written as some scalar multiple of v 1, that we already know. And, in fact, there will be collinear vectors. So, from here we are done that we are written the one vector as a linear combination of the remaining one.

So, it from here we are done, and we have shown that the vector v 2 is a linear combination of the previous vector. So, that is that was the v 1 and this is we are done ok. So these things we are here. So, from here we can say that let so, after doing this one, I reached that let s k is a set which is linearly independent.

So, that is linearly dependent we are taking this s k and s k minus 1 is linearly independent ok. So, in this case my set is linearly dependent means the vector belongs to these are linearly dependent, s k minus 1 is the set which is linearly independent, it mean the vector belongs to these are linearly independent.

Now, from here, now since s k is L D. So, we have we can write alpha 1 v 1 plus alpha 2 v 2 alpha k v k, that is equal to 0 we have with at least 1 alpha k that is not equal to 0. That is why it is the meaning of linearly dependent ok. So, at least 1 alpha k will be there which is non-zero.

So, then we can have this condition. Now, so, and we are choosing alpha k is not equal to 0. So, from here I just I can write this on as 1, 1 alpha i is nonzero ok. So, this is there. Then, we write surely alpha k is not equal to 0, why? Because, if alpha k is 0 then the because since v 1, v 2 up to v k minus 1 are linearly independent. So, these vectors are linearly independent. So, that is why this alpha k cannot be 0, if alpha k is 0, that is also 0.

So, they become linearly independent. So, from here I can write that, I can write from here minus alpha k v k, I can write as alpha 1 v 1 plus alpha 2, v 2, alpha k minus 1 v k minus 1. And, from here I can write my v k as minus alpha 1 alpha k, v 1 minus alpha k minus 1 over alpha k v k minus 1 ok.

And, from here we can say that, v k belongs to the span of v 1, v 2 upto v k minus 1 and this is what we wanted to prove ok. So, the moral of this theorem is that, when we have a large number of vectors, then we start with the first vector, then taking first two vector, then first three vectors and keep checking once we reached a linearly dependent set.

So, once we if we hit a set which will linearly dependent then the whole set will be linearly dependent, otherwise it will be linearly independent.

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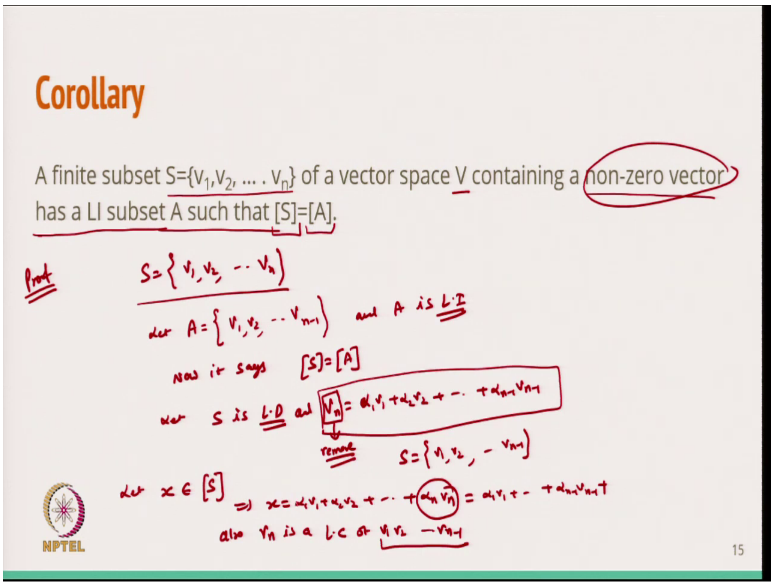

So after this one, now we can also one corollary, that is finite subset I am taking a finite subset S, which contain the vector v 1, v 2 up to v n of the vector space V containing a non-zero vectors has a LI subset A such that a span A is equal to span of S. So, this is what we are going to discuss ok. So, the condition is that we have a set S. So, this is what we I have a set S v n, so, n number of elements are there ok for the vector space V.

Now, it contains a non zero vector, because I am saying that it can be 0 also v 1, v 2 up to v n it can be 0 also. So, we are saying that it contain a non-zero vector ok. So, all this containing a non-zero vectors has a LI subset A such that this one.

Now, so, what we are going to discuss here is that, now let I take a subset A ok. So, that subset contains few vectors. So, maybe I just take v 1, v 2, up to v n minus 1 suppose I take these vectors ok. And, A is linearly independent, so A is linearly independent that is sure ok.

So, I am taking one subset of S and which is linearly independent. Now, it says that S the span of S is equal to the span of A, but how we are going to discuss this one. Now, we know that let I just take let I the set S is linearly dependent, suppose, the set S is linearly dependent ok.

So, now, from here or maybe I can so, linearly dependent. So, I can write that is and v n can be written as a linear combination. So, I can write alpha 1 v 1, plus alpha 2, v 2, alpha n minus 1 v n minus 1, henceforth this is happening. Because is the suppose this the vector v n can be written as a linear combination of the previous one.

Now, from here now the v n is a linear combination of this one. So, I can remove this vector, because it is already the linear combination of this one. So, then I can remove this one. So, it remove then the remaining vector I have a set and that set is a v 1, v 2 up to v n minus 1. Then; obviously, now what I am going to do now; now I will let take any x belongs to suppose I take from span of S.

So, let us I take the x belongs to the span of S, which implies that x can be written as and yes I am taking just one element less just for the so, sorry it is coming from S. So, x can be written as alpha 1 v 1, alpha 2 v 2, alpha n v n. So, I can write like this way. So, it is a linear combination of this vector.

Now, also v n is a linear combination of v 1, v 2, v n minus 1. So, it is also a linear combination of v 1, v 2, v n. So, that can be written as alpha 1 v 1, alpha n minus 1 v n minus

1 plus, now it is the this one and that is already there v n is written as this one. So, I can write from here maybe I just changed the notation I can write. So, let us write this as I just write it as a 1, so, a 1, v 1 a 2 v 2 a n v n.

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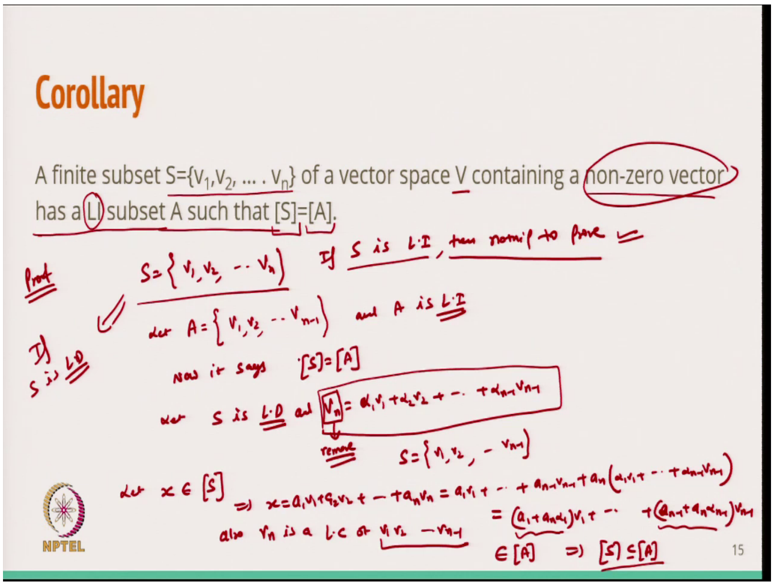

So, this can be written as a 1, v 1 up to a n minus 1 v n minus 1 plus a n and v n, I have wrote it. So, it is alpha 1 v 1, alpha n minus 1 v n minus 1. And, this one can be written as a 1 plus a n alpha 1, v 1 up to a n minus 1, plus a n alpha n minus 1 v n minus 1. So, it is a scalars again.

So, it from here I can say that x belongs to and this is the linear combination of the vectors belongs to the set S. Because, the only condition is that the as a LI subset A, because A is A LI so, this is written. As a LI subset A, so, that we have to keep in mind ok.

So from here this become the linear combination of the set belong the vector belongs to the set A. So, I can say that it means the x belongs to the span of A ok. So, I taken the x from the span of S and that belongs to this one, which implies that this is a subset of this one ok. Similarly, I can take the converse and using these two then I can take element.

So, and that we already know that I have taken the set A, that is a subset of S. So, definitely the span of A belongs to the span of S. So, that is already true. So, in this case from here I can write also A is a subset of S ok.

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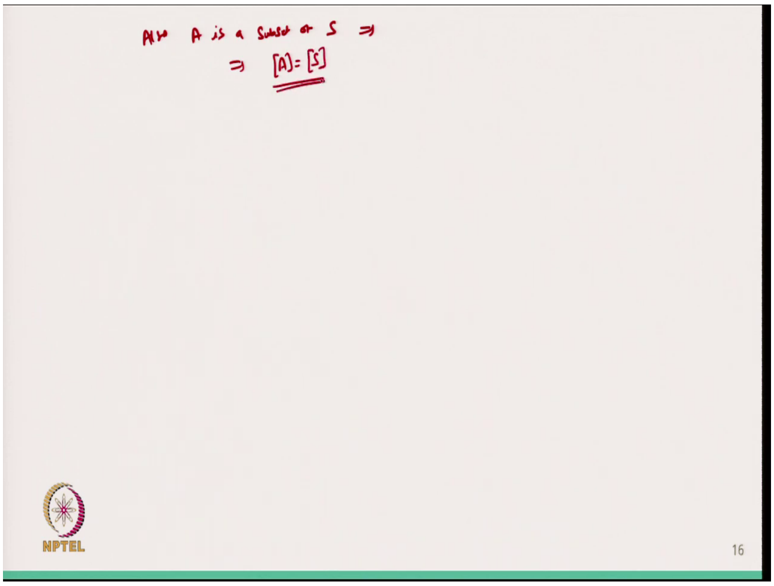

So, then I can choose element and A is LI vector. So, this one I can take, because I can from here I can take that, if S is LI, then nothing to prove. Because, if S is LI and A is a subset of that, that we already seen, that we from the previous theorems we have seen, that if I have the LI set, then it is subset is also LI.

So, a is a subset of a then it will be LI. So, nothing to prove the only thing is that when S is L D ok. So, from here I am taking that if S is linearly dependent. So, if S is linearly dependent then we have check taken a subset of S that is A, which is LI and then I approved this one.

Now, I just take the converse of that one and from there. So, this can be proved with the previous theorem also in which we have taken this way. So, we can even use this theorem also in proving that one.

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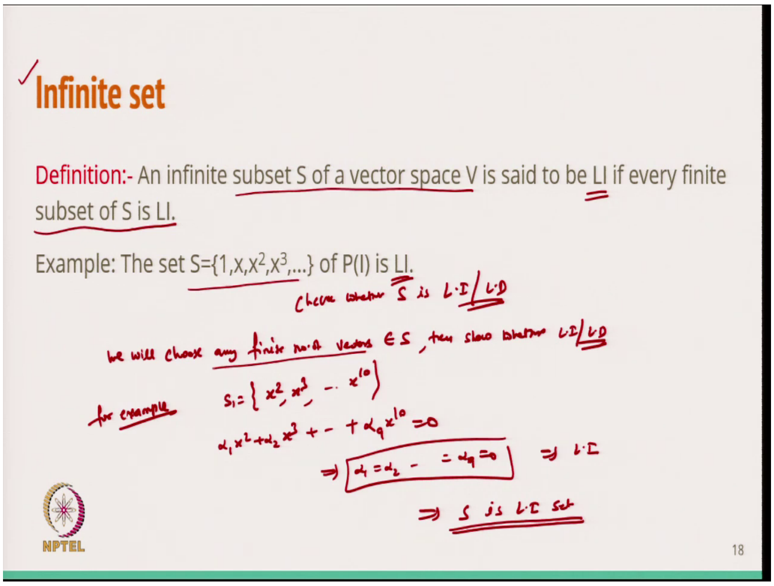

Now, so, after this one we want to define some another term that, is what is going to happen if we have a infinite number of sets, infinite sets. Suppose, I have a infinite subsets of vector space V, like we have a vector space of all the polynomials. And, in that polynomials suppose I take a set as like this one, which contains 1, X, X square, X cube, X 4 all the powers of X. So, it this can be this S is the infinite number of containing the infinite number of vectors belongs to the set of polynomials.

So, an infinite subset S the vector space V is said to be linearly independent, if every finite subset of S is LI. So, for example, if I take this one and I want to check whether S is LI or LD then what is going to because ultimately we are not going to take the linear combination of all the infinite number of elements. So, for this one what we are going to do is that, we will choose any finite number of vectors belongs to S and then show whether it is LI or LD ok.

So, any finite number of sets, you choose you take any finite number of sets finite number of vectors show that, suppose it is coming LI then you can take any other vectors that is also coming LI only, then you can say that this is the LI vector.

So, like this one. So, in this case this is a set I am taking. So, I just choose for example, I choose a set maybe s 1 as x square x cube x 10 upto 10. Then, what I will do, I will take the linear combination alpha 1 x square plus alpha 2x cube upto. So, alpha 9 x 10, I put this equal to 0. And, then so this is a polynomial 0 polynomial and from here we will take the coefficients equal. And, then we will find out that this alpha is are 0 and from here I can say that alpha 1, upto alpha 2, upto alpha n 9, they are coming 0 in this case.

So this set of vectors are LI. Similarly, we choose any vectors that is suppose coming also LI. So, if from here I can say that this is a LI, and then from here I can say that S is linearly independent set ok. So, this way we can do when we have a set containing infinite number of vectors.

Now, so, after this one I just take one more example, it is a different type of example to check whether the set of vectors are linearly independent or dependent ok.

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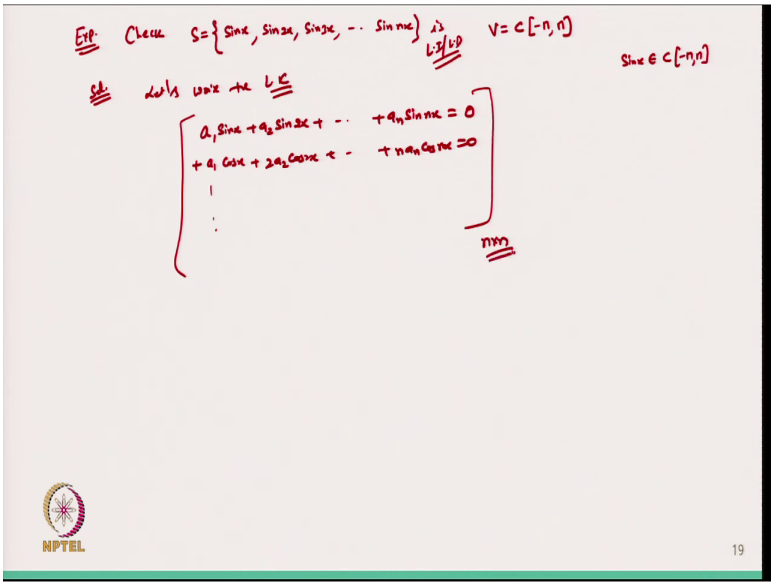

So, example check the set of vectors I am taking, suppose I take the vector s x, sin x, sin 2x, sin 3x, sin n x. So, these are the vector I am taking from the vector space V that is set of all the continuous function from minus pi to pi ok.

So, I am taking the continuous function from minus pi to pi. So, this is the vector space, I am taking and all these vectors are coming from this one ok. So, for example, sin x belongs to C minus pi to pi. So, x is coming from here. So, all these vectors belongs to this one.

Now, I want to check this set is LI or LD ok. Now, what is going to happen I? So, this is my solution. So, we will take the linear combination. So, let us write the linear combination. So, linear combination will be a 1 sin x plus a 2 sin 2x upto a n sin nx and that is coming equal to 0 function. So, this is the same way we have done in the previous example where we had only 3 vectors, x sin x and cos x.

But, now we have a n number of vectors. And, now if I suppose do the same thing again and again taking the derivatives, then I will get n cross n matrix ok. So, maybe I can take the derivative of this one and I can write here minus a 1 cos x plus 2 times a 2 cos 2x and n a n cos nx equal to 0, one time derivative I have taken.

Similarly, I can take another derivative, but in that case it is not going to give. So, suppose I am taken sin x then cos x, then minus sin x, then minus cos x like this one, I can keep going. So, I will get this type of set of equations.

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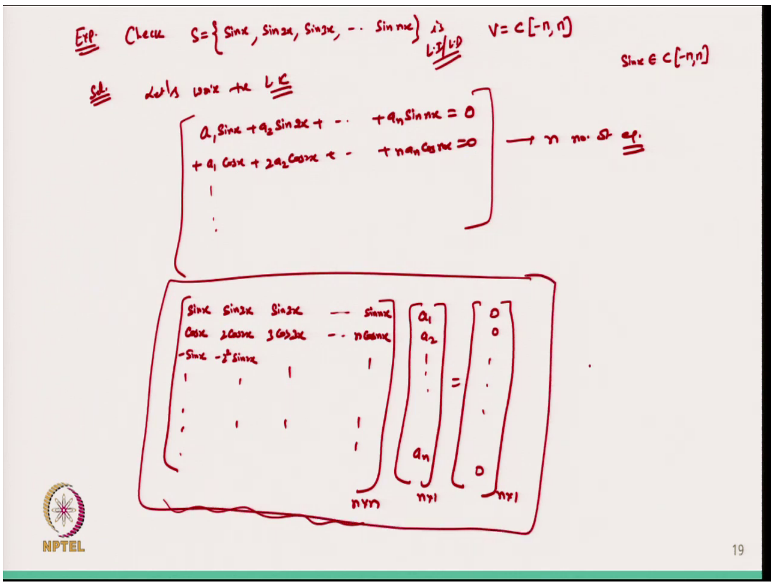

And, from there we are not we are going to have a matrix that is maybe n cross n. So, in this case I am going to have n number of equations ok. So, n number of equations we are going to have in this case, now the question comes that how we can solve? Because, if I write this as.

So, you will see that I will get a matrix of this form a 1, a 2, upto a n and this is I am going to have 0, 0, 0 and here I am going to have sin x, sin 2x, sin 3x or maybe sin nx, then cos x, 2 cos 2x, 3 cos 3x, n cos nx ok. Then, minus sin x, then minus 2 square sin 2x like this one.

So, n time derivatives I am going to take and this matrix is going to be n cross n and this is n cross 1 and this is n cross 1. Now, the question is that because now it is very difficult to take the determinant of this matrix. So, how we can find out the determinant? And, in this case if I put the value of x equal to 0, then definitely this row becomes 0. And, in that case always I will say that the 1 row is 0 so, the determinant 0.

So, I can say that this is a linearly dependent ok. So, these type of problems how to solve, that we are going to discuss ok. So, let me stop here today. And, so, today we have discussed few more facts about that one. And, then we have started with the example in which we have a n number of functions, that is coming from the vector space of continuous function from minus pi to pi.

And, then we check that, how we stuck that how we can find out that the given set of vectors are linearly independent or dependent. So, you can think about this one and we will discuss the solution of this in the next lecture.

Thanks for watching thanks very much.