Scientific Computing using Matlab

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Lecture 63

Convergence and Zero Stability for LMM

Welcome to all of you in the next class of this course. So, first let us recall what we have done in the last lecture. In the last lecture, we have seen Linear Multi Step methods, how to drive Linear MultiStep methods. And in fact, if any Linear Multi Step is given to us, we can also see how we can prove that the Linear Multi Step method is consistent or not. We have also done other kinds of analysis, which means how to find out the order of the Linear MultiStep method in the last lecture.

We have also seen that the Linear Multi Step method is a generalization of a Taylor series method in what sense it is called generalization of a Taylor series method. Because the Linear Multi Step method connects the value of y(x) with its derivative only, while in Taylor series methods we have higher order derivatives term more than first order is also there. So, in that sense Linear Multi Step method is a generalization of a Taylor series method which we have already discussed in the last lecture.

But, particularly in the last lecture, we have seen a special case of a Linear MultiStep method, a special case in the sense that we had focused only on Two Step methods. But now the question comes up how we can generalize this to the k step method. That is what we are going to discuss in today's lecture.

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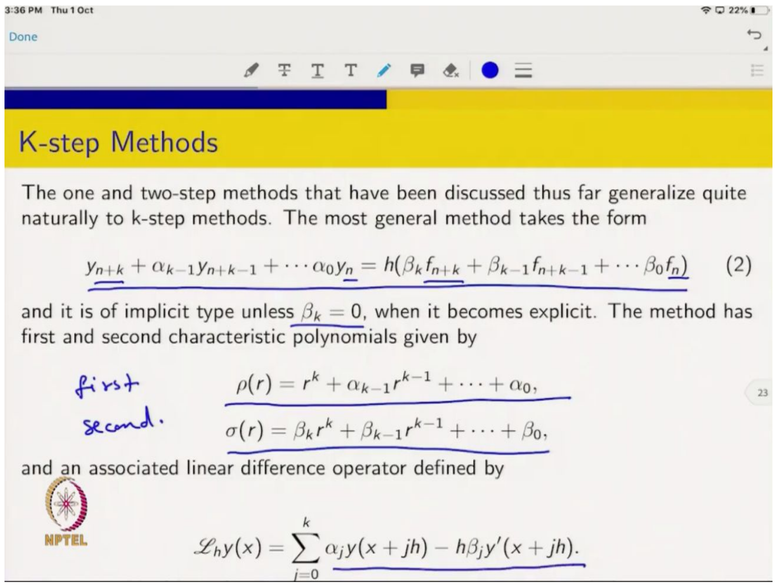

So, k-step methods, the One and Two step methods that have been discussed so far. Generalize quite naturally to k-step methods. That is what I am presenting now, the most general method takes the form of the following way. So, this you can see here it is n here it is n+k. Similar things are here at n+k.

So, they are how many when there are two terms involved means, we go from y0 to y1 and then we call it as a One step method means Two terms One step method. So, they are here in n+k one terms, k step methods that are quite natural as well. And as we have already seen that in one step as well as in two steps, we have been talking about implicit and explicit.

So, here again, if $\beta_k = 0$ then it will be called explicit method otherwise, it is of implicit type. The method has first and second characteristic polynomials also in the same fashion which we have already done in case of a Two step Linear Multi Step method. So, this will be my first characteristic polynomial $\rho(r) = r^k + \alpha_{k-1}r^{k-1} + \cdots + \alpha_0$.

And $\sigma(r) = \beta_k r^k + \beta_{k-1}r^{k-1} + \cdots + \beta_0$ this will be my second characteristic polynomial. So, this is first and this is Second and at the same time, let me also introduce what will be the associated linear difference operator in this case. So, the associated linear difference operator just becomes the difference of the left hand side and right hand side this way.

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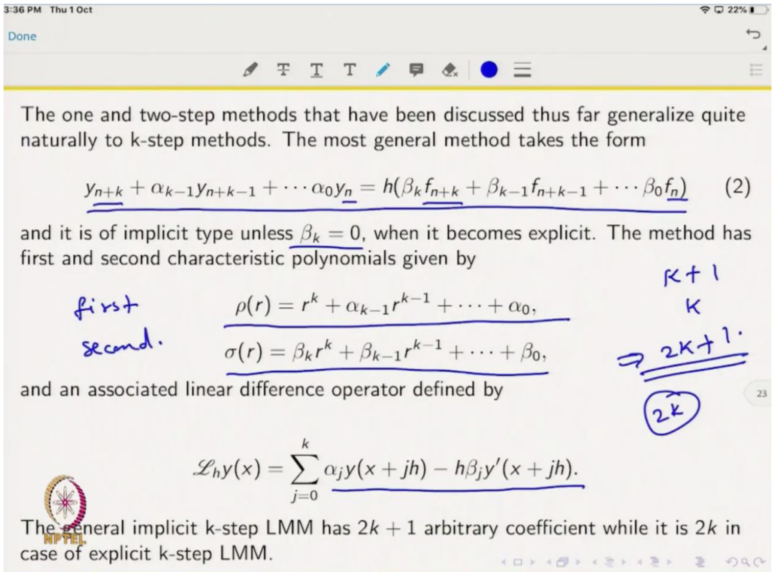

So, as we get the general implicit k-step, Linear Multi Step method has 2k+1 arbitrary coefficient of course, in the right hand side how many coefficients are the k+1 in the left hand side there are k coefficients, so total number of coefficients are 2k+1 in case of a implicit and in case of explicit $\beta_k = 0$, so of course, it will be 2k. That is what we can observe very easily.

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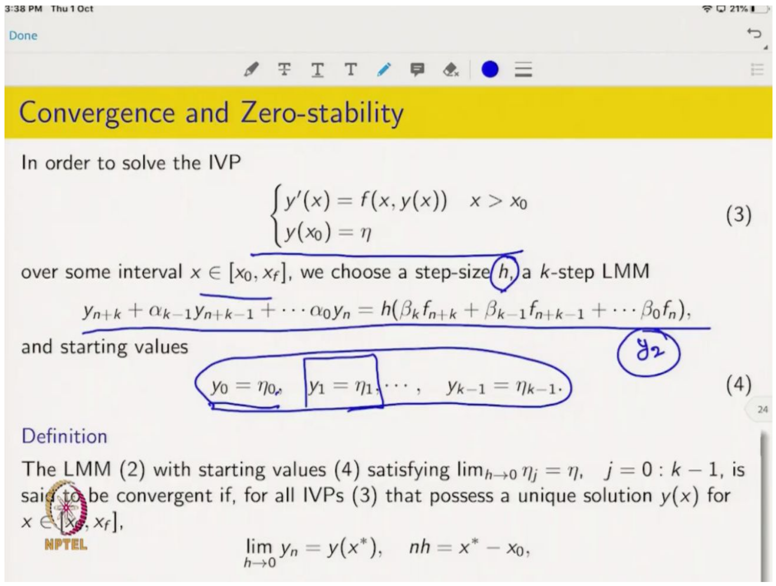

Now, let me talk about Convergence and Zero stability. Convergence of a Linear Multi Step method we have already seen earlier in case of Euler method as well as we have proved one theorem that convergent LMMs are always consistent. Zero stability is the concept which will be new for today's lecture. So, in order to solve the following Initial Value Problem IVP for some interval we choose a step size h okay and k-step Linear Multi Step method will take the following form with the following starting values.

So, why are we defining these many starting values? So, here is the point which you can observe very carefully: what is the disadvantage of Two step, Three step or Multi Step method because they are not self starting they are not self starting initial condition is given to us at $y_0 = \eta$. So, to come from y_0 to y_1 we have to use some One step methods otherwise there is no way out.

So, if we are starting with Two step methods, so we should know y_0 and y_1 only then we can determine y_2 and that was also the drawback of one scheme, which we saw in case of an Adam Bashforth method, which was Two step methods. So, that is not the self starting method. At that point of time I said I will discuss later what is the disadvantage of Multi Step methods. So, in a nutshell you can say that the Linear MultiStep method of more than one step is not self starting, means they are not self starting. So, for initial from initial time to come to y1 you have to use another single step method.

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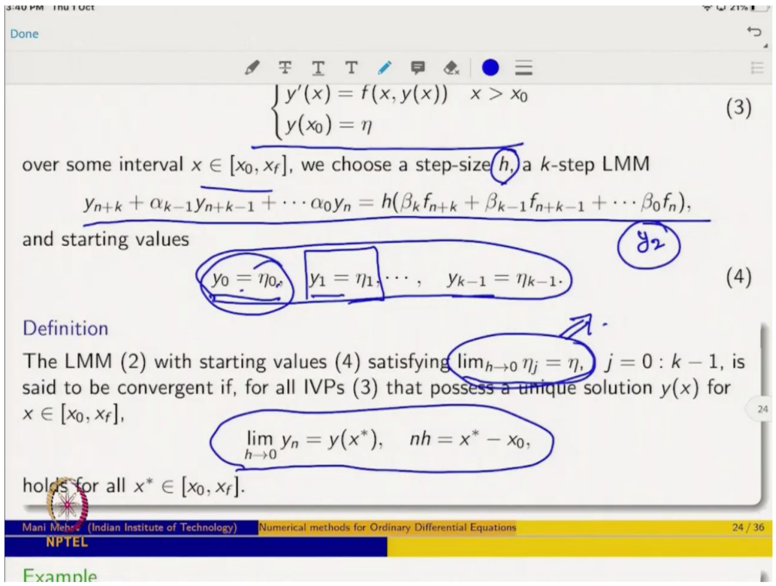

So, now, let me see the definition of a Convergence of a k-step method. So, the Linear MultiStep method with the starting values for satisfying this means all, all the approximations which you would have made at this at this of course, this is exact because you do not make any approximation to the initial conditions set to be convergent provided all the approximation converges to the initial conditions, that is what we have seen.

If for all IVP three convergent for all that poses a unique solution for all x. So, this condition is the same which we have seen earlier also, when we were looking at the definition of a convergence of a single step method, the only additional condition is here that all the initial approximations which you are making should also converge. At $j \rightarrow 0$ here as j, so, η_0, η_1, η_2 all should converge.

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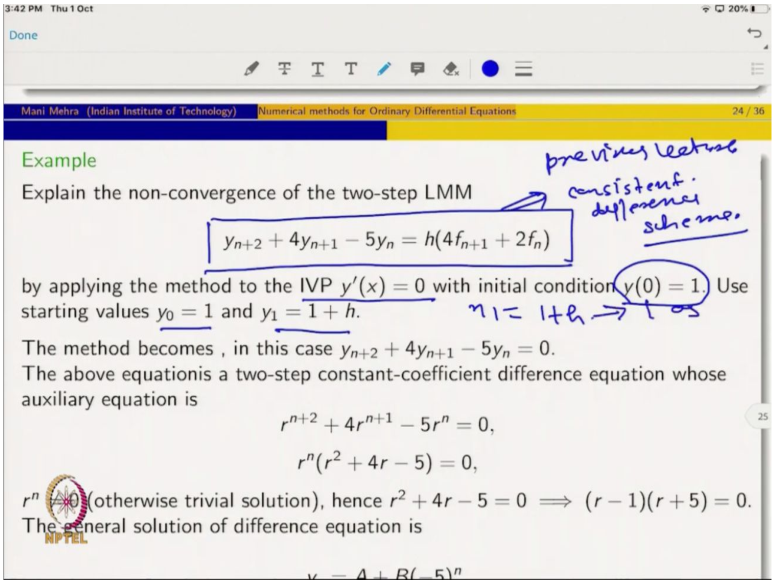

Now, let us discuss one another example, where we can prove that this Linear MultiStep method is not convergent, how can we prove? So, That is what we are going to see here the example is in front of us, if you recall your previous lecture, you can observe that in previous lecture for the same example, we have proved that this method is consistent we have proved in the previous lecture, that it is an example of a consistent differences scheme, consistent differences scheme.

So, what I am going to say here, if it is a consistent difference scheme, but still I will prove that it is not the convergent scheme. So, in the last lecture, we have proved that the convergence method is consistent. Consistency is a necessary condition for convergence, but converse is not true. That is what we are going to show you with the help of the following example. This scheme is consistent with what we have already proved in the last lecture and we will see what else which is not true, because of that it is not convergent.

So, we are applying this method to the following IVP with the initial condition that we are using the following starting values. So, of course, one assumption you could make that $\eta_1 = 1 + h$ is also tending to1 as h tending to 0.

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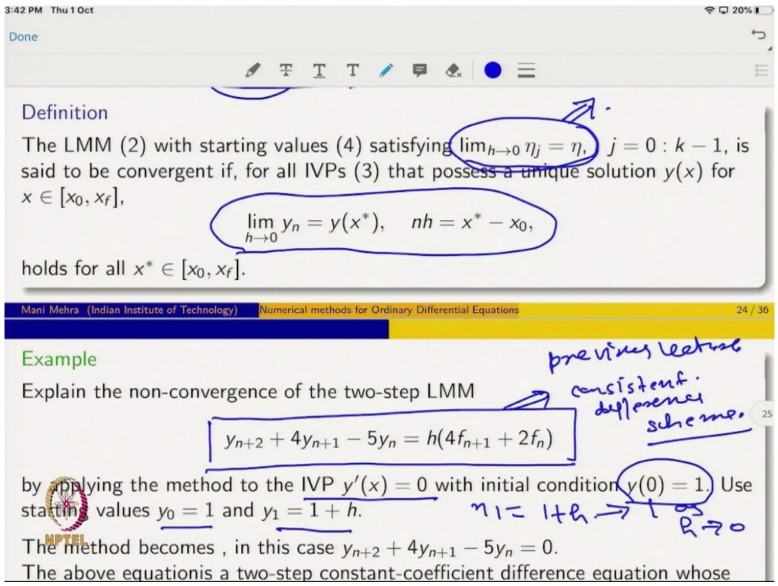

So, the initial assumption which we made here is true.

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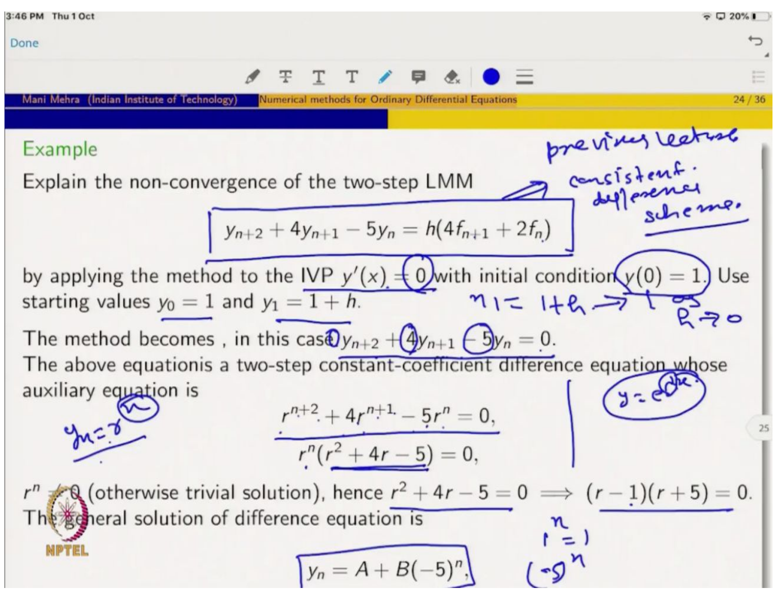

That is the first point to check after that you will start checking other things. The method in this case is this, why? Because the right hand side for this initial value problem is 0. So, That is why it will become $y_{n+2} + 4y_{n+1} - 5y_n = 0$ because f_{n+1} is also 0 f_n is also 0.The above equation is a two step constant coefficient difference equation whose auxiliary equation is this. What do you mean by this?

I am going to explain to you. So, this is basically constant coefficient difference equation like, so far you must have seen constant coefficient differential equation when you call it as a constant coefficient differential equation when the coefficients are constant in the similar way, when the coefficient of this term, this term and this coefficient of this term is 1 it is a constant this is called constant coefficient difference equation.

And how can you recall how you solve differential equations? We solve a differential equation because we substitute initially. I am concentrating only on homogeneous difference, because this is an example of a homogeneous differential equation. So, similar way I will also ask, ask you to recall how you solve a homogeneous differential equation.

You substitute $y = e^{\lambda x}$ in such a way that it is a non trivial solution of course, and then you try to find out λ. So, here why you substitute $e^{\lambda x}$ there, because the derivative of exponential is again in the form of exponential. So, in case of a difference equation what we do we substitute the solution $y_n = r^n$ That is what we are going to do.

So, which is a non trivial solution, so, here our aim is to find out n. Earlier in the case of a differential equation our aim was to find out this λ. So, we substitute this value here, so this will become $r^{n+2} + 4r^n - 5r^n = 0$ Let me take out r^n common then this will become this r^n is 0 otherwise, we will end up with a trivial solution that is what I have already said then let us factorize this polynomial, so which will be this.

So, the general solution of a difference equation will be this. So because r is 1, so, if I substitute here, so, one solution is 1^n which is 1 another solution is $(-5)^n$. So, and then we take a linear combination of these two linearly independent solutions. The same thing which we do when we try to find out the solution of a homogeneous differential equation. So, the concept remains the same.

So, finally, the general solution I can call it also as a general solution because right it is a homogeneous differential equation, otherwise I would have added this is the origin of a homogeneous equation plus a particular solution like the same way we were doing for in differential equations. So, this is a general solution of that difference equation where A and B are arbitrary constants.

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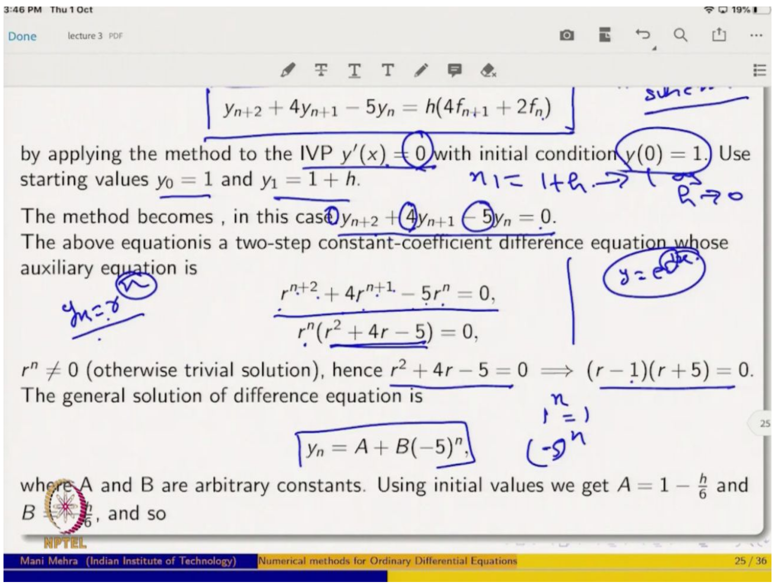

So, now, the question is how to find out these arbitrary constants of course, there are two initial values which were given to us $y_0 = 1$ and $y_1 = 1 + h$.

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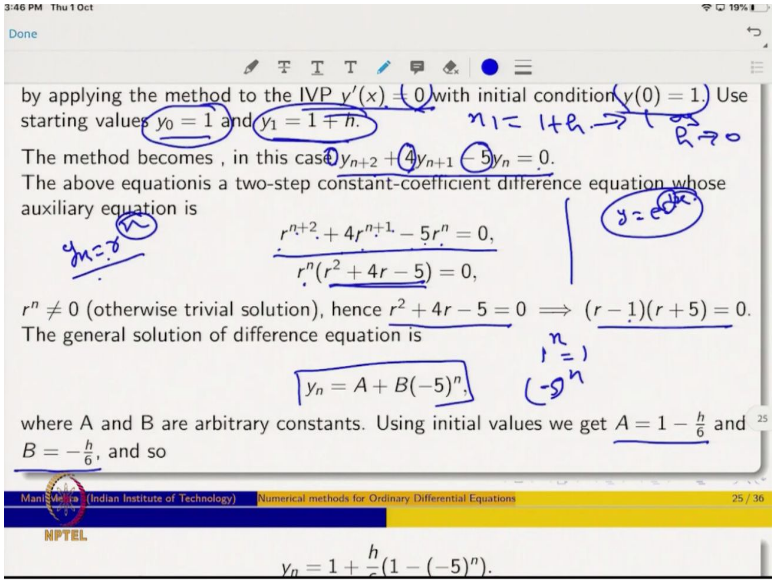

So, this, using these initial values this and this you can see, we can get the value of A as follows and B as follows.

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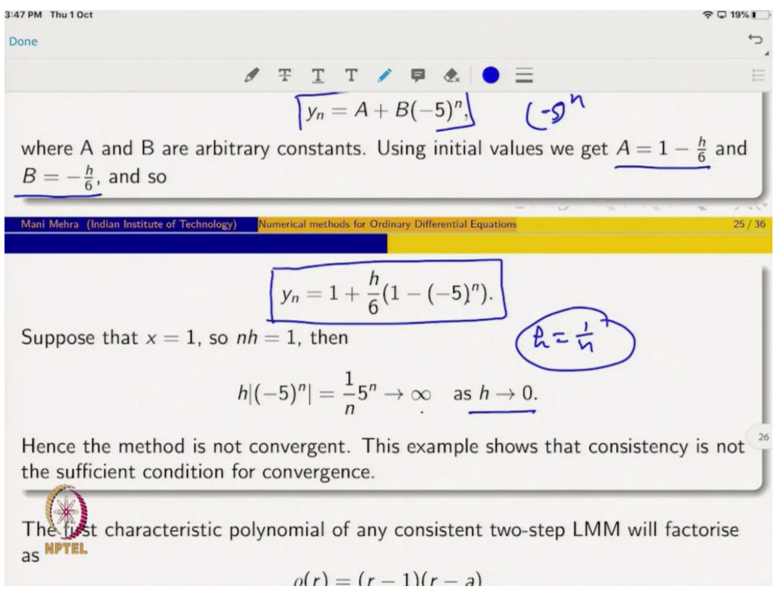

So, finally, the solution to the difference equation will be this. So, now, if I substitute x = 1 , so, basically nh = 1 in that case, this h if you take insight, so nh is 1 so basically in that case, h will become 1/n. So, this will tend to infinity as h tends to 0. So, what we have observed is that the solution of the difference equation is tending to infinity.

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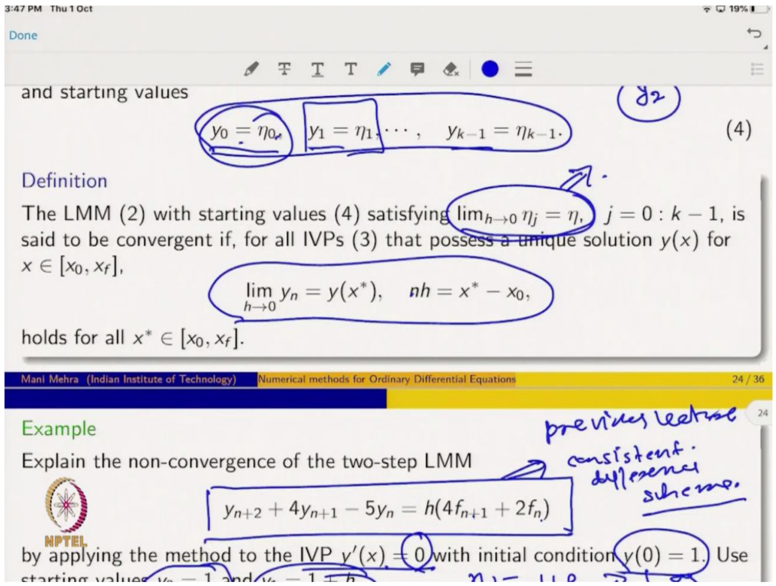

So, you are not getting the convergence, hence the method is not convergent, because the exact solve method should be called convergent if this condition is satisfied, not the infinity that we already know.

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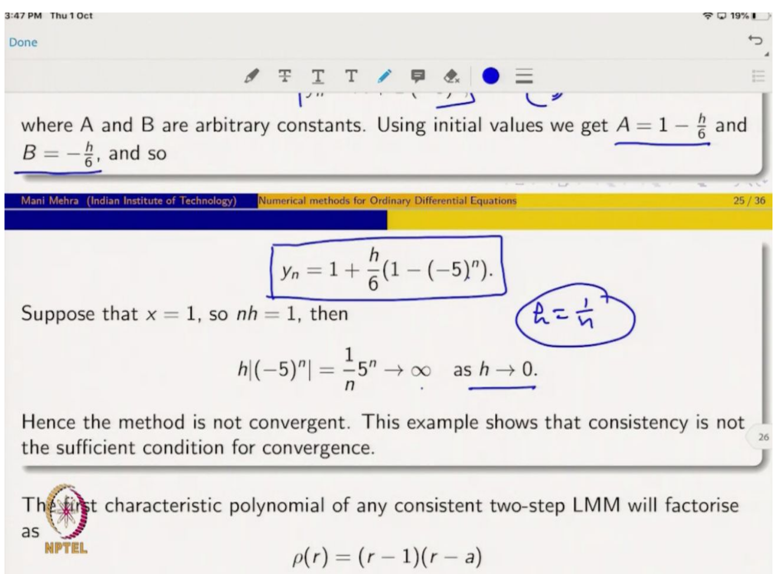

So, this example shows that the consistency is not the sufficient condition for convergence, because, in the last lecture, we have seen that this difference equation was consistent, but now, we have seen that but it is not convergent.

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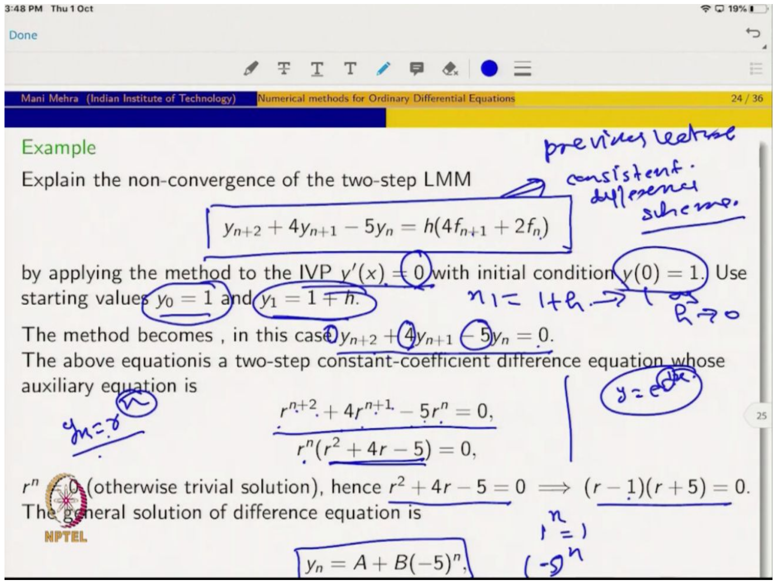

Because what because you can easily get the exact solution of this initial value problem.

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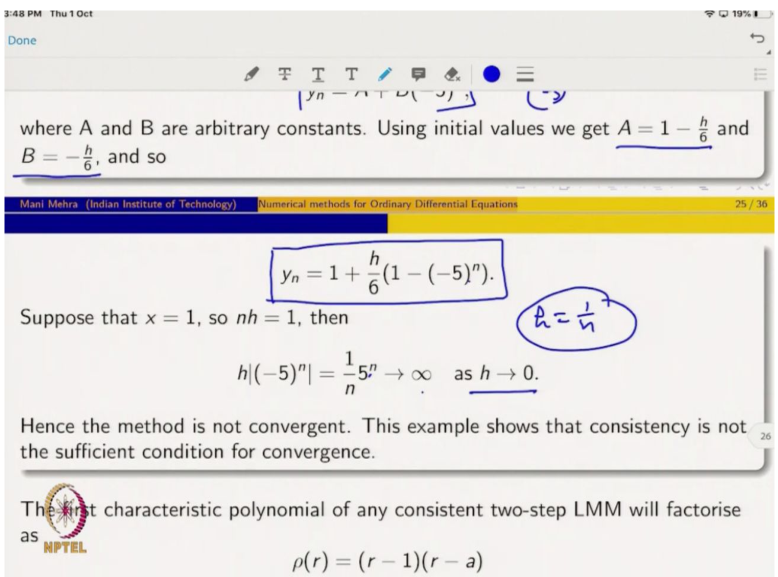

And you can see that $h \to 0$ the solution of this difference equation is not converging to the solution of the differential equation. So, this is an example where you can see that consistency is not the sufficient condition for convergence. So now, we have to observe y why it is not convergent? It is basically if you observe very carefully we are not getting convergence because

5^n tends to infinity as $h \to 0$. And it is happening because 5 is a number which is greater than 1.

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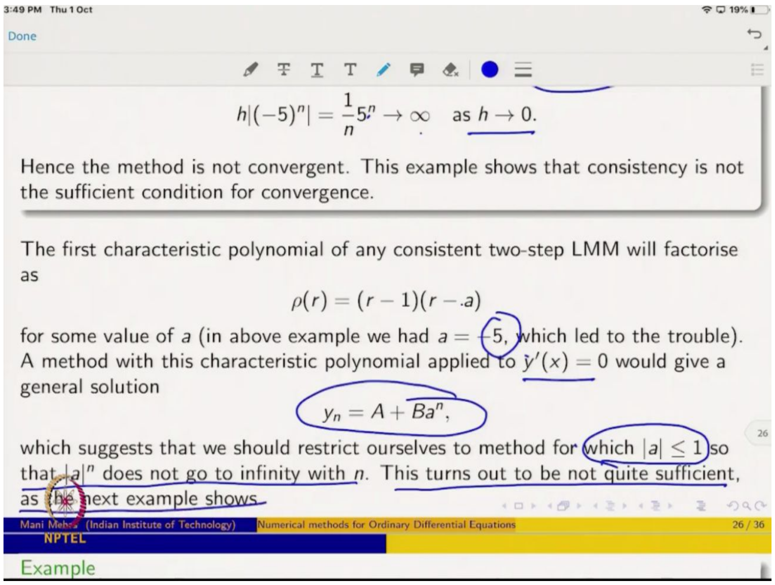

So, that is why keeping that thing in mind what we are doing the first characteristic polynomial of any consistent two step LMM will factorize, of course as such (r-1)(r-a), and in this case, we have seen a = -5, which is leading to the trouble. So, a method with this characteristic polynomial applied to the following initial value problem would give a general solution that is what we have seen.

We suggest that we should restrict ourselves to methods for which mode a should be less than 1 less than equal to 1. So, that mode a does not tends to infinitive with n this turns out to be not quite sufficient as the next example shows. But right now, our solution is that that mode a should be less than equal to 1 in that case, we may get the convergence, but we will see in the next example, there is something more.

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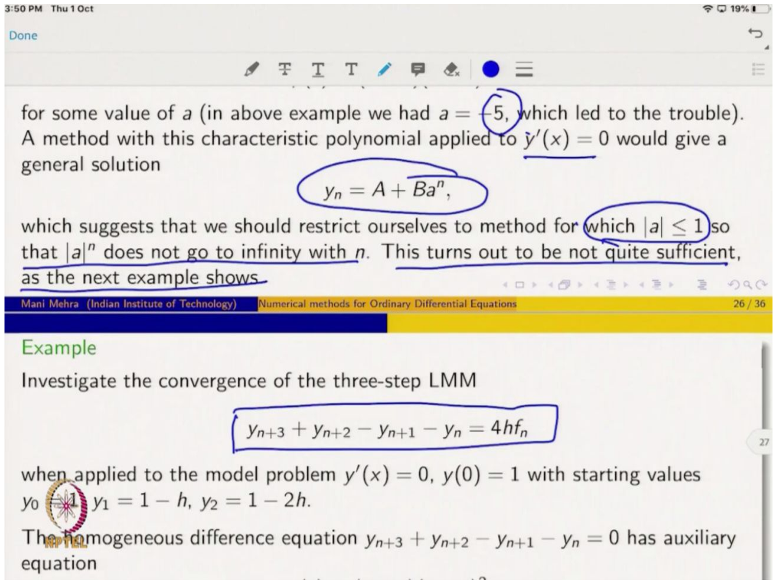

So now, let us see the next example, which is this.

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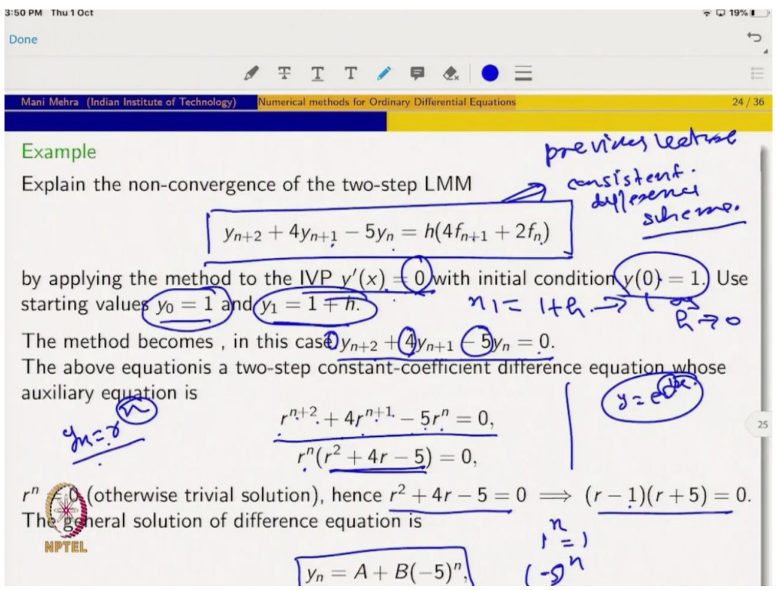

So, if I look at this example, this is an example of a Two step method because n, n+1 and n+2.

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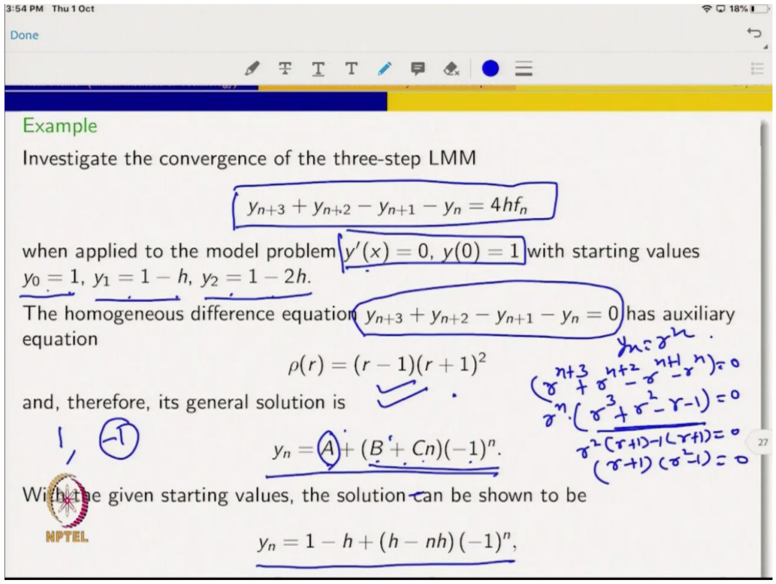

And this is the example of a Three step method. This is the first time when we are looking at any form of a Three step method. And of course, the initial value problem remains the same on which I wanted to apply my numerical method. So of course, this is a Three step method. So I have to take three starting values, $y_0 = 1, y_1 = 1 - h, y_2 = 1 - 2h$ So, of course, I choose this in the following way that h, $h \to 0$, $y_1 \to 1$ That is what is happening.

Similar thing y2 should also tends to 1 as $h \to 0$. That is what the initial face of the convergence results which we should try to prove, of course, I have chosen in such a way. $y_1 = 1 - h, y_2 = 1 - 2h$. And again, because I am applying to this initial value problem, so the homogeneous difference equation will become this. Again, how to find out a solution that I have already made clear in the last example, I will substitute $y_n = r^n$.

And then this will be $r^{n+3} + r^{n+2} - r^{n+1} - r^n = 0$, let me take out r^n common then $r^3 + r^2 - r - 1 = 0$. So this big becomes so what is the factorization of this polynomial? So $r^2(r+1) - 1(r+1) = 0$ which is $(r+1)(r^2-1) = 0$. So, which is the same as this. So, in case of that, this is a first characteristic polynomial.

So, a general solution of this can be written in the following way. So, how? Because if do remember in case of a differential equation, if we have repeating roots we say C+xd and then $e^{\lambda x}$, so similar things we are saying because 1 root is it has three roots, the root 1 which has a simplicity 1 and algebraic multiplicity has a 1 and the root -1 whose algebraic multiplicity is 2.

So, this term corresponds to 1 and this term corresponds to -1. So, here we are writing $(B+Cn)(-1)^n$, of course, the difference equation itself is an established field. So, here I am just trying to tell you the very way to find out a solution in terms of a very basic difference equation because the aim of this course is not that, but of course, when we solve differential equations with the numerical method, we end up with a difference equation. So, we should have some knowledge of a difference equation as well.

With the given is starting values the solution can be shown to be this of course, Now, the question comes up how to find out these coefficient A B C these coefficient I can also find out with these condition $y_0 = 1, y_1 = 1 - h, y_2 = 1 - 2h$

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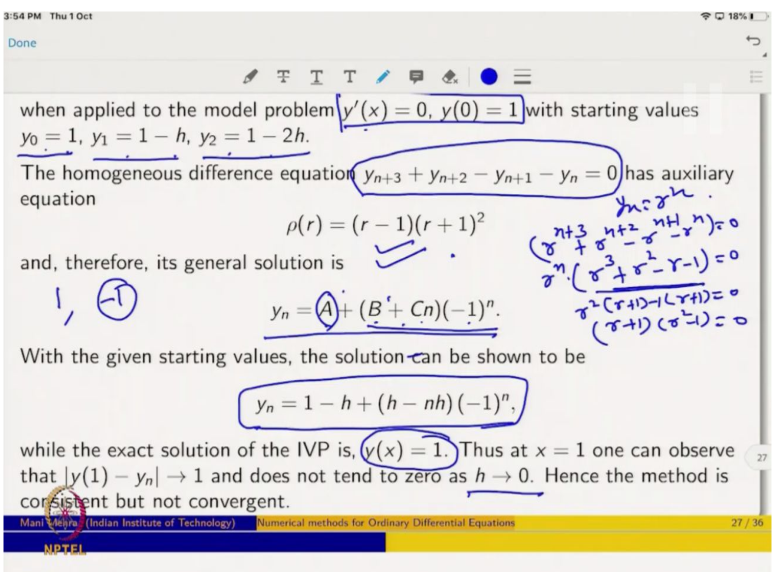

and using those initial conditions finally, the solution of the difference equation will look like this. So, again the solution of exact is, the solution of initial value problem is this exact solution of initial value problem is this. Therefore, at x = 1 we can observe that $y_1 - y_n$ is tending to 1 which you can see because if you will subtract this to 1 you will get the because the other terms will get 0 and does not tends to 0 as $h \to 0$, hence the method is consistent but not convergent.

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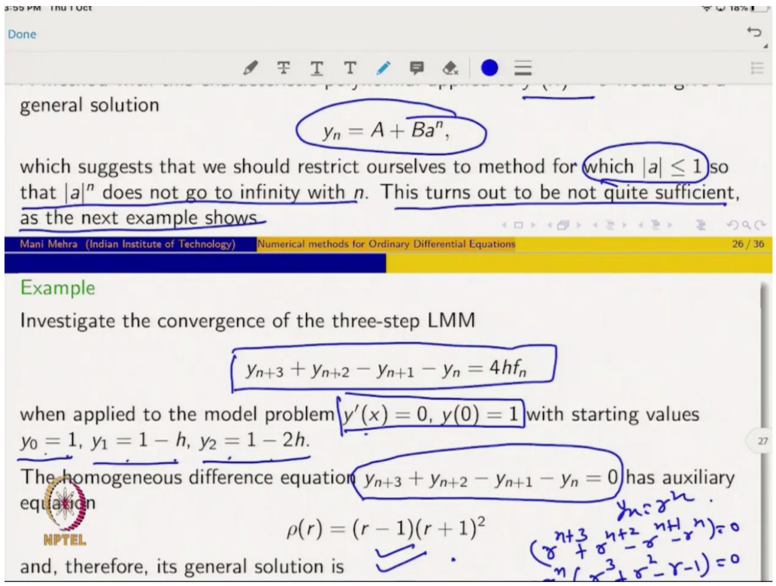

So, the observation from the last example did not go well in the following example as well because we tried to put the condition that mode a should be less than equal to 1.

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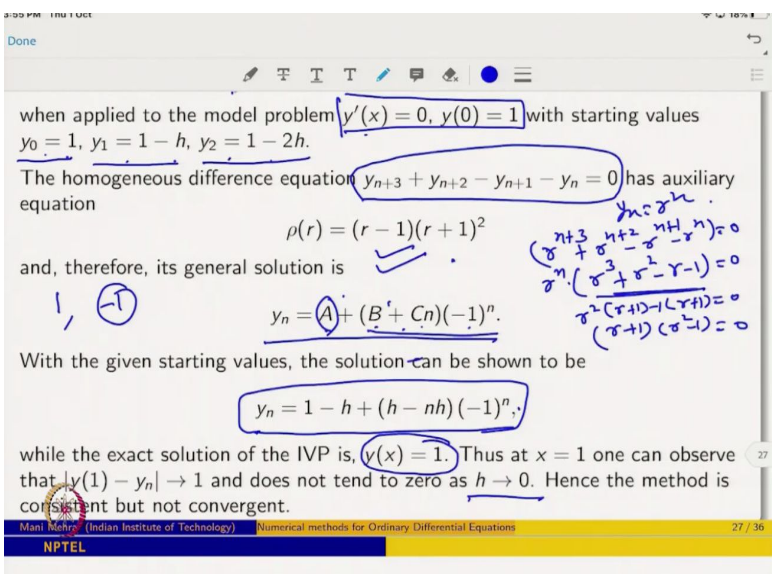

But this is the example where all the roots are 1, but still now, you should observe why this is happening, why it is not tending to 0 as $h \to 0$.

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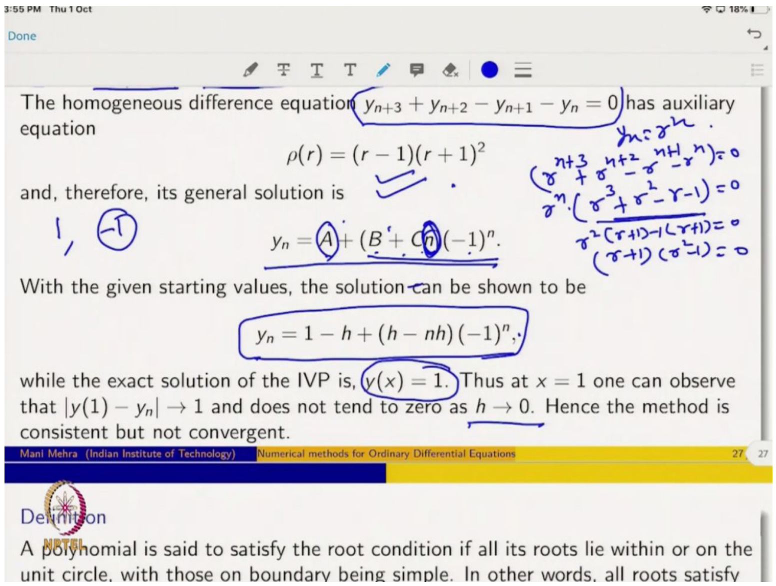

So, again it is happening let because minus 1 has a algebraic multiplicity of 2 and because of that this term has come n, this n term has come and because of that we are not getting the convergence. So, that is observation.

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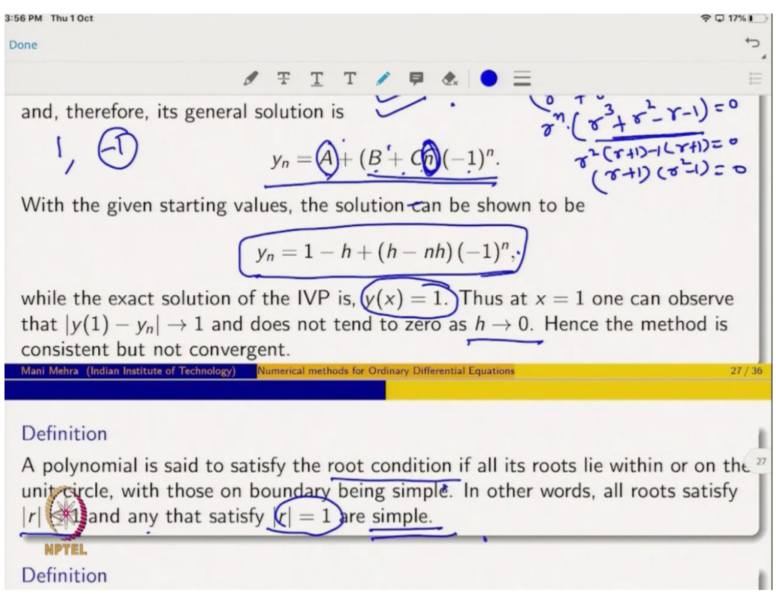

And based on these two observation which we are going to formalize in terms of a definition that a polynomial is set to satisfy the root condition if all the roots lie within or on the unit circle with those on boundary being simple, what do you mean by all the value of all the roots should be strictly less than 1, but if it is at the boundary, it means r = 1 then it should be only simple root means it is multiplicity should be only 1, that is the meaning.

In other words, all roots satisfying $|r| \leq 1$ and any root that satisfies these things is simple. Okay, simple means, whose Algebraic multiplicity is 1 which does not repeat. So, why we are formalizing this definition of a root condition, because it plays an important role later on formalizing the concept of Zero stability and Zero stability is something which is required for a consistent differences scheme to say that it is a convergent difference scheme.

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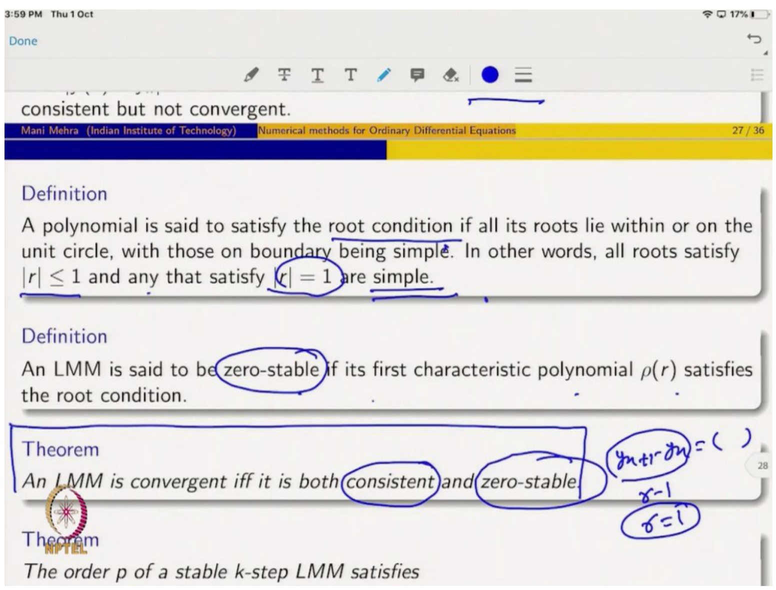

So, the Linear MultiStep method is said to be Zero stable if the first characteristic polynomial row r satisfies that root condition. So, again, we are formalizing the definition of Zero stability with the help of first characteristics polynomial in the following way that if first characteristics polynomial satisfy the root condition, it means that all the roots of your first characteristics polynomial should be less than equal to 1 and if there is any root whose value is 1 it should be only simple.

So, with this notion of a Zero stability, we can prove one theorem of course, in this course, we are not looking at the proof of this theorem, which you can see in any advanced book on numerical methods for ordinary differential equations or in any advanced course, but as far as this course is concerned, we are not looking at the proof of the theorem, but, the essence of that theorem should be clear to you, once the statement is clear and Linear MultiStep method is convergent if it is both consistent and Zero stability.

So, that is what we would keep saying that something more is required, when you can say that consistent differences scheme is convergent and what that thing more is to Zero stability. So, I hope consistency is clear to you Zero stability is clear to you. And, you know, when we were dealing with the Taylor series method, we have not paid any attention at all to Zero stability, why? Because, if you look at the form of a Taylor series method, it is always in the following way yn minus yn is equal to something.

So, what is the first characteristic polynomial in this case? r-1, so, this means root is this, so it always satisfies the root condition, that is why we were not bothering at that point of time.

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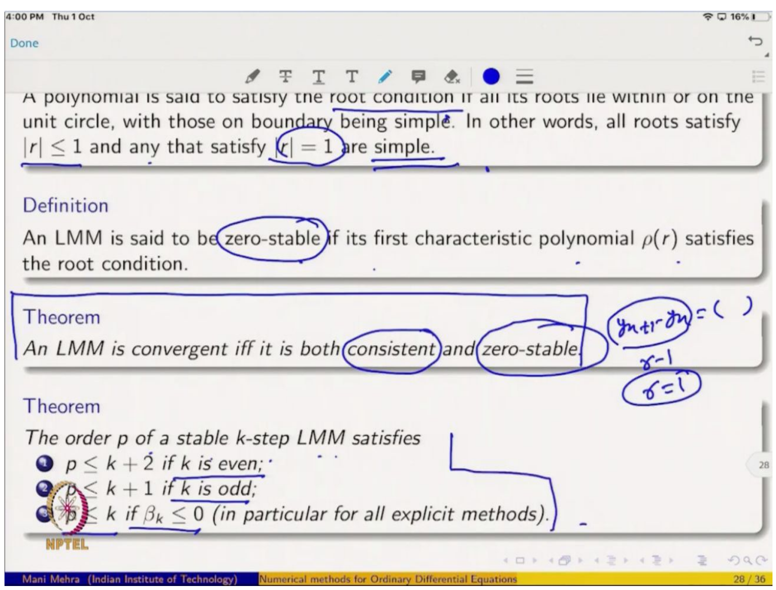

So now, when another theorem, which says that the order p of a stable k step Linear Multi Step method will satisfy the following inequality, $p \leq k+2$ if k is even, $p \leq k+1$ if k is odd, .

moreover $p \leq k$ if $\beta_k \leq 0$ In particular, for all explicit methods, because in case of explicit method $\beta_k = 0$, that is what we have already seen.

So, this theorem is telling that there is a relationship between the order of the method and the step of the method. It is not that once you say that with the k step method, I can get the desired order. No, it is not the case because this theorem limits that thing.

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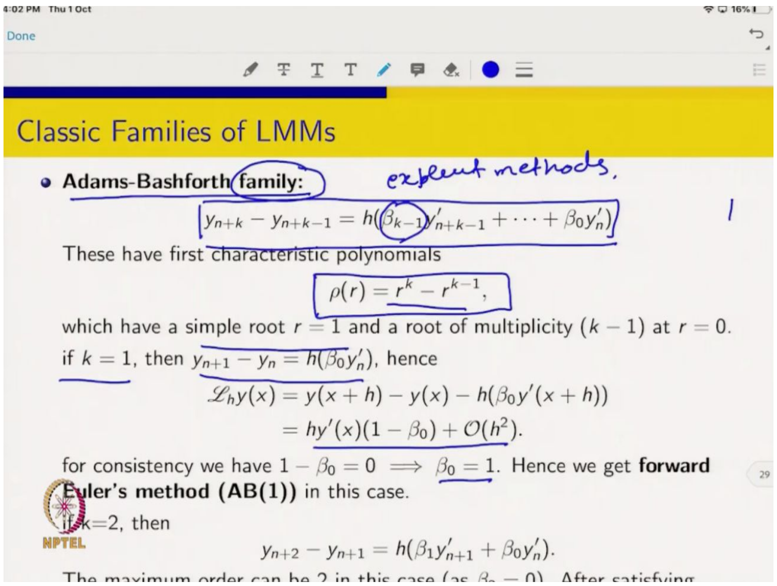

Now, let me give you the more formal way of introducing this Linear MultiStep method. The first category of the methods which we have already seen is Adam Bashforth family, or some people call it as a formula, methods, rules. So, it is up to you how you want to call it. The form of Adam Bashforth Family will be in the following way. That is what we have already seen. Of course, here $\beta_k = 0$ That is why we are starting with β_{k-1}, because this comes under the category of explicit methods.

The first characteristic polynomial corresponding to Adam Bashforth Family will be this $r^k - r^{k-1}$. So, this has only one simple root r = 1. So, of course, if this is the case Zero stability is always satisfied in the case of an Adam Bashforth Family that is what we wanted to observe. So, if k = 1, this is the case and the associated linear difference operator will be this and for consistency we get $\beta_0 = 1$ hence we get forward Euler.

So, basically we are constructing in such a way that it will be always consistent and we are choosing the coefficient in such a way, let me again repeat my statement. We are making this template in such a way that first order characteristics polynomial will satisfy the root condition; it means Zero stability is always satisfied.

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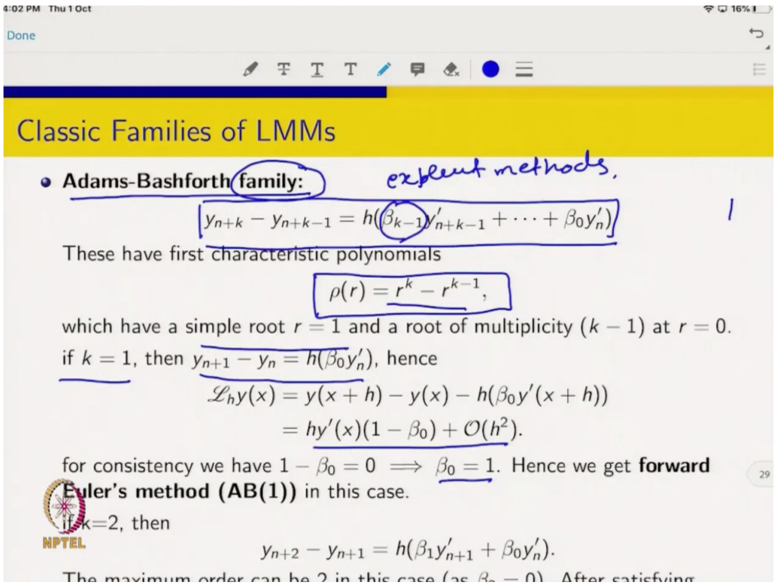

And we are choosing the coefficient in such a way that the scheme is consistent. So, it means we get a forward Euler method which we can from the construction itself we can say it is consistent and Zero stability. So, if you have proved these two things you can say it is convergent also. Though in the first lecture of this part of the course, we have already proved separately also how you can prove the convergence of the Euler method without going into the consistency and Zero stability concept.

So, now you can prove in two ways one is to prove directly another is with the help of consistency and Zero stability and then with the help of the following theorem which you have seen here,

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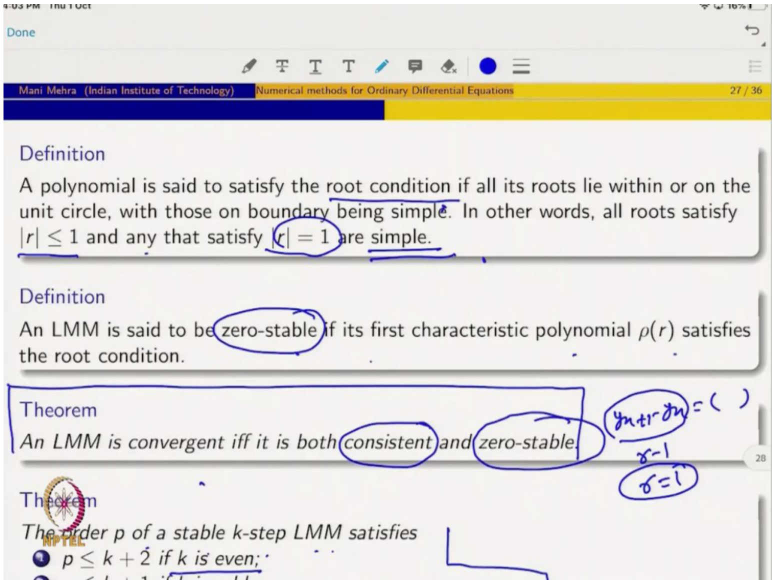

That Linear Multistep Method is convergent if and only if, if it is both consistent and Zero stable.

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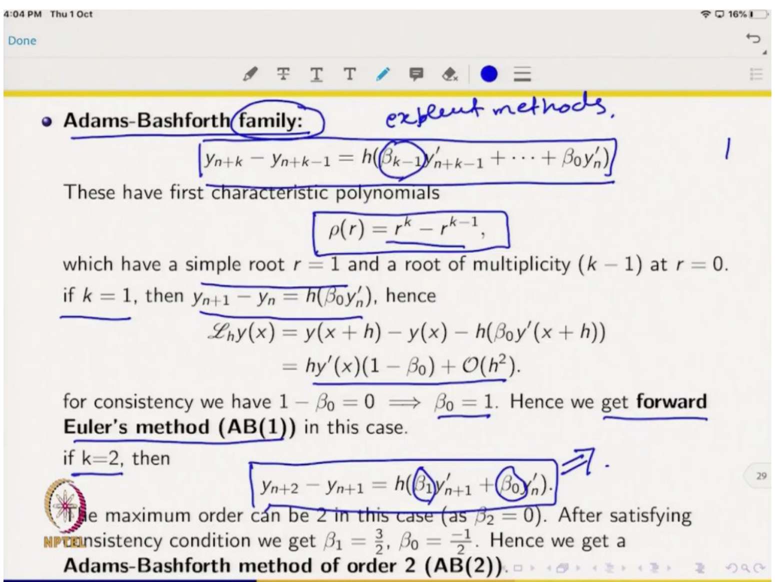

So now, let me try for k = 2, then the our difference equation will be this and if you remember from the previous lecture, we have separately derived this kind of a difference equations, and we proved we found β_0, β_1 in such a way that the method should be consistent and the first characteristic polynomial of this difference equation is automatically satisfying the root conditions, that is what we have in fact said in a more general case, so in a particular case, of course, that condition will always be true.

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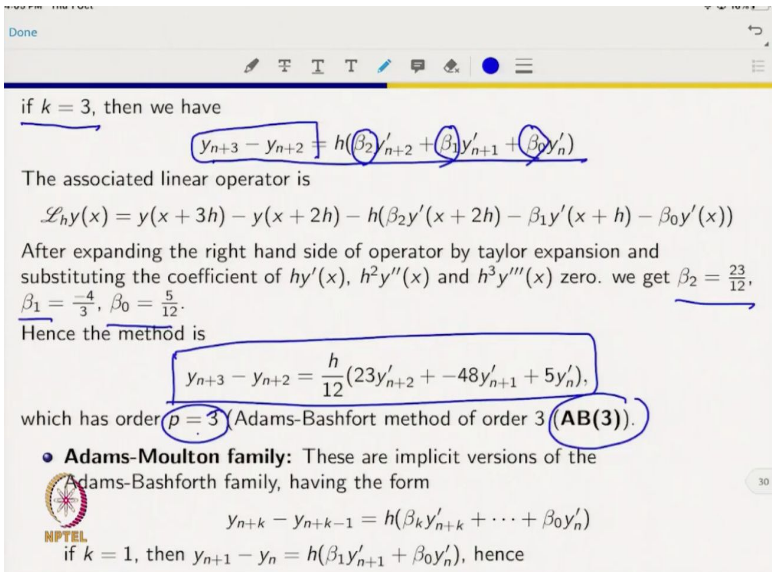

Now, if I say k = 3, then the difference equation will be this and we will find out these coefficients. So, that the resulting method is consistent, the first characteristic polynomial is automatically satisfying the root conditions. So, if we get the following coefficients and finally, the difference equation of a Three step Adam Bashforth method will be this. So, this is the first time we are looking at the exact form of a Three step Adam Bashforth method, which has an order p.

Order also you can determine with the order of the consistency. This is also called AB(3) method, that is what this is just an abbreviation AB(1), AB(2), AB(3).

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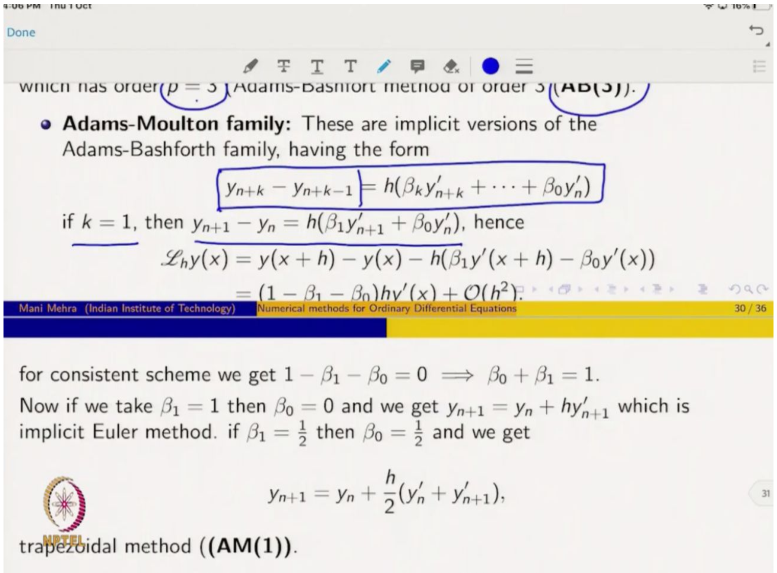

Now, let me come to the next family which we call it as the Adams Moulton family. These are the implicit version of Adam Bashforth family having the following form. So, again we will work out for some special cases if k = 1 the difference equation will be this and hence, we will find out the coefficient in such a way that the resulting differences in becomes consistent while the stability polynomial in this case will be again satisfying the root conditions which you can observe from left hand side. So, we get this implicit Euler method.

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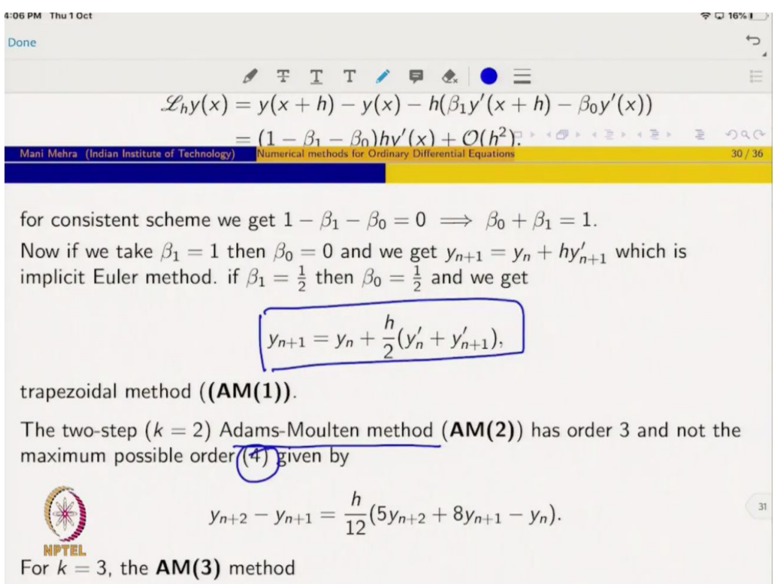

So, finally, we get this method which we call it as AM(1). One step methods, the two step Adams Moulton methods has ordered 3 not the maximum possible order k. How are you determining the maximum possible order k?

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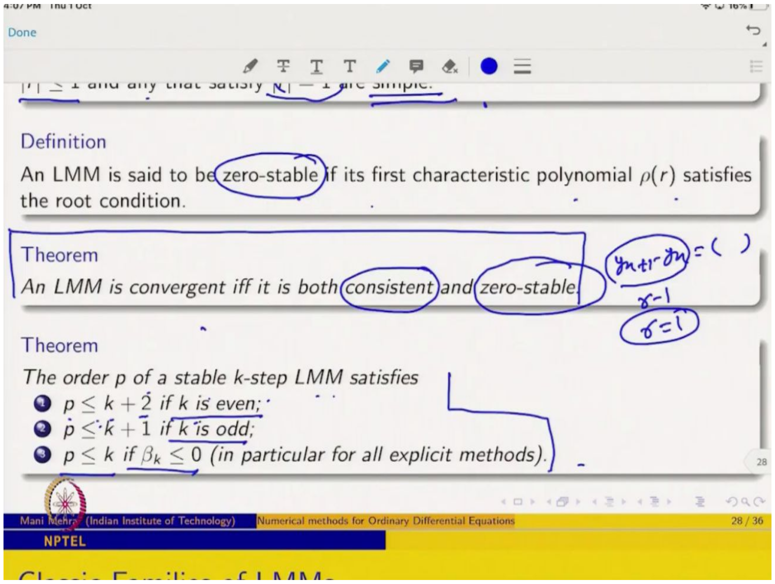

Because, if it is a two step you cannot see from the following theorem if it is to step means k is even and k is even two steps so k is 2, so p should be less than equal to 4. So, the order of the method should be either 4 or less than but so maximum is 4.

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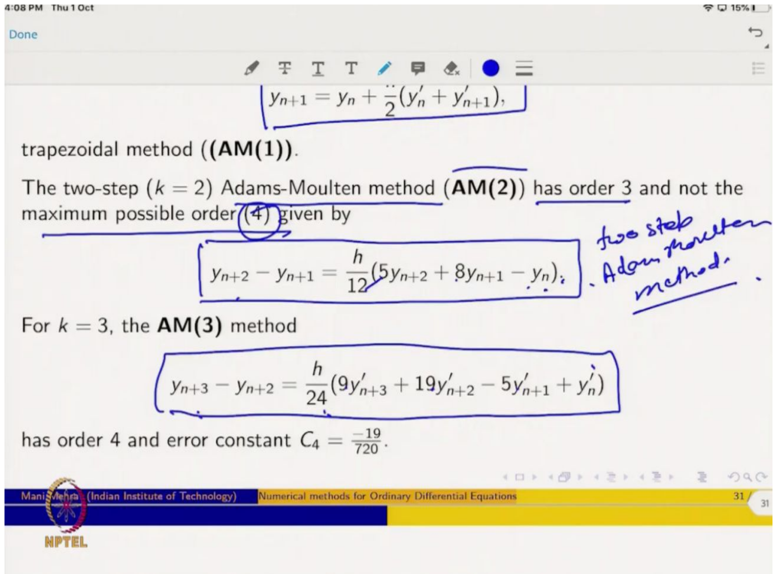

So, but in this case we are getting it should be three has order 3 not the maximum possible order 4 and the difference equation is given by the following formula. So, again this is a Two step Adams Moulton method for k = 3 the difference equation will become. So, you can take both the things you can take as exercise and try to find out the coefficient because I am directly giving you the difference equation how it will look like. So, both of them are exercises. you have to prove how these coefficients are 5/12, 8/12 - 1/12 in this case 9/24, 19/24 -5/24 and 1/24.

So, these two things you can take as exercise to prove what will be the Two step Adams Moulton method and what will be the Three step Adams Moulton method. In fact, if you want more exercise, you can also prove for k = 4, 5, 6 it is up to you. But what is the disadvantage of if you keep adding the steps there will be a problem it will not be self starting till that point, if it is a Three step, so you have to start you have to initially choose three values y0, y1 y2, so that is the problem. And that is the cost we are paying to not to compute higher order derivatives. So, of course, we have to pay some cost.

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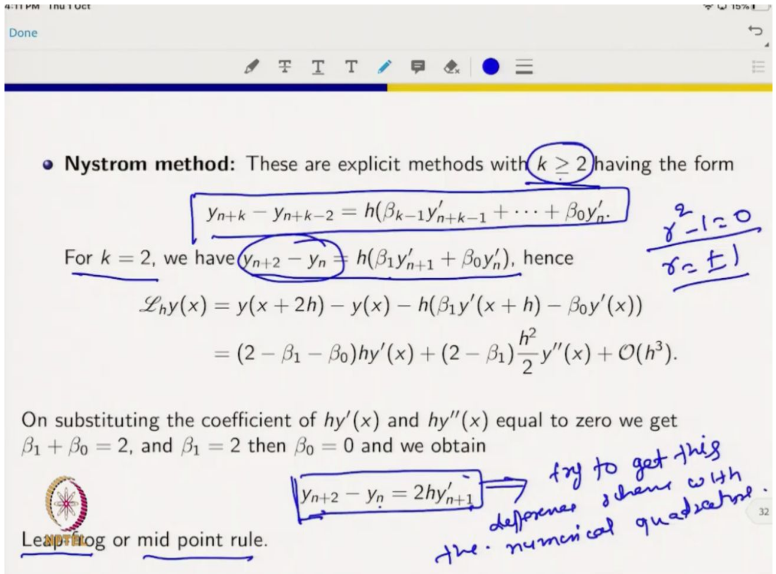

And the next is Nystrom method, these are explicit methods with k greater than or equal to 2 having the following form. So, again as I have already mentioned k should be greater than or equal to 2. So, for k = 2 we will have the following form. So, again if you look at the first characteristics polynomial will be this. So, the roots of this polynomial will be this, so it automatically satisfies the root conditions.

And coefficient will be determined in such a way that the scheme is consistent. So finally, I end up with this. Which is again a different method in the sense we have not seen so far and it is called Leap Frog or Midpoint rule. This is an example of a two step method again because n, n+1 and n+2 it is explicit because the left hand side term involves n+2 while in the right hand side there are no terms which involve any value at n+2.

And why is it called a Midpoint? Because if you try to find out these differences schemes in the numerical quadrature you will be able to justify, so, this is again you can take it as exercise, you try to obtain this differences scheme with the help of numerical quadrature formula. So, try to let me write down here and try to get this difference scheme with the help of numerical quadrature. Because in the last lecture, you can recall I have derived a couple of his schemes with the alternate wave that first numerical quadrature formula and that is how I also justified why the Trapezoidal method was called the Trapezoidal method.

In the same way you can justify this also why it is called the Midpoint rule, so that you can take it as an exercise.

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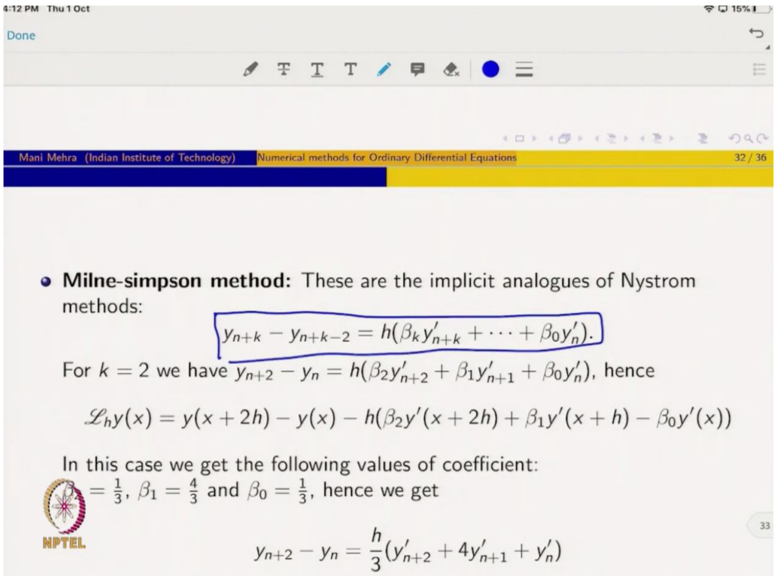

So, the next is Milne Simpsons method. These are again implicit analogous of Nystrom methods like initially we started with Adam Bashforth, Adams Moulton where the implicit version of Adam Bashforth. Similar to how we just said Nystrom methods and then these are the implicit versions of a Nystrom method. So, the differences scheme will look like in the following form.

So again, from the left hand side, you can observe that first characteristics polynomial we will be automatically satisfying the root condition because there is no difference in the left hand side

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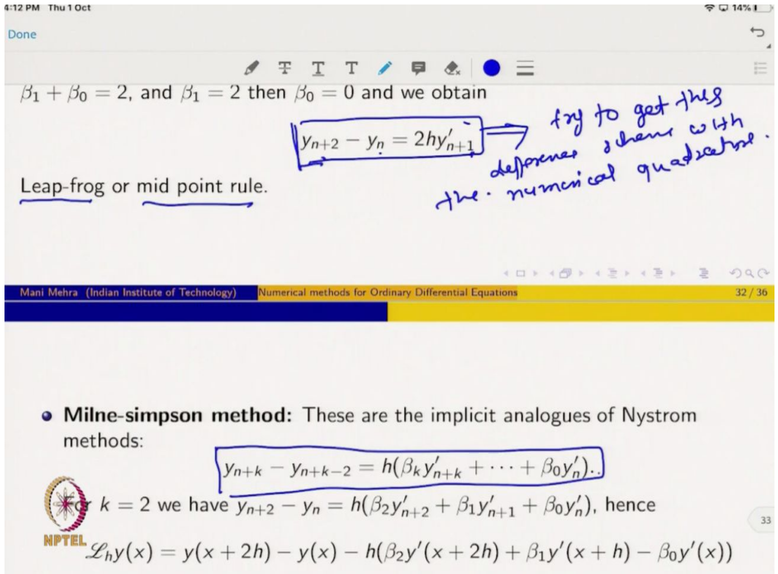

because this is just the implicit analogues of the Nystrom method. So the difference will lie in the right hand side.

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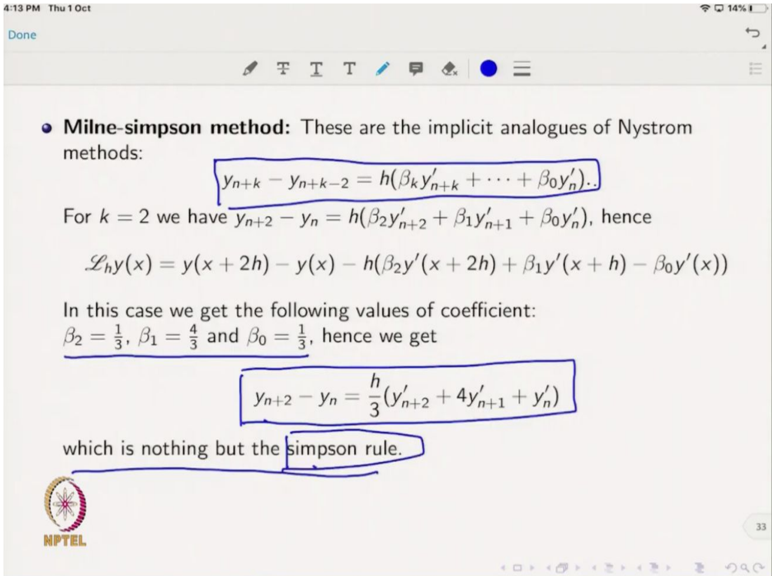

So in fact, I do not need to see anything about the Zero stability and then the coefficient will be chosen in such a way that the resulting differences scheme will become consistent. So finally, for k = 2 we get the following values of coefficient after collecting the terms. Because in all these examples, you can try to find out if you can take it as exercise and try to find out these coefficients the same way in the last lecture. I have done a couple of examples.

So, hence we get the following difference equations. So, there are, so to determine these coefficients you have to choose your order accordingly, which is nothing but the Simpson rules. So, here again let me see why it is called Simpson rule that also you can justify once you start driving the same formula with the help of numerical quadrature. So, again point which I am mentioning here or you can say you can guess whether it is related to Simpson rule of numerical quadrature or not, that is what you can check.

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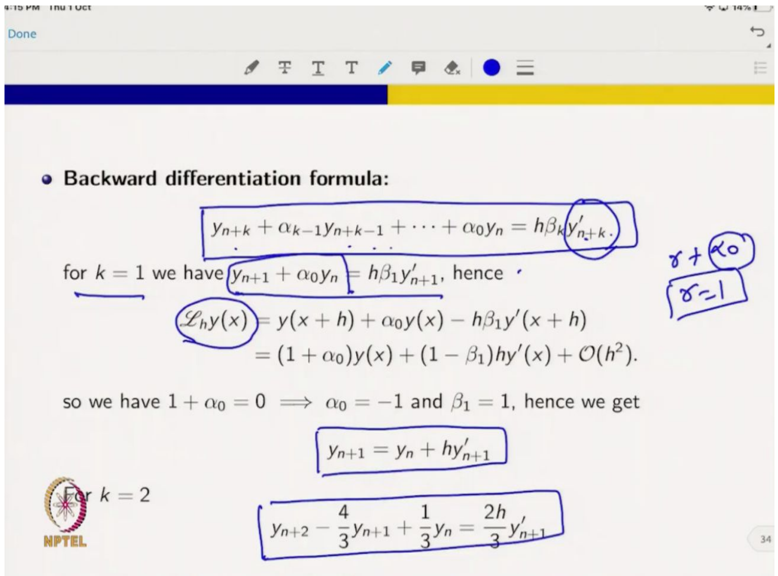

And the next is Backward differentiation formula. So, in Backward differentiation formula the scheme will be in the following way. So, for k = 1 we have this and hence associated linear difference operator we will write, so that we can determine the coefficients in such a way that the scheme is consistent and what will be the form of the first characteristic polynomial that also you can observe in the following case.

So, that root is $r + \alpha_0$. So, α_0 We are choosing from here also $\alpha_0 = -1$ so root is 1. So, means it satisfied the Zero stability as well and the scheme will be this for k is equal to this, you end up with this. So, basically in the Backward differentiation formula we are using on the value of y'_{n+k} we are not using the other values that is the difference if you observe from here to the previous cases.

And while in the left hand side we involve y_{n+k}, y_{n+k-1}, y_n So, that is a different template altogether. So, basically we have seen four categories and in all four categories, we have seen different templates. So, sometimes for the same order, suppose you wanted to work with the third order numerical method. So, you may end up with the different templates. So, in that case you have to choose whether you want to use explicit method, implicit method.

So far I have shown you one disadvantage of implicit method when we were working with the set of differential equations and I also explained or you can choose among single step methods, two step methods though I have also shown you the advantage, disadvantage of multi step methods.

So, means it is up to you how you choose which you want. So, with this, I am stopping at this point of time and in the next lecture, I will try to show you some of the examples how it works numerically with the help of Matlab software. Thank you very much for your attention.