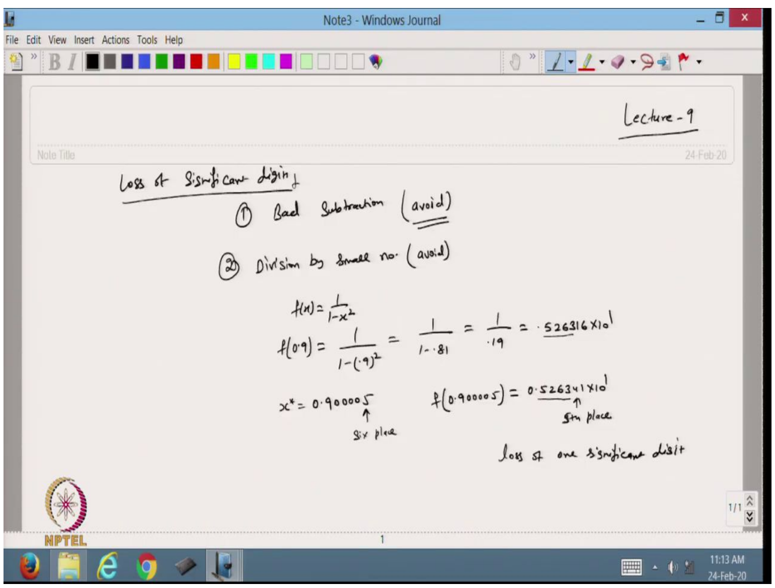
Scientific Computing Using Matlab
Professor Vivek Aggarwal
Professor Mani Mehra
Department of Mathematics
Indian Institute of Technology Delhi
Delhi Technological University
Lecture 9
Iterative Method for solving nonlinear equations

Hello viewers. Welcome back to this course. So today we will start with lecture 9. So in the previous lecture we have discussed how the errors grow when we multiply or additions or division by two numbers. And then we have given one example where we have a loss of a significant digit.

(Refer Slide Time: 0:45)



So another example I can say is how we lose a significant digit. So in this case, the first one was the bad subtraction. So we should avoid this one because whenever we have two numbers of the same magnitude and we have to do the subtraction in the calculation, we always should avoid this one because we have seen that we lose a significant digit in that case. And the second one is that division by a small number. So in that case also we should avoid, avoid in the calculation.

Because for example, suppose I have a number function,

$$f(x) = \frac{1}{1 - x^2}$$

and I want to calculate the value at 0.9. So in this case, what will happen?

$$f(0.9) = \frac{1}{1 - 0.9^2}$$

So that gives me 1 over 1 minus 0.81. So that is the value we are going to have. So it is 0.19. So if I calculate the value of this, this is equal to 0.526316×10^1 .

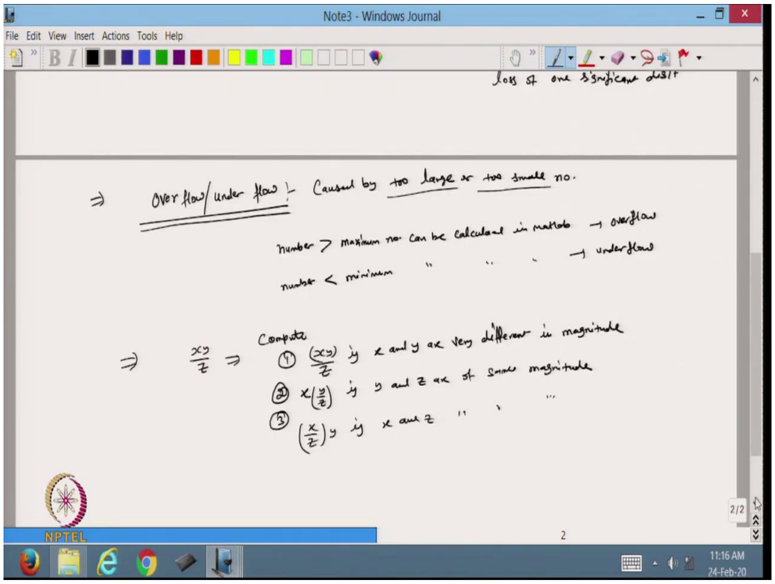
So that is the value we are going to find out or if we do the if we calculate the value 1 by 1 by 9 in our calculation or in our calculator. So in this case, now suppose I take the x, another x and I approximate this number with the help of 0.900005. So let us introduce an error at the sixth position. So this is the error I have introduced at the sixth position.

And then I want to calculate what is the value of 0.900005.

$$f(0.900005) = \frac{1}{1 - 0.900005^2}$$

So in this case, if I do the same calculation and I calculate with the help of a calculator, then this value comes 0.526341×10^1 . So in this case, you can see that I have started with my calculation and in this case I have the same number, 5263 but here earlier it was 1 and the value is coming 4 here.

So in this case, what is happening? I have introduced the error at the sixth place and in my answer, I get the error at the fifth place. So in this case, this number was introduced and this was this number has the significant digit, that was the five significant digits, but here only we have four significant digits because the error was introduced at the fifth place. So that is also a loss of one significant digit. So in the calculation also, we should avoid the division by small numbers.
(Refer Slide Time: 4:27)



Then after doing this one, there is another word used in the calculation, and that is called the overflow or underflow, so overflow or the underflow. So this is caused by too large or too small numbers. So in this case, what is that we are dealing with is the two large numbers or very, very small numbers. So that is called the overflow.

So, because every computer has some limitations, so suppose we are dealing with the Matlab and I have a number, that number is greater than the maximum number we can calculate in Matlab. So suppose I have a number greater than this one. So then this is called overflow and if a number smaller than the minimum number in the Matlab can be calculated, that is called underflow.

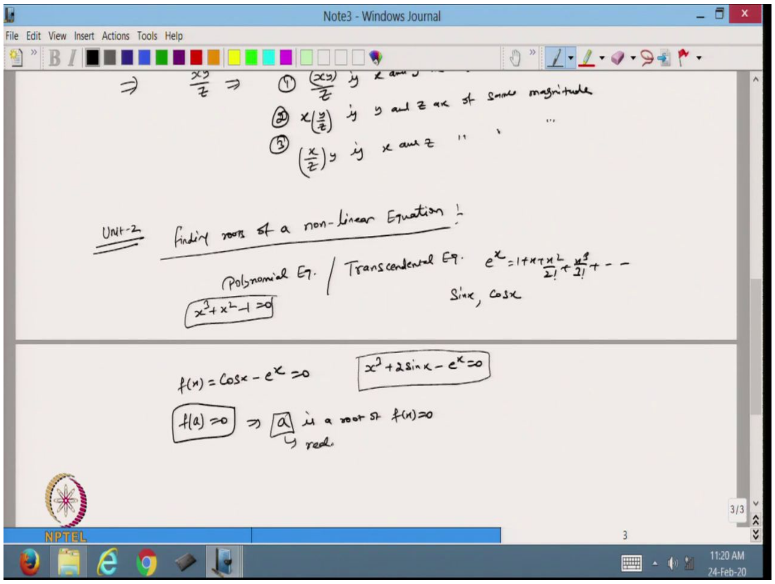
So that is the two types of difference we can say the error that is called the overflow and the underflow. So suppose, so generally we should avoid the bad subtraction, we should avoid the division by a small number and suppose I want to calculate this number ,

$$\frac{xy}{z}$$

So this is the multiplication of two numbers and then division by another number. So this number we can calculate or compute. The first one I can write this as x y by z if x and y are very different in magnitude.

I can compute this one as $x(y/z)$, so this is possible if y and z are of the same magnitude. So they are very close to the magnitude or I can write the third one is I can write $(x / z) y$. So that is also possible if x and z are of the same magnitude. So the same number we are having is $(xy)/z$ but this number can be calculated in the various forms depending on what is our x and y and z.

So in these cases, the calculation is done based on what is the magnitude of the number.
(Refer Slide Time: 8:10)



So this is a type of small errors that are introduced in the calculation for the computation. Now we start with our next unit so I can say that my unit two and we will start with finding roots of a nonlinear equation. So in this case we have two types of equations, so it can be a polynomial equation or it can be a transcendental, transcendental equation. So polynomials like I have

$$x^3 + x^2 - 1 = 0$$

so this is the cubic polynomial. Transcendental means we have a function that is a transcended function like we have an exponential.

So transcendental functions are those functions which can be represented as an infinite series. So exponential we can write down as a power series, we know that it can be written as

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

So this is called a transcendental function or it can be a sin x, it can be a cos x. So like this one, this function, wherever we deal with such a type of function that is called a transcendental function.

And the equations suppose I want to calculate what is the root of

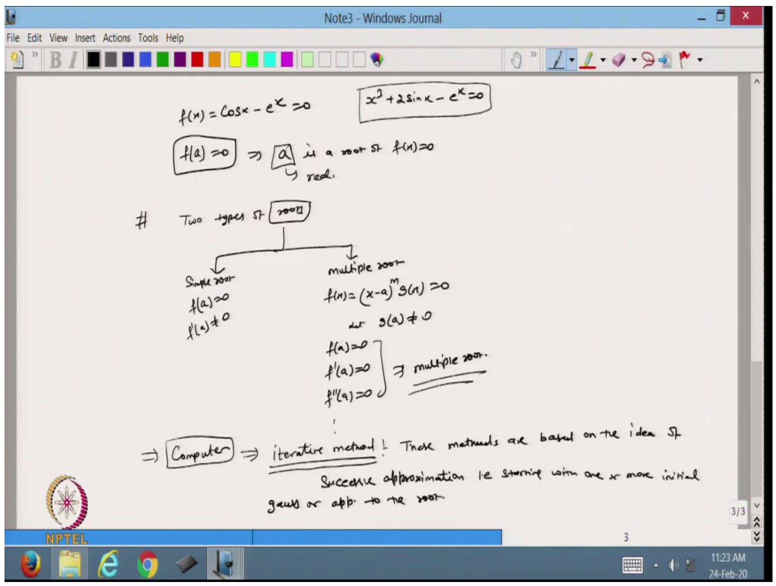
$$f(x) = \cos(x) - e^x = 0$$

So I want to find root x. So that is called a transcendental equation or I can have another equation x cube plus 2 sin x minus e raise to power x equal to 0.

$$f(x) = x^3 + 2\sin(x) - e^x = 0$$

So in this case also I want to find the value of x such that this is satisfied by that x. So this is the x we want to find and that we know that is called the root of the equation. So suppose if I get a value a here and that becomes 0, then I can say from here that a is a root of the equation f(x)=0. So now in this case, we are dealing with the real roots, so real means that is a real number. So how can we find that?

(Refer Slide Time: 10:59)



So roots are also different types. So there are two types of roots. So this is I can classify as simple root and this is multiple root. Simple root means that my $f(a) = 0$, $f'(a) \neq 0$.

So that is called a simple root but multiple root means that suppose I have my function

$$f(x) = (x-a)^m g(x)$$

So I can factorize my function into this form and that is equal to 0.

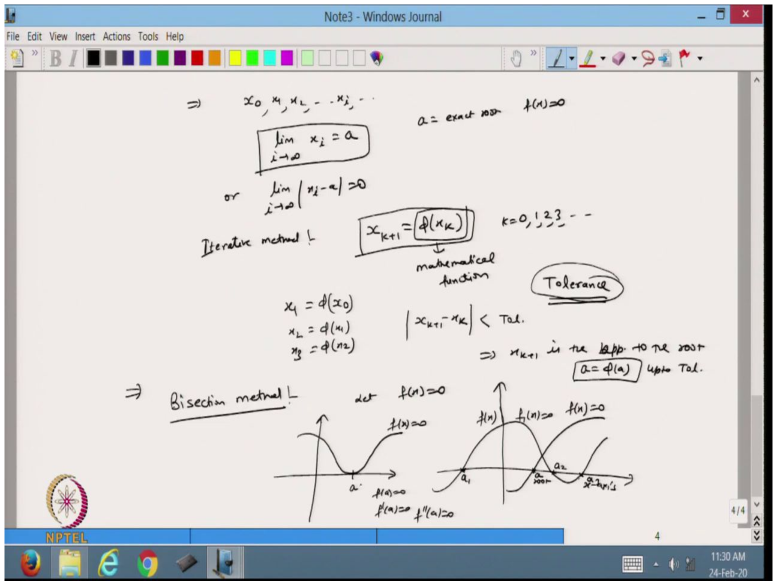
And let $g(a) \neq 0$. So in this case I, from here I can see that

$$f(a) = 0,\ f'(a) = 0,\ f''(a) = 0, \ldots,\ f^{(m)}(a) = 0.$$

So from here I can say this is a multiple root. So that is called a multiple root. So in finding the roots, the roots may be a simple root or a multiple root. So let us find out first to deal with simple roots.

So in this case we are going to deal with the computer and we know that in the computer we always give you the solution, either by direct method or iterative method. So we will start with the iterative method. Iterative method means that over the period of iterations, your accuracy is increasing. So that is called the iterative method. So you can write that these methods are based on the idea of successive approximation. Successive approximation that is starting with the, starting with one or more initial guess or approximation to the root.

(Refer Slide Time: 14:10)



So in this case we start this with the guess, initial guess, and then suppose I start with the initial guess that is x_0. Suppose x_1 is the next guess, x_2, x_3. So in this case, I get the sequence of approximation, and in this case, I will say that limit my I can say this as x_i so I can say that

$i \longrightarrow \infty, x_i \longrightarrow a$

So where a is my exact of the equation f(x) = 0 and all this x_0, x_1, x_2 are the approximation

$i \longrightarrow \infty, (x_i\text{-}a) \longrightarrow 0.$

Okay, so what will I do? So that is called my iteration, iterative method. So the iterative method, I can write a capital letter so the iterative method can be written in this form, so I have my

$X_{k+1} = \Phi(x_k)$, k=0,1,2,3, ...

So this is the form of the iterative method. So and what about this? This is my mathematical function, this one is a mathematical function, I will start with my, some approximation initial guess x_0. I will find the value of this, then I will call it x_1. So that is another approximation to the root of the equation or the method.

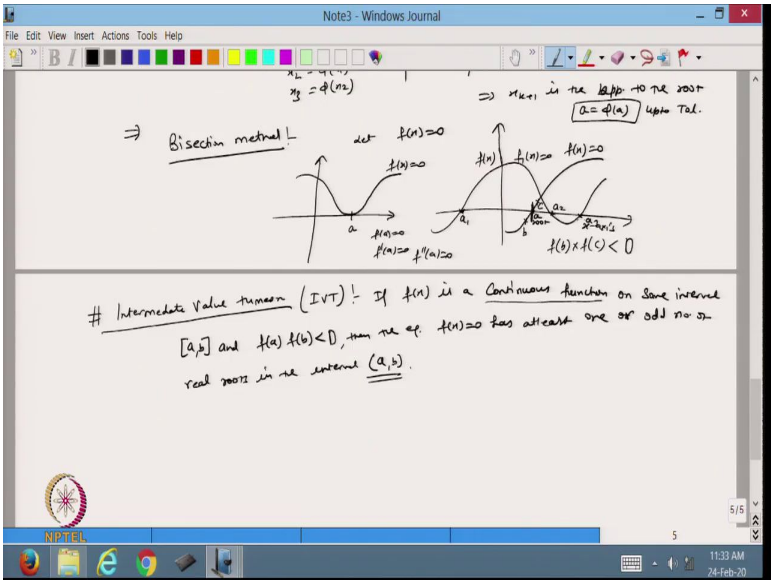
Then I will use this x_1 and with the help I will find x_2. Then I will find x_2, then I will find x_3 and so on. I will keep going like this one and I will try to find out what my (x_{k+1}-x_k) $\leq$ tol. So then I will say that no change is happening now if I get the new value of x and then we call that x_{k+1} is an approximation to the root.

Because in the, when I will reach the root, I will get my $a = \Phi(a)$. So that is the value of a root, because at that value, I will get the same value. So $x_k \simeq a$ is the approximation to the root up to the given tolerance. So that is the tolerance we have. So this is a tolerance and this is given by me to the computer because this process will keep going and we have to terminate this process. So this process will be terminated based on how much tolerance we need. So that is called my iterative process.

Now I will start with the method, the first method we are going to start and that is we call it the bisection method. So before this bisection method, I just want to discuss that let my f(x)= 0 is my equation.
So in this case that is my root. So this is my root of this equation, so wherever this function intersects with the x axis. So this is the x axis and this is my function value, the function f(x) this is y axis. So wherever this function intersects the x axis that is called the root or suppose I have another equation like this one. So in this case, I have multiple roots so x equal to this.

So in this case suppose this is for this another function, I can say that a_1 then a_2 then a_3. So in this case it is not a multiple. This equation and another function I can have $f_1(x) = 0$ is another equation and it has three roots here. So a_1, a_2 and a_3 are the real roots. My function f(x) has only one root and that is a and these all are the simple roots, so this is a simple root. Suppose I want to show that another equation in, suppose my function is there and like this function is like this and this is f(x)= 0.

So in this case this is my function a, so that it is a multiple root because if you see from here then this function will intersect with the x axis and in fact it stays here, x axis for a small time. So in this case my function, if I see that $f(a) = 0,\ f'(a) = 0,\ f''(a) = 0$
So in this case my a is the root of this equation, but it is a, it is called the multiple root of this equation. So that is how we can find the graphically we can show the root of the equation.
(Refer Slide Time: 20:57)

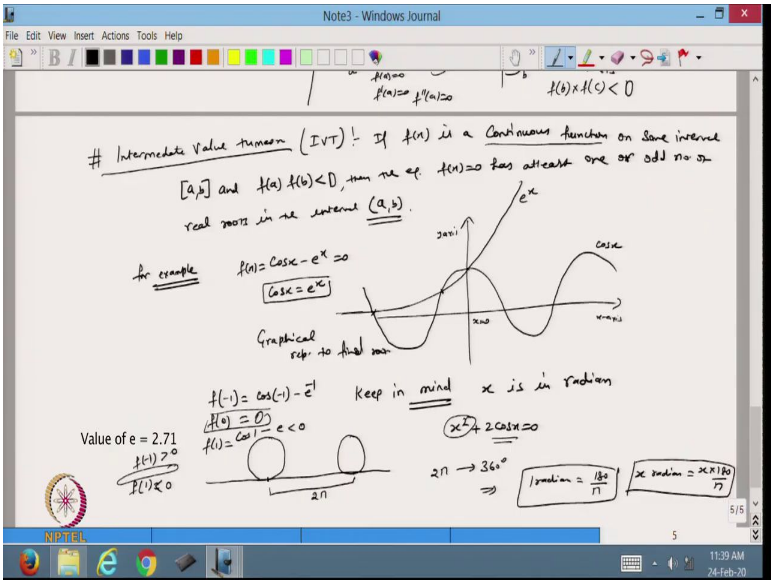


So in this case, you can see that either, so this is my polynomial. Suppose If I take this polynomial so it has an odd number of roots here. It is one root only here. So from here, I can write one theorem and that is called the intermediate value theorem. So that is the intermediate value theorem, I would call it IVT. So what is this?

This theorem says that if f(x) is a continuous function we find on some interval [a,b] and my f(a) f(b) < 0, like here we can see it from here. This is the root. So suppose at this value, if I take a point here, some point here and some point here. Here f value is suppose I call it b and c.

So in this case I can say that my f(b)<0 and I here multiply by f(c) that is the value positive. So it will always be less than 0, negative. So in this case, if I have f(b) f(c)<0 ,then I can say that there is a root lying there in between this one. So in the same case, if the function is continuous, so that is my sufficient thing I should have the f(x) is the continuous function on some interval closed interval ab.

And suppose f(a) f(b)<0 then the equation f(x)=0 has at least one or odd number of real roots in the interval. So there is an open internal. So that is the interval we have. So we should have a close interval and then we will have at least one real root that is lying inside the [a b]. So that is why we go for the open interval. So this is my intermediate value theorem. So this is a very helpful theorem because based on this one only we will try to find out where the root lies.
(Refer Slide Time: 24:10)



So for example, suppose I have this equation cos(x) - $e^x = 0$ and I want to find the root. So there are two ways to find out the root. I will write this equation as cos x = e^x. So in this case what I do, I will plot this. So suppose my cos x, I know that this is the value of cos x like this one.

So this is my cos 0 - $e^0 = 0$. $e^0=1$. So it is, I can call it like this one. But this is the function I can plot, so from here, you can see that this is my x = 0 and this is x-axis and this is y-axis. So this is my cos x and this is an exponential function. So in this case, I know that at least at x = 0 my cos 0 = 1 and $e^0=1$.

So this is the one of the roots here. But this is also another root we can define, this is another root we can define. So in this case I can have multiple places where we can have the roots of this equation. So using this one, plotting the two graphs of the left side function and the right hand function and see where they intersect, that is a place where we can have the root.

Another way is that, so this is a graphical representation, representation to find root. So another is that what I will do, I will try to find out what is the value at maybe minus 1. So this will be cos(-1) - e^{-1} . So this cos(-1), so whenever we do such a type of calculation, you can just keep in mind, keep in mind that x is in radian.

Because in this case, if my x is a real number, so whenever we are dealing with some trigonometric function and some x like I have a $x^2 + 2\cos x = 0$ so in this case, it is automatically that x will be in the radian because here x is a real number. So I cannot take x as a degree. So it is understood that x will be in the radian form.

So and that we know what is the way we can find this radian, that is we know that we have to convert whenever we are dealing with the real number, we have to convert my degree if the angle is given in the degree to the radian form, because we know that, suppose I have a unit circle and this is my suppose x-axis, so if I move this unit circle and rotating this in the 360 degree, so suppose after rotating, it will reach here.

So the distance covered it will be 2π. So I know that the $2\pi = 360^0$. So I can say that the 1 radian is equal to 180/π. So if I have to do the calculations somewhere, my x radian will be x into 180/π. So this one we have to convert to deal with any x which involves a trigonometric function.

So in this case what we will do? We will find out this value then after at 0. So in this case maybe this would be 0. So by chance I go to the root here but I can find the value of f(1). So it will be cos(1)- e. So I know that the value of e is 2.71828, and cos cannot be more than 1. So this is definitely less than 0. And so if I, from here I will get that my value of f_1 is supposed to be negative.

And this value is if you see f (-1), it can be positive also. So if it is positive and it is negative, then I can from here I can say that there will be a root line between -1 and 1 and that is 0 coming in this. So by using this way with the help of the intermediate value theorem, I can say that in which sub interval the root may lie and we will look for that root in that sub interval.

So in this lecture we have discussed about that first topic to find the root of a nonlinear equation, that is, how we can say that in which sub interval the root may lie of a given equation of x equal to 0 using the intermediate value theorem and we will take the help of this one to find out the root for a given equation in the next lecture. So thanks for watching. Thanks very much.